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Abstract

This paper analyzes the determinants of asset replacement investment decisions with maintenance and operating cost uncertainty governed by a mixed modified fractional Brownian motion. It addresses an important issue in investment decision-making and offers an innovative approach using mixed modified fractional Brownian motion. The contingent claims method from the real options literature providing techniques needed to incorporate uncertainty into replacement investment decisions. Following Mauer et Ott [4], we note that the optimal time of replacement of the fixed assets in a firm depends on the policy of optimal replacement which is a critical level of the costs of maintenance and exploitation. The optimal replacement policy is obtained as a function of the present average value of the maintenance and operating costs of the assets. By assuming that the cost of operating and maintenance assets follow a Mixed Modified Fractional Brownian Motion (MMFBM) [6], the optimal replacement policy is minimal and therefore it encourages the firm to replace more of its assets and spend less their maintenance, depending on the MMFBM parameters such as Hurst coefficient. At the end we notice that when the Hurst parameter increases, the optimal replacement policy and the maximum value of the present average value function of asset costs decrease.

Keywords : Real Option, Replacement; Maintenance Costs; Operational Costs; Mixed Modified Fractional Brownian Motion; Investment Decision; Hurst parameter; Optimal replacement policy.

1 Introduction

The asset curve in the financial market describes the relationship between the local market trend and market volatility. This is a most important concept in pricing options in a market. Investors have a lot of interest in researching this curve and fundamental data, hence researching how to optimize investments on these actions has become an important question in the literature. So, calculating the optimal time between replacements of fixed assets in businesses has been and will continue to be a very relevant issue in asset valuation. In the financial market, there are several finance models including cooperative finance which was created in 1958 thanks to the various works and achievements made by Modigliani and Miller [19].They established an independent investment and financing decisions of a company within the framework of perfect capital markets. In their analysis, they demonstrated that the firm is a portfolio of investment projects, generating random income on which each investor has the same information. Over time, financial markets have evolved and become increasingly complex and risky. In finance, investment consists of immobilizing capital, that is to say, incurring an immediate expense, with the aim of obtaining a gain over several successive periods. This expense can be incurred by the company for different reasons: launching new products, increasing production capacity, improving the quality of products and services, reducing production costs, etc... Thus we distinguish several types of of investments according to their objectives,namely modernization or productivity investments which allow the company to increase its production by introducing modern and refined equipment; replacement or renewal investments which aim to maintain the firm at its current level; capacity or expansion investments which make it possible t increase the production capacity of the company by, for example, adding production units, whether of an already existing product(quantitative expansion) or of a new product (qualitative expansion)etc... The last two types of investments are in practice very useful for entrepreneurs because we always seek to determine the best period to replace worm out assets and this is what is common in all production plants.Investors are not often indifferent to the presence of uncertainty. Since it is difficult to account for uncertainties, existing methods for analyzing the determinants of investment decisions lead to erroneous results. McDonald and Siegel [17] show that increasing uncertainty also increases the option value of waiting to invest and this discourages current investments.

In 2013, Mauer and Ott [4] analyze the determinants of replacement investment decisions in a contingent claims model with uncertainty linked to the maintenance and operating cost. However, they propose a model dependent on the maintenance and operating cost (C_t) which follows a standard Brownian motion (B_t) representing the disturbance of the share price in the financial market. This model is defined by:

$$dC_t = \mu C_t dt + \sigma C_t dB_t, \tag{1}$$

where μ et σ are respectively the instantaneous drift and volatility of the process $\{C_t,\ t >$

0}.

In their analyzing, they show that cost uncertainty increases the replacement expectation value and subsequently discourages replacement investments.

However, the model proposed by Mauer and Ott [4] does not make it possible to property minimize the maintenance and operating costs of assets in companies an despite everything the interest of the process which drives their model, it presents limits in particular because that it does not have any long-term dependence(i.e does not allow us to properly model the behavior of long-term phenomena).

In this context, the problem of taking into account cost uncertainty assets, the notion of long memory an self-similar processes in investment is a mojor concern and and essential criterion for success and analysis replacement investment decisions. It is with this in mind that this article.

It will be a question for us in this paper to propose a model whose process which governs the cost of maintenance and operating has the properties of self-similarity and long memory.

This article uses the contingent claims method from the literature on real options which provides the techniques necessary to incorporate uncertainty into replacement investment decisions.

The objective is to determine the optimal replacement policy and to deduce the asset replacement cycle in a company in order to make a comparison of results with those found by Mauer and Ott [4].

2 Preliminaries on investment strategies in a company

This section is essentially based on recalling a few notions concerning investment strategies and the contingent claim method in the litterature.

Definition 1 ***(Investment)*** *Investment is a commitment of funds intended to acquire assets(tangible/intangible)with the aim of obtaining a satisfactory future income.*

Definition 2 ***(Flow)*** *The flow is the set of real and monetary exchanges occurring between the various agents of economic life.*

Definition 3 ***(Depreciation)*** *Depreciation is an accounting term defining the loss of value of a company's fixed asset.*

Definition 4 ***(Portfolio)*** *A portfolio is the set of financial assets (options, stocks, bonds, etc) held by investors.*

Definition 5 ***(Discount rate)*** *The discount rate is the weighted average cost of capital of the investment; It is the average of cost of the different financing used by the company.*

Definition 6 ***(Discounted Average Cost Function Function)*** *We call Discounted Average Cost Function, the total cost of obtaining the service required to operate equipment.*

Proposition 1 *The Discounted Average Cost Function of equipment is given by:*

$$F_m(x) = \mathbb{E}\left[\int_0^\infty e^{-lt} x_t dt | x_0 = x\right], \tag{2}$$

where x_0 is the state of the asset at time zero, which may or not be the initial state, x the cost of maintaining the equipment? and l the appropriate risk-adjusted discount rate associate with the cost x.

Definition 7 ***(Contingent claim method)*** *The contingent claim method is considered a generalization of option pricing theory with the aim of specifying a framework within which all contingent claims can be valued.*

It is based on three simple principles:

- The value of liabilities derives from assets;
- Liabilities have different seniority and therefore have different risks linked to their seniority;
- The existence of an element of randomness to how the vale of assets changes over time.

3 Mixed modified fractional Brownian motion

This section recalls the Mixed Modified Fractional Brownian Motion(MMFBM) which is the modification of the mixed fractional Brownian motion and the basic notions of options theory. These different notions recalled in this part are contained in [6], [29] and [14].

Definition 8 *[6]*
Let $H \in]\frac{1}{2}, 1[$ *and* $\lambda > 0$. *A process MMFBM parameters* a, b, λ *and* H, *denoted* $M^{H,\lambda,a,b} = (M_t^{H,\lambda,a,b})_{t\in[0,T]} = (M_t^{H,\lambda})_{t\in[0,T]}$ *is a linear combination of process* B *and process* $B^{H,\lambda}$, *defined on* $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ *by*

$$\forall\ t \in [0,T],\ M_t^{H,\lambda} := M_t^{H,a,b,\lambda} = aB_t + bB_t^{H,\lambda}. \tag{3}$$

In (3), *the process* $B^{H,\lambda}$ *is the mixed fractional Brownian motion and the parameter* λ *is the adjustment coefficient for the time between two quotes on the financial market.*

Using the differentiation formula for Fractional Brownian motion introduced by [15], we have:

$$dB_t^{H,\lambda} = \overline{\mu}\phi_t^\lambda dt + \lambda^{\overline{\mu}} dB_t, \tag{4}$$

which leads to the following lemma:

Lemma 1 *[29]*
Let $H \in]\frac{1}{2}, 1[$. *For any* $\lambda > 0$, *the Mixed modified fractional Brownian motion (MMFBM) is the continuous* $\mathcal{F}_t$*-semi-martingale with the following decomposition :*

$$M_t^{H,\lambda} = b\int_0^t \overline{\mu}\phi_s^\lambda ds + \left(a + b\lambda^{\overline{\mu}}\right) dB_t. \tag{5}$$

where

$$\phi_t^\lambda = \int_0^t (t - s + \lambda)^{\overline{\mu}-1}\, dB_s. \tag{6}$$

The following lemma studied the convergence of the process defined by (3) in space $L^2(\Omega)$.

Lemma 2 *[6]*
The process $M_t^{H,\lambda}$ *converges towards* M_t^H *in space* $L^2(\Omega)$ *for any* $t \in [0;T]$, *when* $\lambda \to 0$.

Definition 9 *[6] Let* $H \in]\frac{1}{2}, 1[$ *and* $\lambda > 0$. *A MMFBM is a process of a model in which the price of the underlying asset noted* S_t^λ *is described by the SDE*

$$dS_t^\lambda = \mu S_t^\lambda dt + S_t^\lambda \sigma dM_t^{H,\lambda},\ S_0^\lambda > 0, \tag{7}$$

where $M_t^{H,\lambda}$ *is defined by* (3).

Remark 1 *For* $H \in]\frac{1}{2}, 1[$ *and* $\lambda > 0$, *the model defined by* (7) *is without arbitrage in the market* $\mathcal{M}$.

Theorem 1 *[6] Let $H \in]\frac{1}{2}, 1[$ and $T > 0$, under the assumptions $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$, the equation (7) admits a unique solution for any $t \in [0, T]$.*

Theorem 2 *[6]*
Let $H \in]\frac{1}{2}, 1[$ and $\lambda > 0$. The solution of equation (7) is:

$$S_t^{\lambda} = S_0^{\lambda} \exp \left\{ \mu t - \frac{1}{2} \left(a + b\lambda^{\overline{\mu}} \right)^2 \sigma^2 t + \sigma M_t^{H,\lambda} \right\}. \tag{8}$$

4 Decision to replace a process with memory

4.1 Motivation

The evolution of financial markets has revealed probability theory as the fundamental tool of financial mathematics. In this paper, we analyze the determinants of replacement investment decisions with uncertainty in maintenance and operating costs governed by a mixed modified Fractional Brownian motion.

Consider a firm F operating an asset A which produces a fixed level of production for a maintenance and operating cost C that varies over time. Let C_N denote the initial cost of maintaining and operating the asset. C is a measure of deterioration of the asset which evolves over time in the form of a geometric mixed modified Fractional Brownian motion whose equation is given by:

$$dC_t = \mu C_t dt + \sigma C_t dM_t^{H,\lambda}, \tag{9}$$

where:

- μ is the instantaneous drift rate;
- σ is the instantaneous volatility rate;
- $dM_t^{H,\lambda}$ is the increment of the mixed modified Fractional Brownian motion;
- H the Hurst parameter belonging to the interval $]\frac{1}{2}, 1[$.

The model (9) is obtained by substituting model (10) for model (12). This new basic model highlights:

- maintenance and operating cost;
- salvage value;
- tax effects.

4.2 Old results oriented

In 2019, Djeutcha et al. [6] proposed a self-similar process defined by:

$$M_t^{H,\lambda} := M_t^{H,a,b,\lambda} = aB_t + bB_t^{H,\lambda}, \tag{10}$$

where $t \in [0,T]$; a, b, λ are constants; H is Hurst coefficient parameter; B_t is standard Brownian motion and $B_t^{H,\lambda}$ is the modified Fractional Brownian motion.

Lemma 3 *[6] The geometric Fractional Brownian $C = (C_t)_{t\in[0,T]}$ defined by:*

$$C_t^0 = C_N \exp\left\{\mu t - \frac{1}{2}\left(a + b\lambda^{\overline{\mu}}\right)^2 \sigma^2 t + \sigma M_t^{H,\lambda}\right\}, \tag{11}$$

is solution of the FSDE (9)*, where* $\overline{\mu} = H - \dfrac{1}{2}$.

The two constants σ and μ are deterministic functions defined on $\mathbb{R}_+^*$.

In 2013, Mauer and Ott [4] propose a model dependent on the maintenance and operating cost (C_t) which follows a standard Brownian motion (B_t) representing the disturbance of the stock price in the financial market. This model is defined by:

$$dC_t = \mu C_t dt + \sigma C_t dB_t, \tag{12}$$

where μ and σ respectively the drift and the instantaneous volatility of the process $\{C_t,\ t > 0\}$.

In their analysis, they show that cost uncertainty increases the expected replacement value and subsequently discourages replacement investments.

However, the model proposed by Mauer and Ott [4] does not make it possible to properly minimize the costs of maintenance and operating assets in companies and despite all the interest in the process that drives their model, it has limitations, particularly due to the fact that it does not have long-term dependence(i.e it does not allow the behavior of long-term phenomena to be well modeled).

4.3 New results

In the model (9), we assume that all assets have the same maintenance and operating cost and we admit that the firm concentrates on operating stochastically equivalent assets. In this condition, the remaining tax book value of asset A is given by the following result:

Proposition 2 *The remaining tax book value of asset A which depreciates exponentially over time at depreciation rate δ is given by:*

$$V_r = P(1-\varphi)e^{-\delta t}, \tag{13}$$

or P is the initial cost of a new asset and φ the investment tax credit.

Proof *:* The net purchase price of asset A is given by:

$$p_n = P - \varphi P = P(1-\varphi),$$

which here represents the decreasing annuity.

Given that the asset depreciates exponentially at rate δ, then the declining rate at time $t > 0$ is given by:

$$t_d = e^{\delta t}.$$

So, the remaining tax book value of asset is then:

$$\begin{aligned} V_r &= \frac{p_n}{t_d} \\ &= \frac{P - \varphi P}{e^{\delta t}} \\ &= P(1-\varphi)e^{-\delta t}. \end{aligned}$$

Thus $V_r = P(1-\varphi)e^{-\delta t}$.

Given that C is the measure of deterioration of asset A (i.e. it characterizes the state of A)then, the fundamental result below gives the time of passage from the initial measure of deterioration to the measure of deterioration at time t.

Theorem 3 *Using the equation* (11) *of lemma* (3)*, the first expected transition time of the asset deterioration measure from of asset level C_N to level C_t is given by:*

$$E[\,\bar{t}\,] = Z^{-1} ln\left(\frac{C_t}{C_N}\right), \tag{14}$$

with $Z = \mu - \frac{1}{2}\sigma^2(a + b\lambda^{\overline{\mu}})^2$.

Proof *:* It is a question for us of determining the average time spent by the process C, when it passes from C_N to C_t.

Now the first time of passage from C_N to C_t of the process C is given by:

$$\inf\left\{t > 0; C_t^0 \geq C_t\right\},$$

where C_t^0 is the solution of the equation (9).

So, we have:

$$\begin{aligned}
\inf\left\{t>0; C_t^0 \geq C_t\right\} &= \inf\left\{t>0; C_N \exp\left\{\mu t - \frac{1}{2}\left(a+b\lambda^{\overline{\mu}}\right)^2 \sigma^2 t + \sigma M_t^{H,\lambda}\right\} \geq C_t\right\} \\
&= \inf\left\{t>0; \exp\left\{\mu t - \frac{1}{2}\left(a+b\lambda^{\overline{\mu}}\right)^2 \sigma^2 t + \sigma M_t^{H,\lambda}\right\} \geq \frac{C_t}{C_N}\right\} \\
&= \inf\left\{t>0; \mu t - \frac{1}{2}\left(a+b\lambda^{\overline{\mu}}\right)^2 \sigma^2 t + \sigma M_t^{H,\lambda} \geq \ln\left(\frac{C_t}{C_N}\right)\right\} \\
&= \inf\left\{t>0; \mu t - \frac{1}{2}\left(a+b\lambda^{\overline{\mu}}\right)^2 \sigma^2 t + \sigma \mathbb{E}(M_t^{H,\lambda}) \geq \ln\left(\frac{C_t}{C_N}\right)\right\} \\
&= \inf\left\{t>0; \mu t - \frac{1}{2}\left(a+b\lambda^{\overline{\mu}}\right)^2 \sigma^2 t \geq ln\left(\frac{C_t}{C_N}\right)\right\} \\
&= \inf\left\{t>0; \left(\left(a+b\lambda^{\overline{\mu}}\right)^2 \sigma^2\right) t \geq ln\left(\frac{C_t}{C_N}\right)\right\} \\
&= \inf\left\{t>0; t \geq \frac{ln\left(\frac{C_t}{C_N}\right)}{\mu - \frac{1}{2}\left(a+b\lambda^{\overline{\mu}}\right)^2 \sigma^2}\right\} \\
&= \inf\left\{t>0; t \geq (\mu - \frac{1}{2}\left(a+b\lambda^{\overline{\mu}}\right)^2 \sigma^2)^{-1} ln\left(\frac{C_t}{C_N}\right)\right\} \\
&= \inf\left\{t>0; t \geq Z^{-1} ln\left(\frac{C_t}{C_N}\right)\right\},
\end{aligned}$$

with $Z = \mu - \frac{1}{2}\left(a+b\lambda^{\overline{\mu}}\right)^2 \sigma^2$.
Then we have:

$$E[\,\bar{t}\,] = Z^{-1} ln\left(\frac{C_t}{C_N}\right).$$

Hence the result.

An immediate consequence of the above result makes it possible to obtain the approximate tax book value of asset A which is defined by:

Corollary 1 *The tax book value of asset A obtained in equation* (14) *of theorem* (3) *be approximated by:*

$$V_a = P(1-\varphi)\left(\frac{C_t}{C_N}\right)^{-\frac{\delta}{Z}}. \tag{15}$$

Proof *:* Using the theorem 3 above and replacing in equation (13) of proposition (2) the time t by $E[\,\bar{t}\,]$ obtain in equation (14) of theorem (3) , we have:

$$\begin{aligned} P(1-\varphi)e^{-\delta t} &= P(1-\varphi)e^{-\delta E(\bar{t})} \\ &= P(1-\varphi)e^{-\delta Z^{-1} ln\left(\frac{C_t}{C_N}\right)} \\ &= P(1-\varphi)e^{-\frac{\delta}{Z} ln\left(\frac{C_t}{C_N}\right)} \\ &= P(1-\varphi)e^{ln\left(\frac{C_t}{C_N}\right)^{-\frac{\delta}{Z}}} \\ &= P(1-\varphi)\left(\frac{C_t}{C_N}\right)^{-\frac{\delta}{Z}} \end{aligned}$$

So the approximate tax book value is:

$$V_a = P(1-\varphi)\left(\frac{C_t}{C_N}\right)^{-\frac{\delta}{Z}}.$$

Hence the result.

Remark 2 *: Although Equation* (15) *of corollary* (1) *is an approximation of the true tax book value found at* (13) *in proposition* (2)*, the importance of this approach is that it reduces the number of state variables in the determination problem analysis of the present value function of costs.*

Now, it is a question for us of defining the tax shield for the depreciation of asset A. It is important to know that the tax shield is achieved in an infinitely short time. Hence the following result:

Proposition 3 *Te tax shield for depreciation of asset A in interval* $[t, t+dt]$ *is defined by:*

$$B_f = \tau\delta P(1-\varphi)(\frac{C_t}{C_N})^{-\frac{\delta}{Z}}dt. \tag{16}$$

Proof *:* The depreciation tax shield of asset A in the time interval $[t,\ \ t+dt]$ is equal to the product of the Reference Tax Income(RTI) of asset A in the time interval [t, t+dt] by the depreciation rate δ.

Furthermore, the RTI of asset A in the interval $[t\,t+dt]$ is the product of the remaining tax book value of asset A at time t, the length Δ_t of the interval $[t,\ \ t+dt]$ and the tax τ on its performance.

So, we have:

$$
\begin{aligned}
B_f &= RTI \times \delta \\
&= V_r \times (\tau) \times (\delta) \times \Delta_t \\
&= V_r \times (\tau) \times (\delta)(t + dt - t) \\
&= P(1-\varphi)(\frac{C_t}{C_N})^{-\frac{\delta}{Z}} \times (\tau) \times (\delta) \times (dt) \\
&= \tau\delta P(1-\varphi)(\frac{C_t}{C_N})^{-\frac{\delta}{Z}} dt
\end{aligned}
$$

Hence we obtain:

$$B_f = \tau\delta P(1-\varphi)(\frac{C_t}{C_N})^{-\frac{\delta}{Z}} dt,$$

where τ is the tax levied on the return of A and δ is the depreciation rate.

Since we said above that the after-tax cost C changes over time following a Mixed Modified Fractional Brownian Motion, then when C reaches some unknown level, the firm will stop operating or using the asset A, will sell it on a secondary market and replace it with another stochastically equivalent asset A' with the same initial maintenance and operating cost C_N. In this case, the company's problem is to minimize the after-tax cots operating costs of the asset by determining this unknown level of C denoted $\overline{C}$ which is defined here as the optimal replacement policy.

If we denote for example $V_m(C)$ the average present value of the after-tax costs resulting from the optimal replacement policy, we have according to Mauer and Ott [4], the following result:

Proposition 4 *The average present value of the after-tax costs resulting from the optimal replacement policy is given by:*

$$V_m(C) = \min_{\overline{C}} \left\{ \mathbb{E}\left[\int_0^\infty e^{-\xi t} \left\{ C_t(1-\tau) - \tau\delta P(1-\varphi)\left(\frac{C_t}{C_N}\right)^{-\frac{\delta}{Z}} \right\} dt | C_0 = C \right] \right\}, \tag{17}$$

where ξ is the appropriate risk-adjusted discount rate for the cost, and C_0,state of the asset at time zero, which may or may not be C_N.

Proof *:* The present value function of the after-tax costs of an asset is given by:

$$K(C) = \mathbb{E}\left[\int_0^\infty e^{-\xi t} B(C,t) dt | C_0 = C \right], \tag{18}$$

where $B(C,t)$ is the net cost of operating an asset less the tax effect on this cost.

Determine B(C,t).

The maintenance and operating cost of an asset is given by:

$$C(1-\tau),$$

where C is the net maintenance an operating cost of asset and τ the tax levied on its performance.

Furthermore , the tax effect of asset is given according the equation (16) of proposition (3) by:

$$\tau\delta P(1-\varphi)\left(\frac{C_t}{C_N}\right)^{-\frac{\delta}{Z}}.$$

Ths we have:

$$B(C,t) = C_t(1-\tau) - \tau\delta P(1-\varphi)\left(\frac{C_t}{C_N}\right)^{-\frac{\delta}{Z}}. \tag{19}$$

Replacing (19) in (18), we obtain:

$$K(C) = \mathbb{E}\left[\int_0^{\infty} e^{-\xi t}\left\{C_t(1-\tau) - \tau\delta P(1-\varphi)\left(\frac{C_t}{C_N}\right)^{-\frac{\delta}{Z}}\right\} dt|C_0 = C\right]. \tag{20}$$

Given that company's problem is to minimize the after-tax operating and maintenance costs of asset, the average present value of the after-value costs resulting from the optimal replacement policy is given by:

$$V_m(C) = \min_{\overline{C}} K(C). \tag{21}$$

Replacing (20) in (21), we obtain :

$$V_m(C) = \min_{\overline{C}}\left\{\left[\int_0^{\infty} e^{-\xi t}\left\{C_t(1-\tau) - \tau\delta P(1-\varphi)\left(\frac{C_t}{C_N}\right)^{-\frac{\delta}{Z}}\right\} dt|C_0 = C\right]\right\}. \tag{22}$$

Hence the result. Using the computational techniques proposed by Pindyck[22] to eliminate the appropriate risk- adjusted discount rate for the cost ξ, we have the following result:

Theorem 4 *The average discounted value $V_m(C)$ of the after-tax costs resulting from the optimal replacement policy is solution of the non-homogeneous differential equation defined by:*

$$\frac{1}{2}\sigma_H^2 C^2 V_{CC} + \mu^* C V_C + C(1-\tau) - \tau\delta P(1-\varphi)\left(\frac{C}{C_N^0}\right)^{-\frac{\delta}{Z}} - rV = 0, \tag{23}$$

where $\sigma_H^2 = \left(a + b\lambda^{H-\frac{1}{2}}\right)^2\sigma^2$; $\mu^* = \mu - \eta\rho\sigma$ is the risk-adjusted cost drift rate and r is the risk-free interest rate of asset A.

Proof :

Let asset A whose cost C follows the model equation (9), r_C be the expected return adjusted according to of the risk on C.

We have:

$$r_C = r + \eta\rho\sigma, \tag{24}$$

where η is the market price of the risk; ρ the instantaneous correlation between cost and the systematic pricing factor; r the risk-free interest rate and σ the instantaneous volatility rate.

Consider a portfolio whose holding of the investment opportunity has a value of $V(C)$, and that n units of assets of this portfolio have been sold short with a maintenance and operating price C. The value of this portfolio is given by:

$$\phi = V(C) - nC. \tag{25}$$

Since the expected growth rate of C is strictly less than the expected risk-adjusted return on C, then the short position will require a payment flow between times t and $t + dt$ defined by

$$\varphi(t) = n(r_C - \mu)Cdt, \tag{26}$$

and holding the investment opportunity must involve a payment flow over time denoted $B(C, t)$ which is the net cost of the portfolio less the tax effect on this cost. It is defined by:

$$B(t, C) = C(1 - \tau) - \tau\delta P(1 - \varphi)\left(\frac{C}{C_N}\right)^{-\frac{\delta}{Z}}. \tag{27}$$

The infinitesimal variation of this portfolio is given by:

$$d\phi = dV - ndC - \varphi(t) + B(C, t)dt. \tag{28}$$

According to the fractional lemma of Itô, we have :

$$dV = \left[\left(a + b\lambda^{H-\frac{1}{2}}\right)^2 \frac{\partial^2 V}{\partial^2 C}\sigma^2 C^2 + \mu C\frac{\partial V}{\partial C}\right] dt + \sigma C\frac{\partial V}{\partial C} dM_t^{H,\lambda}. \tag{29}$$

Replacing (29), (26) in (28), we have:

$$\begin{aligned}
d\phi &= dV - ndC - \varphi(t) + B(t, C)dt \\
&= \left[\left(a + b\lambda^{H-\frac{1}{2}}\right)^2 \frac{\partial^2 V}{\partial^2 C}\sigma^2 C^2 + \mu C\frac{\partial V}{\partial C}\right] dt + \sigma C\frac{\partial V}{\partial C} dM_t^{H,\lambda} \\
&- n(\mu Cdt + \sigma CdM_t^{H,\lambda}) - n(r_C - \mu)Cdt + B(t, C)dt.
\end{aligned}$$

So, we obtain :

$$
\begin{aligned}
d\phi &= \left[\left(a+b\lambda^{H-\frac{1}{2}}\right)^2 \frac{\partial^2 V}{\partial^2 C}\sigma^2 C^2 + \mu C\frac{\partial V}{\partial C}\right]dt + \sigma C\frac{\partial V}{\partial C}dM_t^{H,\lambda} \\
&- n(\mu C dt + \sigma C dM_t^{H,\lambda}) - n(r_C - \mu)C dt + B(t,C)dt. \qquad (30)
\end{aligned}
$$

To eliminate the risk of portfolio, we set:

$$n = \frac{\partial V}{\partial C}. \qquad (31)$$

Replacing (31) in (30), we obtain:

$$
\begin{aligned}
d\phi &= \left[\left(a+b\lambda^{H-\frac{1}{2}}\right)^2 \frac{\partial^2 V}{\partial^2 C}\sigma^2 C^2 + \mu C\frac{\partial V}{\partial C}\right]dt + \sigma C\frac{\partial V}{\partial C}dM_t^{H,\lambda} \\
&- n(\mu C dt + \sigma C dM_t^{H,\lambda}) - n(r_C - \mu)C dt + B(t,C)dt \\
&= \left[\left(a+b\lambda^{H-\frac{1}{2}}\right)^2 \frac{\partial^2 V}{\partial^2 C}\sigma^2 C^2 + \mu C\frac{\partial V}{\partial C}\right]dt + \sigma C\frac{\partial V}{\partial C}dM_t^{H,\lambda} \\
&- \frac{\partial V}{\partial C}(\mu C dt + \sigma C dM_t^{H,\lambda}) - \frac{\partial V}{\partial C}(r_C - \mu)C dt + B(t,C)dt. \qquad (32)
\end{aligned}
$$

On the other hand, the return of portfolio is defined by:

$$\mathbb{E}[d\phi]. \qquad (33)$$

Replacing (32) in (33), we obtain:

$$
\begin{aligned}
\mathbb{E}[d\phi] &= \mathbb{E}\left[\left[\left(a+b\lambda^{H-\frac{1}{2}}\right)^2 \frac{\partial^2 V}{\partial^2 C}\sigma^2 C^2 + \mu C\frac{\partial V}{\partial C}\right]dt\right] + \mathbb{E}\left[\sigma C\frac{\partial V}{\partial C}dM_t^{H,\lambda}\right] \\
&- \mathbb{E}\left[\frac{\partial V}{\partial C}(\mu C dt + \sigma C dM_t^{H,\lambda})\right] - \mathbb{E}\left[\frac{\partial V}{\partial C}(r_C - \mu)C dt\right] + \mathbb{E}\left[B(t,C)dt\right] \\
&= \left[\left(a+b\lambda^{H-\frac{1}{2}}\right)^2 \frac{\partial^2 V}{\partial^2 C}\sigma^2 C^2 - r_C C\frac{\partial V}{\partial C} + B(t,C) + \mu C\frac{\partial V}{\partial C}\right]dt.
\end{aligned}
$$

Thus, we obtain:

$$
\begin{aligned}
\mathbb{E}[d\phi] &= \left[\left(a+b\lambda^{H-\frac{1}{2}}\right)^2 \frac{\partial^2 V}{\partial^2 C}\sigma^2 C^2 - r_C C\frac{\partial V}{\partial C}\right]dt \\
&+ \left[B(t,C) + \mu C\frac{\partial V}{\partial C}\right]dt. \qquad (34)
\end{aligned}
$$

Furthermore, the portfolio risk being eliminated , it becomes risk-free and its return earns a constant interest rate proportional to its value i.e

$$\mathbb{E}[d\phi] = r\phi dt. \qquad (35)$$

By substitution of (34) for (35), we have:

$$\mathbb{E}[d\phi] = r\phi dt$$

i.e

$$\left(a+b\lambda^{H-\frac{1}{2}}\right)^2\frac{\partial^2 V}{\partial^2 C}\sigma^2 C^2 - r_C C\frac{\partial V}{\partial C} + B(t,C) + \mu C\frac{\partial V}{\partial C} = r(V - C\frac{\partial V}{\partial C}).$$

i.e

$$\left(a+b\lambda^{H-\frac{1}{2}}\right)^2\frac{\partial^2 V}{\partial^2 C}\sigma^2 C^2 + (\mu-\eta\rho\sigma)C\frac{\partial V}{\partial C} + B(t,C) - rV = 0. \tag{36}$$

Replacing (19) in (36), we have:

$$\left(a+b\lambda^{H-\frac{1}{2}}\right)^2\frac{\partial^2 V}{\partial^2 C}\sigma^2 C^2 + (\mu-\eta\rho\sigma)C\frac{\partial V}{\partial C} + B(C,t) - rV = 0$$

i.e

$$\left(a+b\lambda^{H-\frac{1}{2}}\right)^2\frac{\partial^2 V}{\partial^2 C}\sigma^2 C^2 + (\mu-\eta\rho\sigma)C\frac{\partial V}{\partial C} + C_t(1-\tau) - \tau\delta P(1-\varphi)\left(\frac{C_t}{C_N}\right)^{-\frac{\delta}{Z}} - rV = 0$$

i.e

$$\left(a+b\lambda^{H-\frac{1}{2}}\right)^2\frac{\partial^2 V}{\partial^2 C}\sigma^2 C^2 + (\mu-\eta\rho\sigma)C\frac{\partial V}{\partial C} + C_t(1-\tau) - \tau\delta P(1-\varphi)\left(\frac{C_t}{C_N}\right)^{-\frac{\delta}{Z}} - rV = 0$$

Hence

$$\frac{1}{2}\sigma_H^2 C^2 V_{CC} + \mu^* C V_C + C_t(1-\tau) - \tau\delta P(1-\varphi)\left(\frac{C_t}{C_N}\right)^{-\frac{\delta}{Z}} - rV = 0,$$

with $\sigma_H^2 = \sigma^2\left(a+b\lambda^{H-\frac{1}{2}}\right)^2$ and $\mu^* = \mu - \eta\rho\sigma$.

The general solution of equation (23) is given by the following fundamental result:

Theorem 5 *The solution to non homogeneous differential equation* (23) *is the discounted average value of the after-tax costs $V_m(C)$ resulting from the optimal replacement policy. It is defined by:*

$$V_m(C) = K_1 C^{\beta^+} + K_2 C^{\beta^-} + \frac{C(1-\tau)}{r-\mu^*} + \frac{\theta C^{\varepsilon}}{\Phi(\varepsilon)}, \tag{37}$$

where K_1 et K_2 are constants and β^+ and β^- are respectively defined by:

$$\begin{cases} \beta^- = \frac{1}{2} - \frac{\mu^*}{\sigma_H^2} - \sqrt{\left[\left(\frac{\mu^*}{\sigma_H^2} - \frac{1}{2}\right)^2 - \frac{2r}{\sigma_H^2}\right]} \\ \beta^+ = \frac{1}{2} - \frac{\mu^*}{\sigma_H^2} + \sqrt{\left[\left(\frac{\mu^*}{\sigma_H^2} - \frac{1}{2}\right)^2 - \frac{2r}{\sigma_H^2}\right]} \end{cases} \tag{38}$$

Proof *:* The equation (23) of the proposition (4) is a differential equation of order 2 with non-homogeneous second member.

Let

$$\begin{cases} \theta = -\tau\delta P(1-\varphi)(C_N)^{\frac{\delta}{Z}} \\ \varepsilon = -(\dfrac{\delta}{Z}) \\ \Phi(\varepsilon) = r - \mu^*\varepsilon - \dfrac{1}{2}\sigma_H^2\varepsilon(\varepsilon-1) \\ V^0 = \dfrac{C(1-\tau)}{r-\mu^*} + \dfrac{\theta C^\varepsilon}{\phi(\varepsilon)} \end{cases} \tag{39}$$

Let's check that V^0 is a particular solution of equation (23) of the proposition (4) .

Just check that:

$$\frac{1}{2}\sigma_H^2 C^2 {V^0}_{CC} + \mu^* C {V^0}_C + C(1-\tau) - \tau\delta P(1-\varphi)\left(\frac{C}{C_N}\right)^{-\frac{\delta}{Z}} = rV^0, \tag{40}$$

i.e:

$$\frac{1}{2}\sigma_H^2 C^2 {V^0}_{CC} + \mu^* C {V^0}_C - rV^0 = \tau\delta P(1-\varphi)\left(\frac{C}{C_N}\right)^{-\frac{\delta}{Z}} - C(1-\tau). \tag{41}$$

Then, we have :

$$V_C^0 = \frac{(1-\tau)}{r-\mu^*} + \frac{\varepsilon\theta C^{\varepsilon-1}}{\phi(\varepsilon)} \tag{42}$$

and

$$V_{CC}^0 = \frac{\varepsilon(\varepsilon-1)\theta C^{\varepsilon-2}}{\phi(\varepsilon)} \tag{43}$$

So :

$$\begin{aligned} \frac{1}{2}\sigma_H^2 C^2 {V^0}_{CC} + \mu^* C {V^0}_C - rV^0 &= \frac{1}{2}\sigma_H^2 C^2\left[\frac{\varepsilon(\varepsilon-1)\theta C^{\varepsilon-2}}{\phi(\varepsilon)}\right] - r\left[\frac{C(1-\tau)}{r-\mu^*} + \frac{\theta C^\varepsilon}{\phi(\varepsilon)}\right] \\ &+ \mu^* C\left[\frac{(1-\tau)}{r-\mu^*} + \frac{\varepsilon\theta C^{\varepsilon-1}}{\phi(\varepsilon)}\right] \\ &= \frac{\frac{1}{2}\sigma_H^2\varepsilon(\varepsilon-1)\theta C^\varepsilon}{\phi(\varepsilon)} + \frac{\mu^* C(1-\tau)}{r-\mu^*} + \frac{\mu^*\varepsilon\theta C^\varepsilon}{\phi(\varepsilon)} \\ &- \frac{rC(1-\tau)}{r-\mu^*} + \frac{r\theta C^\varepsilon}{\phi(\varepsilon)} \\ &= \frac{\frac{1}{2}\sigma_H^2\varepsilon(\varepsilon-1)\theta C^\varepsilon + \mu^*\varepsilon\theta C^\varepsilon - r\theta C^\varepsilon}{\phi(\varepsilon)} \\ &+ \frac{\mu^* C(1-\tau) - rC(1-\tau)}{r-\mu^*} \\ &= \frac{\frac{1}{2}\sigma_H^2\varepsilon(\varepsilon-1) + \mu^*\varepsilon\theta C^\varepsilon - r}{\phi(\varepsilon)\theta C^\varepsilon} + \frac{C(\mu^*-r)(1-\tau)}{r-\mu^*} \\ &= -\theta C^\varepsilon - C(1-\tau) \\ &= \tau\delta P(1-\varphi)\left(\frac{C}{C_N}\right)^{-\frac{\delta}{Z}} - C(1-\tau). \end{aligned}$$

Then V^0 is a particular solution of (23).

The homogeneous equation associated with (23) is:

$$\frac{1}{2}\sigma_H^2 C^2 V_{CC} + \mu^* C V_C - rV = 0. \tag{44}$$

Let's solve the equation (44).

Let's

$$\begin{cases} C = e^t \\ U(t) = V(e^t) \end{cases} \tag{45}$$

and

$$\begin{cases} U'(t) = e^t V'(e^t) \\ U''(t) = e^t V'(e^t) + e^{2t} V''(e^t) \end{cases} \tag{46}$$

So, the equation (44) is rewritten as follows:

$$\frac{1}{2}\sigma_H^2 e^{2t} V''(e^t) + \mu^* e^t V'(e^t) - rV(e^t) = 0. \tag{47}$$

Then expressing $V''(e^t)$ and $V'(e^t)$ in terms of $U''(t)$ and of $U'(t)$, we obtain:

$$\begin{cases} V'(e^t) = e^{-t} U'(t), \\ V''(e^t) = e^{-2t}[U''(t) - U'(t)]. \end{cases} \tag{48}$$

By introducing (48) into equation (47), we have :

$$\begin{aligned}
\frac{1}{2}\sigma_H^2 e^{2t} V''(e^t) + \mu^* e^t V'(e^t) - rV(e^t) = 0 \iff & \frac{1}{2}\sigma_H^2 e^{2t}\left[e^{-2t}\left(U''(t) - U'(t)\right)\right] \\
+ & \mu^* e^t (e^{-t} U'(t)) - rU(t) = 0 \\
\iff & \frac{1}{2}\sigma_H^2 \left(U''(t) - U'(t)\right) + \mu^* U'(t) \\
- & rU(t) = 0 \\
\iff & \frac{1}{2}\sigma_H^2 U''(t) - \frac{1}{2}\sigma_H^2 U'(t) + \mu^* U'(t) \\
- & rU(t) = 0 \\
\iff & \frac{1}{2}\sigma_H^2 U''(t) + (\mu^* - \frac{1}{2}\sigma_H^2) U'(t) \\
- & rU(t) = 0.
\end{aligned}$$

So:

$$\frac{1}{2}\sigma_H^2 U''(t) + (\mu^* - \frac{1}{2}\sigma_H^2) U'(t) - rU(t) = 0. \tag{49}$$

The equation (49) is an ODE of order 2 with constant coefficient, function of U which has the characteristic:

$$\frac{1}{2}\sigma_H^2 R^2 + (\mu^* - \frac{1}{2}\sigma^2) R - r = 0. \tag{50}$$

To solve the equation (50), we find the discriminant Δ, then :

$$\begin{aligned}\Delta &= (\mu^* - \frac{1}{2}\sigma_H^2)^2 - 4(\frac{1}{2}\sigma_H^2)(-r) \\ &= \sigma_H^4(\frac{\mu^*}{\sigma_H^2} - \frac{1}{2})^2 - 2r\sigma_H^2 \\ &= \sigma_H^4\left[(\frac{\mu^*}{\sigma_H^2} - \frac{1}{2})^2 - \frac{2r}{\sigma_H^2}\right].\end{aligned} \tag{51}$$

The square root of the discriminant is:

$$\sqrt{\Delta} = \sigma_H^2\sqrt{\left[(\frac{\mu^*}{\sigma_H^2} - \frac{1}{2})^2 - \frac{2r}{\sigma_H^2}\right]}. \tag{52}$$

So the solutions of equation (50) are given respectively by R_1 and R_2 defined by:

$$\begin{aligned}R_1 &= \frac{-(\mu^* - \frac{1}{2}\sigma_H^2) - \sigma_H^2\sqrt{\left[(\frac{\mu^*}{\sigma_H^2} - \frac{1}{2})^2 - \frac{2r}{\sigma_H^2}\right]}}{\sigma_H^2} \\ &= \frac{1}{2} - \frac{\mu^*}{\sigma_H^2} - \sqrt{\left[(\frac{\mu^*}{\sigma_H^2} - \frac{1}{2})^2 - \frac{2r}{\sigma_H^2}\right]},\end{aligned} \tag{53}$$

and

$$\begin{aligned}R_2 &= \frac{-(\mu^* - \frac{1}{2}\sigma_H^2) + \sigma_H^2\sqrt{\left[(\frac{\mu^*}{\sigma_H^2} - \frac{1}{2})^2 - \frac{2r}{\sigma_H^2}\right]}}{\sigma_H^2} \\ &= \frac{1}{2} - \frac{\mu^*}{\sigma_H^2} + \sqrt{\left[(\frac{\mu^*}{\sigma_H^2} - \frac{1}{2})^2 - \frac{2r}{\sigma_H^2}\right]}.\end{aligned} \tag{54}$$

Then the solution of equation (49) is the continuous function with respect to t defined by:

$$U^1(t) = K_1 e^{R_1 t} + K_2 e^{R_2 t}, \tag{55}$$

where K_1 anK_2 are constants.

Given that $C = e^t$; we have: $t = ln(C)$, we deduce that the solution of (49) is the function of the form:

$$\begin{aligned}V^1(C) &= K_1 e^{R_1 ln(C)} + K_2 e^{R_2 ln(C)} \\ &= K_1 e^{ln[(C)^{R_1}]} + K_2 e^{ln[(C)^{R_2}]} \\ &= K_1 C^{R_1} + K_2 C^{R_2} \\ &= K_1 C^{\beta^+} + K_2 C^{\beta^-},\end{aligned} \tag{56}$$

with $R_1 = \beta^+$ et $R_2 = \beta^-$.

The general solution pf equation (23) is the sum of the particular solution and the solution of the homogeneous equation (49). i.e

$$V(C) = K_1 C^{\beta^+} + K_2 C^{\beta^-} + \frac{C(1-\tau)}{r-\mu^*} + \frac{\theta C^{\varepsilon}}{\Phi(\varepsilon)}. \tag{57}$$

Our objective now is to determine the constants K_1 and K_2 of (37) and then to calculate the optimal replacement policy $\overline{C}$.

For this, we adopt the approach of Dixit [5] and Sick [24], by imposing three boundary conditions on V(C) of equation (37) namely:
$(\mathcal{C}_1)$: When replacing asset A, the average present value of after-tax costs before replacement is equal to the average present value of after-tax costs after replacement plus the price of a new asset less the value of post-tax recovery of the old asset i.e:

$$V(\overline{C}) = V(C_N) + P(1-\varphi) - \left[S(\overline{C}) - (\tau)\left(S(\overline{C}) - P(1-\varphi)\left(\frac{\overline{C}}{C_N}\right)^{-\frac{\delta}{Z}}\right)\right], \tag{58}$$

where

- $V(\overline{C})$ is the average present value of efter-tax costs before replacement,

- $V(C_N)$ is the average present value of after-tax costs after replacement,

- $P(1-\varphi)$ is the price of a new asset,

- $\left[S(\overline{C}) - (\tau)\left(S(\overline{C}) - P(1-\varphi)\left(\frac{\overline{C}}{C_N}\right)^{-\frac{\delta}{Z}}\right)\right]$ is the salvage value of the old asset.

  $(\mathcal{C}_2)$: To exclude the case that an old asset may have a lower maintenance an operating cost than a stochastically equivalent new asset, it makes sense to place a reflective barrier at the initial level of process C.

  The reflective barrier on C is satisfied when $V_m(C)$ satisfies the following equation:

  $$\frac{\partial V}{\partial C}|_{C=C_N} = 0. \tag{59}$$

  $(\mathcal{C}_3)$: To determine the optimal policy $\overline{C}$ which makes it possible to minimize the discounted average value of after-tax costs $V(C)$, we pass equation (37) through the smooth bonding boundary condition which consists of deriving the first boundary condition.

So we obtain:

$$\frac{\partial V}{\partial C}|_{C=\overline{C}} = \frac{\delta}{Z}(\tau)P(1-\varphi)(\frac{1}{C_N})\left(\frac{\overline{C}}{C_N}\right)^{-\frac{\delta}{Z}-1} - S'(C)\left(1-\tau\right). \quad (60)$$

Remark 3 • *It should be noted that equations* (58) *and* (59) *are sufficient for the calculation of* K_1 *and* K_2 *based on the optimal replacement policy* $\overline{C}$*,*

- *According to the principle of contingent claim method, we set the resale price of an asset in the secondary market*

$$S(C) = \frac{\varrho}{C},$$

where ϱ *is the systematic pricing factor of the asset.*

Substituting equation (37) into the previous equations (58), (59) and (60), we obtain the following system:

$$\begin{cases} \beta^+ C_N^{\beta^+-1}K_1 + \beta^- C_N^{\beta^--1}K_2 + \dfrac{(1-\tau)}{r-\mu^*} + \dfrac{\varepsilon\theta C_N^{\varepsilon-1}}{\Phi(\varepsilon)} = 0 \\ \\ K_1(\overline{C})^{\beta^+} + K_2(\overline{C})^{\beta^-} + \dfrac{(\overline{C}-C_N)(1-\tau)}{r-\mu^*} + \dfrac{\theta\left((\overline{C})^{\varepsilon} - (C_N)^{\varepsilon}\right)}{\Phi(\varepsilon)} - C_N^{\beta^+}K_1 - C_N^{\beta^-}K_2 \\ +\dfrac{\varrho(1-\tau)}{\overline{C}} + P(1-\varphi)\left(-1+\tau(C_N)^{\frac{\delta}{Z}}(\overline{C})^{-\frac{\delta}{Z}}\right) = 0 \\ \\ \beta^+\overline{C}^{\beta^+-1}K_1 + \beta^-\overline{C}^{\beta^--1}K_2 + \dfrac{(1-\tau)}{r-\mu^*} + \dfrac{\varepsilon\theta\overline{C}^{\varepsilon-1}}{\Phi(\varepsilon)} - \tau\dfrac{\delta}{Z}P(1-\varphi)(C_N)^{\frac{\delta}{Z}}(\overline{C})^{-\frac{\delta}{Z}-1} \\ -\dfrac{\varrho(1-\tau)}{(\overline{C})^2} = 0 \end{cases} \quad (61)$$

The system (61) is a nonlinear system with three unknowns K_1, K_2 and $\overline{C}$.

Accepting the parameters r; α; σ; η; ρ; α^*; C_N; P; φ; τ; δ; a; b; λ and H, solving the system (61) gives us the full value of the discounted average cost function $V_m(C)$ as well as that of the optimal replacement policy $\overline{C}$. Considering only the equations (58) and (59), we have the following system:

$$
\begin{cases}
\beta^+ C_N^{\beta^+-1} K_1 + \beta^- C_N^{\beta^--1} K_2 + \dfrac{(1-\tau)}{r-\mu^*} + \dfrac{\varepsilon\theta C_N^{\varepsilon-1}}{\Phi(\varepsilon)} = 0 \\
\\
\left[(\overline{C})^{\beta^+} - C_N^{\beta^+}\right] K_1 + \left[(\overline{C})^{\beta^-} - C_N^{\beta^-}\right] K_2 + \dfrac{(\overline{C}-C_N)(1-\tau)}{r-\mu^*} + \dfrac{\theta\left((\overline{C})^{\varepsilon} - (C_N)^{\varepsilon}\right)}{\Phi(\varepsilon)} \\
+\dfrac{\varrho(1-\tau)}{\overline{C}} + P(1-\varphi)\left(-1+\tau(C_N)^{\frac{\delta}{z}}(\overline{C})^{-\frac{\delta}{z}}\right) = 0
\end{cases}
\tag{62}
$$

By fixing $\overline{C}$, the system (62) becomes a linear system of two equations with two unknowns K_1 and K_2.

By substitution, we have:

$$
\begin{aligned}
K_1 \;=\; & \frac{(\frac{1-\tau}{r-\mu^*})\left(\beta^- C_N^{\beta^--1}(C_N-\overline{C}) - (C_N^{\beta^-} - \overline{C}^{\beta^-})\right)}{\beta^- C_N^{\beta^--1}(\overline{C}^{\beta^+} - C_N^{\beta^+}) + \beta^+ C_N^{\beta^+-1}(-\overline{C}^{\beta^-} + C_N^{\beta^-})} \\
+ \; & \frac{\frac{\theta}{\Phi(\varepsilon)}\left(\beta^- C_N^{\beta^--1}(-C_N^{\varepsilon} + \overline{C}^{\varepsilon}) + \varepsilon C_N^{\varepsilon-1}(C_N^{\beta^-} - \overline{C}^{\beta^-})\right)}{\beta^- C_N^{\beta^--1}(\overline{C}^{\beta^+} - C_N^{\beta^+}) + \beta^+ C_N^{\beta^+-1}(-\overline{C}^{\beta^-} + C_N^{\beta^-})} \\
+ \; & \frac{\beta^- C_N^{\beta^--1}\left(\varrho(\overline{C})^{-1}(-1+\tau) - \tau P(1-\varphi)(C_N)^{-\varepsilon}(\overline{C})^{\varepsilon}\right)}{\beta^- C_N^{\beta^--1}(\overline{C}^{\beta^+} - C_N^{\beta^+}) + \beta^+ C_N^{\beta^+-1}(-\overline{C}^{\beta^-} + C_N^{\beta^-})}
\end{aligned}
\tag{63}
$$

and

$$
\begin{aligned}
K_2 \;=\; & \frac{(\frac{1-\tau}{r-\mu^*})\left(-\beta^+ C_N^{\beta^+-1}(C_N-\overline{C}) + (C_N^{\beta^+} - \overline{C}^{\beta^+})\right)}{\beta^- C_N^{\beta^--1}(\overline{C}^{\beta^+} - C_N^{\beta^+}) + \beta^+ C_N^{\beta^+-1}(-\overline{C}^{\beta^-} + C_N^{\beta^-})} \\
+ \; & \frac{\frac{\theta}{\Phi(\varepsilon)}\left(-\beta^+ C_N^{\beta^+-1}(-C_N^{\varepsilon} + \overline{C}^{\varepsilon}) - \varepsilon C_N^{\varepsilon-1}(C_N^{\beta^+} - \overline{C}^{\beta^+})\right)}{\beta^- C_N^{\beta^--1}(\overline{C}^{\beta^+} - C_N^{\beta^+}) + \beta^+ C_N^{\beta^+-1}(-\overline{C}^{\beta^-} + C_N^{\beta^-})} \\
+ \; & \frac{\beta^+ C_N^{\beta^+-1}\left(-\varrho(\overline{C})^{-1}(-1+\tau) + \tau P(1-\varphi)(C_N)^{-\varepsilon}(\overline{C})^{\varepsilon}\right)}{\beta^- C_N^{\beta^--1}(\overline{C}^{\beta^+} - C_N^{\beta^+}) + \beta^+ C_N^{\beta^+-1}(-\overline{C}^{\beta^-} + C_N^{\beta^-})}.
\end{aligned}
\tag{64}
$$

Replacing the K_1 and K_2 values in equation (37), we obtain the complete solution of equation (23) dependent on model parameters and the optimal replacement policy.

Hence the result. The time between replacements of assets is function of $\overline{C}$ and it is given by the following result:

Proposition 5 *The optimal average time between replacements of assets (measured in years) is given by:*

$$\overline{T} = \frac{ln(\overline{C}) - ln(C_N)}{(\mu - \frac{1}{2}\sigma_H^2)} - \frac{1}{2}\left(\frac{\sigma_H}{\mu - \frac{1}{2}\sigma_H^2}\right)^2 \left[1 - \left(\frac{\overline{C}}{C_N}\right)^{(1-\frac{2\mu}{\sigma_H^2})}\right]. \tag{65}$$

Proof *:* The time between replacements is given by:

$$\overline{T} = E[\,\bar{t}\,] - \omega, \tag{66}$$

where ω is a value which allows $\overline{T}$ to be adjusted taking into account the reflective barrier.

In effect, the time between replacements $\overline{T}$ is identical to the first passage time calculated in (14) of theorem (3) with the only difference of the value ω seen as the probability distribution of the first passage.

Let us determine the of ω.

The new value of ω is defined by:

$$\begin{aligned}
\omega &= \frac{1}{2}\frac{\sigma_H^2}{(\mu - \frac{1}{2}\sigma_H^2)^2}\left[1 - \exp\left(-\frac{2}{\sigma_H^2}(\mu - \frac{1}{2}\sigma_H^2)^2 E[\,\bar{t}\,]\right)\right] \\
&= \frac{1}{2}\frac{\sigma_H^2}{(\mu - \frac{1}{2}\sigma_H^2)^2}\left[1 - \exp\left(-\frac{2}{\sigma_H^2}(\mu - \frac{1}{2}\sigma_H^2)^2 Z^{-1}\left(ln(\overline{C}) - ln(C_N)\right)\right)\right] \\
&= \frac{1}{2}\frac{\sigma_H^2}{(\mu - \frac{1}{2}\sigma_H^2)^2}\left[1 - \exp\left((\frac{-2\mu}{\sigma_H^2} + 1) ln\left(\frac{\overline{C}}{C_N}\right)\right)\right] \\
&= \frac{1}{2}\left(\frac{\sigma_H}{\mu - \frac{1}{2}\sigma_H^2}\right)^2 \left[1 - \left(\frac{\overline{C}}{C_N}\right)^{(1-\frac{2\mu}{\sigma_H^2})}\right].
\end{aligned}$$

Thus:

$$\omega = \frac{1}{2}\left(\frac{\sigma_H}{\mu - \frac{1}{2}\sigma_H^2}\right)^2 \left[1 - \left(\frac{\overline{C}}{C_N}\right)^{(1-\frac{2\mu}{\sigma_H^2})}\right]. \tag{67}$$

By replacing (14) of theorem 3 and (67) in (66), we obtain:

$$\begin{aligned}
\overline{T} &= E[\,\bar{t}\,] - \omega \\
&= Z^{-1} ln\left(\frac{\overline{C}}{C_N}\right) - \frac{1}{2}\frac{\sigma_H^2}{(\mu - \frac{1}{2}\sigma_H^2)^2}\left[1 - \exp\left(-\frac{2}{\sigma_H^2}(\mu - \frac{1}{2}\sigma_H^2)^2 E[\,\bar{t}\,]\right)\right] \\
&= \frac{ln(\overline{C}) - ln(C_N)}{(\mu - \frac{1}{2}\sigma_H^2)} - \frac{1}{2}\left(\frac{\sigma_H}{\mu - \frac{1}{2}\sigma_H^2}\right)^2 \left[1 - \left(\frac{\overline{C}}{C_N}\right)^{(1-\frac{2\mu}{\sigma_H^2})}\right].
\end{aligned}$$

Hence the result.

4.3.1 Parameters of the discounted average cost function

The data used in this subsection are a combination of parameters from the calibration of the (9) model and the parameters proposed by Mauer and Ott [4].

In this paper, we extend the process used by Mauer and Ott to make a comparison of the results.

We summarize this data in the following table which will be readjusted as our work progresses depending on the type of need.

Parameters	value
Risk-free interest rate	$r = 0.07$
Interest rate of the nerlying	$\mu = 0.15$
Volatility of the underlying	$\sigma = 0.10$
Market price at risk	$\eta = 0.40$
Correlation between cost and systematic pricin factor	$\rho = 0$
Adusted interest rate of the underlying	$\mu^* = 0.15$
Initial cost of maintenance and operating	$C_N = 1$
Purchase price of a new asset	$P = 10$
The systematic pricing factor	$\varrho = 8$
Resale price of an ol asset	$S(C) = 8C^{-1}$
Credit investment rate	$\varphi = 0$
Rate tax	$\tau = 0.30$
Depreciation rate	$\delta = 0.50$

Table 1: Basic Parameters

Table 1 shows a set of basic parameter values proposed by Mauer and Ott [4].

In this table, the annualized risk-free interest rate r is 7% and te annualized drift rate and cost volatility rate σ are 15% and 10%, respectively. We estimate the market price of risk η to be 0.40. We assume that the correlation between cost and the systematic pricing factor ρ is zero and therefore the risk-adjusted drift rate α^* is equal to 15% per year. We set the initial cost of maintaining and operating a new asset C_N equal to 10. The basic tax parameters include an investment tax credit φ=0, a corporate tax rate τ=30% and a depreciation rate δ=50%.

An investment tax credit of zero and a corporate tax of 30% are consistent with current US tax law. The authors assume that the systematic pricing factor ϱ is equal to 8. Thus, a newly purchased asset could be sold immediately at 80% of its purchase price and when the cost of maintenance and operating increases, this salvage value decreases.

We take Mauer and Ott's table 1 above and complete it wit three new parameters, hence the following new table:

Parameters	value
Risk-free interest rate	$r = 0.07$
expected return rate of the underlying	$\mu = 0.15$
Volatility of the underlying	$\sigma = 0.10$
Market price at risk	$\eta = 0.40$
Correlation between cost and systematic pricing factor	$\rho = 0$
Adjusted interest rate of the underlying	$\mu^* = 0.15$
Initial cost of maintenance and operating	$C_N = 1$
Purchase price of a new asset	$P = 10$
The systematic pricing factor	$\varrho = 8$
Resale price of an ol asset	$S(C) = 8C^{-1}$
Credit investment rate	$\varphi = 0$
Rate tax	$\tau = 0.30$
Depreciation rate	$\delta = 0.50$
Hurst parameter	H=0,75
The coefficient of adustment of the time between two quotations on the financial market	$\lambda \in \{\frac{1}{2^4}, \frac{1}{2^3}, \frac{1}{2^2}, \frac{1}{2^1}\}$
Parameters of MMFBM	a= 1.75 ,b=0.5

Table 2: Basic parameters

The new table 2 presents a set consisting of basic parameter values proposed by Mauer and Ott [4] and those of three adjusted parameters.We take te Hurst parameter H equal to 0.7, the adjustment coefficient of the time between two quotes on the financial market is $\lambda \in \{\frac{1}{2^4}, \frac{1}{2^3}, \frac{1}{2^2}, \frac{1}{2^1}\}$ and we choose the parameters a and b respectively 1.75 and 0.5.

The objective now is to determine the replacement optimal policy $\overline{C}$ for each value of $\lambda \in \{\frac{1}{2^4}, \frac{1}{2^3}, \frac{1}{2^2}, \frac{1}{2^1}\}$ when we replace the parameters of equation (37) by the values given in table 2.

Thus , by replacing the parameters of system (61) by the values given in table 2, we obtain the following system to the first value of λ:

$$
\begin{cases}
(-3.25+\sqrt{14.0625})K_1+(-3.25-\sqrt{14.0625})K_2+\dfrac{1-0.3}{0.07-0.15}+\dfrac{1267500}{60229}=0\\[2ex]
K_1(\overline{C})^{(-3.25+\sqrt{14.0625})}+K_2(\overline{C})^{(-3.25-\sqrt{14.0625})}+\dfrac{(\overline{C}-1)(1-0.3)}{0.07-0.15}\\
+\dfrac{25350}{4633}\left((\overline{C})^{-\frac{50}{13}}-1\right)-K_1-K_2+\dfrac{8(1-0.3)}{\overline{C}}+10(1-0)\left(-1+0.7(\overline{C})^{-\frac{0.5}{0.13}}\right)=0\\[2ex]
(-3.25+\sqrt{14.0625})\overline{C}^{(-4.25+\sqrt{14.0625})}K_1+(-3.25-\sqrt{14.0625})\overline{C}^{(-4.25-\sqrt{14.0625})}K_2\\
+\dfrac{(1-0.3)}{0.07-0.15}+\dfrac{1267500}{60229}\overline{C}^{-\frac{50}{13}-1}-0.3\dfrac{0.5}{0.13}(10)(1-0)(\overline{C})^{-\frac{0.5}{0.13}-1}-5.6(\overline{C})^{-2}=0.
\end{cases}
\tag{68}
$$

ie:

$$
\begin{cases}
(-3.25+\sqrt{14.0625})K_1+(-3.25-\sqrt{14.0625})K_2+\dfrac{740496.25}{60229}=0\\[2ex]
K_1(\overline{C})^{(-3.25+\sqrt{14.0625})}+K_2(\overline{C})^{(-3.25-\sqrt{14.0625})}-K_1-K_2-8.75\overline{C}\\
+\dfrac{57781}{4633}(\overline{C})^{-\frac{50}{13}}+5.6(\overline{C})^{-1}-\dfrac{63572.25}{4633}=0\\[2ex]
(-3.25+\sqrt{14.0625})K_1(\overline{C})^{(-4.25+\sqrt{14.0625})}+(-3.25-\sqrt{14.0625})K_2(\overline{C})^{(-4.25-\sqrt{14.0625})}\\
+\dfrac{7443150}{782977}(\overline{C})^{-\frac{63}{13}}-5.6(\overline{C})^{-2}-8.75=0.
\end{cases}
\tag{69}
$$

Using the two first equations, we obtain the values of K_1 and K_2 respectively defined in (63) and (64).

By replacing the values of K_1 and K_2 in the third equation, we obtain an equation with a single unknown $\overline{C}$.

Let $\overline{C}$=X, we have:

$$
\begin{aligned}
& 30.625X^{\frac{50}{13}+2\sqrt{14.0625}} - 30.625X^{\frac{63}{13}+2\sqrt{14.0625}} - 8.75(-3.25+\sqrt{14.0625}) \\
+ \quad & 8.75(-3.25+\sqrt{14.0625})X^{\frac{50}{13}+2\sqrt{14.0625}} + \frac{975}{46.33}(-3.25-\sqrt{14.0625})X^{2\sqrt{14.0625}} \\
+ \quad & \frac{975}{46.33}(-3.25+\sqrt{14.0625})X^{\frac{50}{13}+2\sqrt{14.0625}} + 19.6X^{\frac{37}{13}+2\sqrt{14.0625}} + 10.5X^{2\sqrt{14.0625}} \\
- \quad & \frac{975}{46.33}(-3.25+\sqrt{14.0625})X^{\frac{7.75}{13}+2\sqrt{14.0625}} - 30.625X^{\frac{50}{13}} + 30.625X^{\frac{63}{13}} \\
+ \quad & 8.75(-3.25-\sqrt{14.0625})X^{\frac{7.75}{13}+\sqrt{14.0625}} - 8.75(-3.25-\sqrt{14.0625})X^{\frac{50}{13}} \\
+ \quad & \frac{887.25}{46.33}X^{\frac{50}{13}} - \frac{887.25}{46.33} - \frac{975}{46.33}(-3.25-\sqrt{14.0625})X^{\frac{50}{13}} + \frac{975}{46.33}X^{\frac{7.75}{13}+\sqrt{14.0625}} \\
- \quad & 5.6(-3.25-\sqrt{14.0625})X^{\frac{37}{13}+2\sqrt{14.0625}} + \frac{975}{46.33}(-3.25+\sqrt{14.0625})X^{3.25+\sqrt{14.0625}} \\
- \quad & \frac{975}{46.33}(-3.25+\sqrt{14.0625}) + 5.6(-3.25+\sqrt{14.0625})X^{\frac{37}{13}} \\
- \quad & \frac{975}{46.33}(-3.25-\sqrt{14.0625})X^{3.25+\sqrt{14.0625}} - \frac{150}{13}(-3.25+\sqrt{14.0625})X^{3.25+\sqrt{14.0625}} \\
+ \quad & \frac{150}{13}(-3.25+\sqrt{14.0625}) - \frac{150}{13}(-3.25-\sqrt{14.0625})X^{2\sqrt{14.0625}} \\
+ \quad & \frac{150}{13}(-3.25-\sqrt{14.0625})X^{3.25+\sqrt{14.0625}} - 5.6(-3.25+\sqrt{14.0625})X^{\frac{79.25}{13}+\sqrt{14.0625}} \\
+ \quad & 5.6(-3.25-\sqrt{14.0625})X^{\frac{79.25}{13}+2\sqrt{14.0625}} - 19.6X^{\frac{37}{13}} - 10.5 = 0
\end{aligned} \tag{70}
$$

Soving equation (70) manually is extremely tedious. We use **Maple** software to find the approximate value of X and we obtain the value $1,619$.

The Maple code used for this calculation is available in the appendix to this paper.

Thus the optimal replacement policy $\overline{\mathbf{C}} = 1.619$.

Using the 5, we have :

$$\overline{T} = \frac{ln(1.619) - ln(1)}{(0.15 - \frac{1}{2} \times 0.04)} - \frac{1}{2}\left(\frac{0.02}{0.15 - \frac{1}{2} \times 0.04}\right)^2 \left[1 - \left(\frac{1.619}{1}\right)^{(1 - \frac{2 \times 0.15}{0.04})}\right]$$

$$= 2.574.$$

Using the principle of the contingent claim method, we have: For $\lambda = \frac{1}{2^4}$, $S(\overline{C}) = 4,941$. For a another value, we have by using the same method,the following summary table:

λ		New Results	Old Results[4]
$\lambda = \frac{1}{2^4}$	$\overline{C}$	**1.619**	2.736
$\lambda = \frac{1}{2^4}$	$\overline{T}$	**2.574**	6.700
$\lambda = \frac{1}{2^4}$	$S(\overline{C})$	**4.941**	2.923
$\lambda = \frac{1}{2^3}$	$\overline{C}$	**2.720**	3.847
$\lambda = \frac{1}{2^3}$	$\overline{T}$	**3.685**	7.811
$\lambda = \frac{1}{2^3}$	$S(\overline{C})$	**5.152**	3.134
$\lambda = \frac{1}{2^2}$	$\overline{C}$	**3.831**	4.958
$\lambda = \frac{1}{2^2}$	$\overline{T}$	**4.796**	8.922
$\lambda = \frac{1}{2^2}$	$S(\overline{C})$	**6.263**	4.245
$\lambda = \frac{1}{2^1}$	$\overline{C}$	**4.941**	5.149
$\lambda = \frac{1}{2^1}$	$\overline{T}$	**5.807**	9.133
$\lambda = \frac{1}{2^1}$	$S(\overline{C})$	**7.374**	5.356

Interpretation of results and discussions

The results we obtained are different from those obtained by Mauer and Ott [4] who in their article, use Standard Brownian motion to perturb the cost of maintenance and operating of assets in a company. Compared to the results found by Mauer and Ott, the optimal replacement policy is minimal.We deduce from these results that when the maintenance cost is disturbed by a mixed modified fractional Brownian motion, the company replaces more assets and spends less on the maintenance of the latter. We find that when a firm replaces an asset with a new stochastically equivalent asset then it could sell the old asset at almost 50% of its initial purchase price which will allow it to invest in other assets or to increase its economy.

In conclusion, in terms of business management, the mixed modified fractional Brownian motion that we used is better suited to minimize the cost of maintenance and operating of assets compared to that used in the literature.

4.3.2 Evolution of the discounted average cost function

In this subsection, we analyze the evolution of the optimal replacement policy and the discounted average cost function as the Hurst parameter increases.

We summarize in the following table the different values of $\overline{C}$ for $H = 0.75; 0.80$ and 0.90 , as well as those of $V_m(C)$.

The Maple codes used to calculate $\overline{C}$ for these different values of H are in the appendix to this report.

H	$\overline{C}$	K_1	K_2	$V_m(C)$
0.75	1.619	23.82	-3.58	$23.82C^{-3.25+\sqrt{14.0625}} - 3.58C^{-3.25-\sqrt{14.0625}} - 8.75C - \dfrac{25350}{4633}C^{-\frac{50}{13}}$
0.8	1.383	22.08	-2.78	$22.08C^{-3.44+\sqrt{15.568}} - 2.78C^{-3.44-\sqrt{15.568}} - 8.75C - 4.58C^{-\frac{500}{131}}$
0.9	1.203	20.22	-2.02	$20.22C^{-3.66+\sqrt{17.333}} - 2.02C^{-3.66-\sqrt{17.333}} - 8.75C - 3.05C^{-\frac{500}{132}}$

Table 3: Impact of the self-similarity parameter on $\overline{C}$

Table 3 above examines the effect of the self-similarity parameter H on the optimal replacement policy. It is seen that as the self-similarity parameter increases, the optimal replacement policy decreases hence the firm replaces more of its assets.

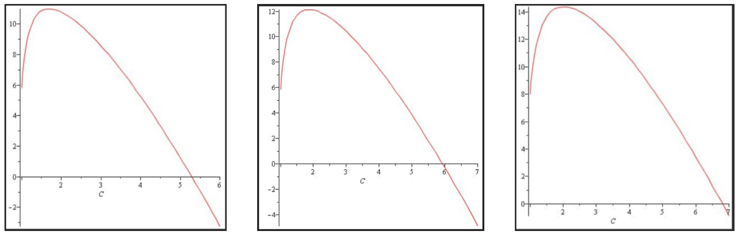
For example, if the Hurst parameter increases by 20%, the optimal replacement barrier decreases by approximately 25 %, resulting in the company spending less on maintenance its assets.

The $2D$ simulation of the function V_m of equation (37) of theorem 5 for the different values of H with Maple allows us to obtain the following figure:

Interpretation of numerical results

Figure 1 examines the evolution of te average discounted value function asset maintenance and operating costs.

Curse (a) shows the evolution of V_m for $H = 0.75$. We notice that in the interval $[1; 1.619[$, the average discounted value function of costs is



(a) $H = 0.75$ (b) $H = 0.80$ (c) $H = 0.90$

Figure 1: The Evolution of V_m for $H = 0.75$, $H = 0.80$ and $H = 0.90$

increasing and reaches its maximum at point $M_1(1.619; 14.5)$ where 1.619 is the threshold value of the maintenance and operating cost and 14.5 is the precise value of V_m at this critical level of C. From $C = 1.619$, the value of the discounted average value function of costs decreases, this is due to the fact that the company replaces the old asset with a new asset which will have an initial maintenance and operating cost equal to $C_N = 1$.

Curve (b) has the same behavior as (a), the only differences are located at the level where it reaches its maximum at point $M_2(1.383; 12.45)$. We deduce that for $H = 0.80$, $V_m(\overline{C}) = 12.45$ where $\overline{C}$ is the policy of optimal replacement for $H = 0.80$.

Curve (c) shows us the evolution of $V(C)$ for $H = 0.90$. We note that it follows the same pace as (a) and (b), except that it reaches its maximum at point $M_3(1.203; 11.58)$ where 1.203 is the barrier optimal replacement when we take $H = 0.90$.

Looking at the analysis of these three curves, it appears that when the self-similarity parameter of the process increases, the optimal replacement policy decreases and the maximum value of the value function discounted average asset costs decrease.

Compared to the result found by Mauer and Ott [4], we conclude that the model proposed in this thesis allows, thanks to its process which has good properties, to minimize the maximum value of the expected value function of asset costs, therefore allows the firm to spend less on the maintenance of its assets, something that all production plants are looking for.

5 Conclusion

In this work, it was a question for us of analyzing the determinants of replacement investment decisions with uncertainty linked to the cost of maintaining and operating the assets. For carry out our analysis, we used the contingent claims method from the real options literature which provides the techniques needed to incorporate uncertainty into replacement investment decisions.

By perturbing the cost of assets by a mixed modified fractional Brownian motion and from our basic model, we obtained the present average value of assets costs. Then, based on this average value, we successively determined the optimal replacement policy $\overline{C}$, the time between the replacements of the assets $\overline{T}$ and the resale value of the assets on the secondary market in order to examine the impact of the parameters of the mixed modified fractional Brownian motion on investment decisions.

The optimal replacement policy being characterized by a critical level (the optimal replacement barrier) at which the company must replace an existing worn-out asset with another stochastically equivalent asset, we note that when the cost of maintaining and operating the assets is governed by a mixed modified fractional Brownian motion, the minimum optimal replacement policy and this encourages new investments. We notice that when we use mixed modified fractional Brownian motion to perturb the cost of maintaining and operating assets in a company, the company spends less on maintaining its assets.

We presented the analytical approach of determining the discounted average value of asset costs without considering the effect of technological uncertainty and tax policy. In our next research, we intend to extend our studies to:

- The inclusion of asset capacity adjustments using the same model,
- The numerical determination of the expected value function of costs using the modified explicit finite difference method Hull et White [9],
- The effect of technological uncertainty and tax policy on replacement investment decisions using the modified fractional mixed model.

6 Compliance with Ethical Standards

6.1 Declaration of Competing Interest

All authors declare that they have no conflicts of interest.

6.2 Ethical approval:

This article does not contain any studies with human participants or animals performed by any of the authors.

6.3 Informed consent

Informed consent was obtained from all individual participants included in the study.

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