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Equilibrium bids for reverse auctions when the budget is announced – Some preliminary results

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Abstract: We consider a budget-constrained reverse auction: offers are accepted until some predefined budget is exhausted. Payments are discriminatory: bidders receive their own bid as payment when they win the bid. Bidders have a private value for their item which is uniformly distributed between 0 and 1; they are risk-neutral and items are indivisible. For small budget and an arbitrary number of bidders, as well as in the case of two bidders, we find an equilibrium bidding function. When possible, we compare with the equilibrium bidding function of target-constrained reverse auctions.

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Key-words: budget-constrained reverse auctions, equilibrium bidding function, target-constrained reverse auction

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Équilibres pour les enchères inversées quand le budget est annoncé – Quelques résultats préliminaires

Résumé : Nous considérons des enchères inversées avec une contrainte budgétaire : les offres sont acceptées jusqu'à la saturation d'une limite de budget donnée. La règle de paiement est à prix discriminants, c'est-à-dire que les enchérisseurs reçoivent, lorsqu'ils remportent l'enchère, des paiements égaux à leurs offres. Les enchérisseurs assignent une valeur à leur objet qui est uniformément distribuée entre 0 et 1; ils sont neutres face au risque et les objets sont indivisibles. Pour les petits budgets et un nombre arbitraire d'enchérisseurs, ainsi que dans le cas de deux enchérisseurs, nous trouvons une fonction d'enchère d'équilibre. Lorsque cela est possible, nous comparons avec la fonction d'enchère d'équilibre des enchères inversées avec une contrainte d'objectif (dans lesquelles l'agence se fixe un objectif et les offres sont acceptées jusqu'à l'atteindre).

Mots-clés : enchères inverses avec contrainte de budget, fonction d'enchère d'équilibre, enchères inverses avec contrainte d'objectif

Contents

1 Introduction

Auctions are often used as an alternative to fixed payments to reduce the asymmetry of information between an auctioneer and the bidders [Ferraro, 2008, Viaggi et al., 2010].

In this paper we are interested in reverse auctions. Contrary to standard selling auctions, in reverse auctions, the auctioneer is the buyer, and the bidders are the sellers. This kind of auction is a widely studied tool used to allocate agri-environmental payments to farmers (conservation auctions) [Schilizzi, 2017, Whitten et al., 2017, Bingham et al., 2021]. A multi-unit reverse auctions can be modeled either with a fixed budget (which is the norm, see the review in [Bingham et al., 2021]) or, alternatively, with a fixed target (also implemented to address conservation issues [Cummings et al., 2004, Janmaat, 2011, DePiper et al., 2013, Squires, 2010]). In the first case, called the budget-constrained auction, the buyer announces a budget and accepts bidders until this predetermined fixed budget is exhausted. The quantity of units and the residual from the budget are determined ex post. In the target-constrained auction the buyer announces the quantity of units and accepts bids until the target is achieved. Here budget is not known and must be determined ex post.

We consider the simple case of a *discriminatory* multi-unit auction, i.e. bid auction where winning bidders are paid their own bid. Each bidder is risk neutral, has only one unit to sell, all units are identical for the buyer and sellers have independent and private costs to produce their unit. In a multi-unit target (selling) auction, an equilibrium bidding strategy exists that is based on maximizing the bidders' expected surplus (see [Harris and Raviv, 1981, Cox et al., 1984]). [Hailu et al., 2005] and [Liu, 2021] extend this result to the reverse auction case. However, to the best of our knowledge, no equilibrium bidding strategy has yet been identified in budget-constrained auctions. Therefore, we cannot determine theoretically which format is the most efficient in a reverse auction, as observed already in [Müller and Weikard, 2002]. The situation does not seem to have evolved since. As pointed out in the report [Latacz-Lohmann and Schilizzi, 2005]:

> "There are no a priori reasons to believe that one auction format is better than the other except, perhaps, that the existence of a budget constraint may have the psychological effect of 'disciplining' bidders, encouraging them to bid closer to their opportunity costs than they might otherwise do."

Some papers investigate performance of these two auction formats using a controlled economic experiments. In [Schilizzi, 2017, Boxall et al., 2017], students play a repeated auction taking the role of farmers (bidders) whose opportunity costs are set randomly. Both experiments find that the target-constrained format outperforms the budget-constrained format in the first round but that after several repetitions the auction performance evens out. [Coiffard et al., 2023] compare the two auction formats in an online (one shot) decontextualized experiment. They obtain participants' bidding strategies and define equivalent target and budget constraints. They find that, on average, the budget-constrained auction format outperforms the target-constrained auction format when taking into account the average number of units purchased.

In this note we make the first steps towards filling the lack of theoretical results in special simple cases. Specifically, we consider a budget-constrained and discriminatory reverse auction with risk-neutral sellers, in which items are indivisible. We exhibit equilibrium bid functions in two main cases: when the budget is "small enough" for any number of bidders, and for any budget when there are two bidders. For both these cases, we sketch a comparison of the efficiency of target-constrained and budget-constrained auctions. We argue that the latter ones tend to be more efficient for the buyer.

In the next section, we present generalities on reverse auctions and their equilibria. Then in Section 3 we recall results concerning the standard target-constrained auction. In Section 4 we introduce the budget-constrained auction and prove the main results. Finally in Section 5, we compare metrics for both types of auctions. In Section 6 we discuss the results and present some extensions.

2 Reverse auctions and their equilibria

We begin by recalling the general setting of reverse discriminatory auctions, where bidders are risk-neutral and objects are indivisible. In particular we recall the concept of equilibrium bid. In the process, we introduce the notation and the main assumptions.

Consider $n \geq 2$ bidders who possess one item, which they are willing to sell to some buyer. Items are assumed to be *indivisible*: a bidder will either sell her whole item or nothing.

For each bidder, the item has a private value, denoted with v. It is assumed that this value is random with some distribution given by its distribution function $F(\cdot)$ with support $\mathcal{V}$. It is also assumed that this distribution has a density $f(\cdot)$. This is a common knowledge: all bidders assume that the other bidders have a private value with this distribution.

For the auction, every bidder produces a bid, denoted with b. The mapping $v \mapsto b$ is called the bid function, denoted generally with $b = B(v)$. For the determination of the bid b, given the value v, each bidder solves an optimization problem. Taking into account the rules of the auction, and assuming that the $n-1$ other bidders use the same bid function $B(\cdot)$ and have some private value distributed according to $F(\cdot)$, she concludes that she will sell her item with some probability $\pi[B](b)$. The reference "$[B]$" stresses the fact that this evaluation depends on the other bidders' assumed behavior captured by $B(\cdot)$.

Since bidders are risk neutral, the expected benefit of the representative bidder, given that she bids b for an item of value v, is then $(b - v)\pi[B](b)$. Her optimization problem is therefore (see *e.g.* [Krishna, 2009, pp. 16–17]):

$$\max_{b \geq 0} \{(b - v)\pi[B](b)\} \tag{1}$$

If the resulting optimization results in the mapping $b = B(v)$, this bidding function is called an *equilibrium bidding function.* This principle leads to the definition of an equilibrium as a fixed point:

Definition 1 (Equilibrium Bid)**.** *A bidding function $B(\cdot)$ is an equilibrium of a reverse auction if $b = B(v)$ solves* (1)*. Equivalently, if*

$$B(v) \in \arg\max\{(b - v)\pi[B](b) \mid b \geq 0\}, \qquad \forall v \in \mathcal{V}$$

or

$$(B(v) - v)\pi[B](B(v)) \geq (b - v)\pi[B](b), \qquad \forall b \geq 0, \forall v \in \mathcal{V}.$$

This definition is as general as possible. In particular it does not impose properties a priori on the function $B(\cdot)$. Some properties will however be imposed when looking for some solutions.

3 Equilibrium bidding function in target-constrained reverse auctions with discriminatory rule payment

In target-constrained reverse auctions with discriminatory rule payment, the rules of the auction are as follows. The buyer sorts the n bids by increasing order, and the q first ones are selected.[1] The target number q is fixed and common knowledge. Each winning bidder is paid their own bid.

It is proved in *e.g.* [Hailu et al., 2005, Liu, 2021] that if B is an equilibrium bid function which is increasing and differentiable, then:

$$B(v) \;=\; \frac{\displaystyle\int_v^{\infty} uF(u)^{q-1}f(u)(1 - F(u))^{n-q-1}\mathrm{d}u}{\displaystyle\int_v^{\infty} F(u)^{q-1}f(u)(1 - F(u))^{n-q-1}\mathrm{d}u}. \tag{2}$$

[1] If there are tie bids, a rule may be necessary to break these ties. However, we shall not consider this situation for target-constrained auctions.

We shall be particularly interested in the case of $q = 1$ items, when values are uniformly distributed over $[0, 1]$. For comparison purposes, we are also interested in the total expected budget $\overline{W}^T_{n,q}$ spent by the buyer for the q items. Accordingly, we state the following result, which is derived for completeness from (2) in Appendix D.1.

Lemma 1. *Assume that values are uniformly distributed over the interval* $[0, 1]$*. Then if* $q = 1$*, an equilibrium bid function for the target-constrained auction is*

$$B^T_{n,1}(v) = \frac{1 + (n-1)v}{n} = \frac{1}{n} + \frac{(n-1)v}{n} \ . \tag{3}$$

The average budget spent by the buyer is, when $q = 1$*,*

$$\overline{W}^T_{n,1} = \frac{2}{n+1}. \tag{4}$$

4 Budget constrained reverse auctions

4.1 Definition of budget-constrained reverse auctions

In budget-constrained reverse auctions with discriminatory rule payment, the rules of the auction are as follows. Prior to the auction, the buyer announces a budget W that is common knowledge. After the bidding, the buyer ranks the n bids in ascending order, and takes the first bids (that is, buys the corresponding items), as long as the sum of these bids is smaller or equal than W. If she would buy one more item, the total sum would be strictly greater than W.

This rule corresponds to items which are indivisible: either they are bought entirely, or not at all. As a consequence, it may happen that some budget is lost. A variant exists where objects are *divisible* (or splittable), in which the remaining budget is used to buy a proportion of the first item which does not fit entirely in the budget (if any is left). The study of this variant is out of the scope of the present paper.

In order to determine the function $\pi(b)$, denote with $V_1, V_2, \ldots, V_{n-1}$ the private values of the $n-1$ other bidders, and $V^{(1)}, V^{(2)}, \ldots, V^{(n-1)}$ the increasing order statistics.[2] Assume the bidding function $B(\cdot)$ of the $n-1$ other bidders is increasing. Then the bids $B(V_i)$ are in the same order than the values. The buyer will choose the first Q items, such as:

$$B(V^{(1)}) + \ldots + B(V^{(Q)}) \ \leq \ W \ < \ B(V^{(1)}) + \ldots + B(V^{(Q)}) + B(V^{(Q+1)}) \ . \tag{5}$$

The representative seller will bid b. Her bid will be accepted by the buyer if the following event occurs:

$$\begin{cases} \quad b < B(V^{(1)}) \text{ and } b \leq W \\ \text{or} \quad B(V^{(1)}) \leq b < B(V^{(2)}) \text{ and } B(V^{(1)}) + b \leq W \\ \text{or} \quad \ldots \\ \text{or} \quad B(V^{(\ell)}) \leq b < B(V^{(\ell+1)}) \text{ and } \sum_{i=1}^{\ell} B(V^{(i)}) + b \leq W \\ \text{or} \quad \ldots \\ \text{or} \quad B(V^{(n-1)}) < b \text{ and } \sum_{i=1}^{n-1} B(V^{(i)}) + b \leq W. \end{cases} \tag{6}$$

This specification is enough if $B(\cdot)$ is strictly increasing and the distribution of V has a density, in which case bids (including the bid b) are all different with probability one. Otherwise, an additional tie-breaking rule must be added. In order to preserve the symmetry of the problem, it will involve typically some drawing at random uniformly among candidates.

[2] We use capital letters for random variables. According to the optimization procedure described in Section 2, the representative bidder assumes that other bidders have random private values with distribution function $F(\cdot)$.

4.2 Some general properties

The following properties generally hold.

Proposition 2. *Any equilibrium bid function $B(\cdot)$ for the budget-constrained auction with non-splittable objects has the following properties:*

i) $B(v) \geq v$ *for all* $v \in \mathcal{V}$ *such that* $\pi(B(v)) > 0$;

ii) *if* $B(v) = v$ *for some* $v \in \mathcal{V}$, *then* $\pi(b) = 0$ *for all* $b > v$;

iii) $B(v) \geq W/n$ *for all* $v \in \mathcal{V}$.

Proof. If $b = B(v)$ is the equilibrium bid of some player, then (see Definition 1) it must hold that

$$(B(v) - v)\pi(B(v)) \geq (b - v)\pi(b), \quad \forall b \geq 0. \tag{7}$$

If $B(v) < v$ and $\pi(B(v)) > 0$, we have a contradiction because the left-hand side is strictly negative, whereas the right-hand side can be made equal to 0 with $b = v$. This proves *i*).

If $B(v) = v$ then the left-hand side is 0, whereas the right-hand side would be strictly positive if there would exist some b with $b > v$ and $\pi(b) > 0$. This proves *ii*).

Assume now that $b^\dagger = B(v^\dagger) < W/n$ for some v. Then for both the bid $b = b^\dagger$ and $b = W/n$, we have $\pi(b) = 1$. Indeed, in the worst case where all other players have a smaller bid, the total budget requested is $\sum_{i=1}^{n-1} B(V_i) + b \leq (n-1)W/n + W/n = W$. The bid is then necessarily winning.

Then (7) implies, by selecting $b = W/n$ in the right-hand side, that: $B(v^\dagger) - v^\dagger \geq W/n - v^\dagger$. We have a contradiction with the hypothesis that $B(v^\dagger) < W/n$. This proves statement *iii*). □

Remark 1. *Given the rule of the auction, it is evident that a bidder whose value v is larger than the budget W cannot have a positive gain $(b - v)\pi(b)$: either they bid $b > W$ and $\pi(b) = 0$, or they bid $b \in (v, W]$ with a chance to win and $(b - v)\pi(b) < 0$. Then it is not clear why they would participate in the auction at all. If not participating is allowed, the buyer will actually face a random number of bidders who will all bid below W. On the other hand, in the analysis it makes no difference whether bidders in this situation actually bid some value $b > W$ or do not bid at all. We shall therefore stick with the idea that there are n bidders who may be constrained to bid above W.*

In the following sections, we exhibit equilibrium bid functions in some specific situations.

4.3 The case of large budget

The first case is the ideal situation where there is enough budget for everyone.

Proposition 3. *If the value v is almost surely bounded by $\bar{v}$ (i.e. $F(\bar{v}) = 1$), and if $W \geq n\bar{v}$, then $B(v) = W/n$ is an equilibrium bid function for the budget-constrained auction. In that case, the number of items bought is n and the budget spent is W.*

Proof. By the same argument in the proof of Proposition 2 *iii*), if $W \geq n\bar{v}$, then any bid $b \leq B(v) = W/n$ is such that $\pi(b) = 1$. If the player would bid $b > W/n$, then since the items are indivisible, $\pi(b) = 0$ and clearly (7) holds. This proves the proposition.

□

4.4 The case of small budget and uniform values

We consider here that there are n bidders, and that the private value of each item is uniformly distributed on $[0, 1]$. The general idea in this section is that when the budget W is "small enough", then *at most one item* will be bought. Indeed, as it turns out, bids can be larger than $W/2$, which implies that the sum of two bids exceeds the budget.

The main result is stated in the following theorem.

Theorem 4. *Assume that $n \geq 2$. Let $W_n \leq 1$ be the positive, non-zero root of the equation*

$$\frac{W}{2} = \frac{1-(1-W)^n}{n} . \tag{8}$$

Then for every $W \leq W_n$, the bid function $B^B_{n,W}(\cdot)$ defined as

$$B^B_{n,W}(v) = \begin{cases} B^T_{n,1}(v) - \dfrac{(1-W)^n}{n(1-v)^{n-1}} & v \leq W \\ v & v \geq W, \end{cases} \tag{9}$$

where $B^T_{n,1}(\cdot)$ is defined in (3), is an equilibrium bid for the budget-constrained auction.

The proof is presented in Appendix A.

Remark 2. *The equilibrium bidding function (9) proposed in Theorem 4 is unique in the class of increasing and piecewise differentiable functions such that $B(W) = W$. Indeed, as apparent from the proof, such functions are solutions to a regular differential equation, and uniqueness follows from the choice of the boundary condition. The choice of $B(W) = W$ as boundary condition is necessary in view of Proposition 2 ii).*

As a matter of fact, the differential equation turns out the same as in the analysis of the target-constrained auction with $q = 1$: this follows from the fact that only one item is bought under the conditions of Theorem 4. Then with the boundary condition $B(W) = W$, the solution with a general distribution $F(\cdot)$ for values would write as:

$$B(v) = v + \frac{\displaystyle\int_v^W (1-F(u))^{n-1}\,du}{(1-F(v))^{n-1}} ,$$

for $v \leq W$. This function will be an equilibrium bid under the condition that $B(0) \geq W/2$.

However, the uniqueness observed here is valid only for values $v \leq W$. We have noted in Remark 1 that any rational bid is losing if $v > W$. The lack of uniqueness in this case can be considered as non-essential.

Some properties of this equilibrium bid are as follows. See Appendix B for the proof.

Proposition 5. *Assume $W \leq W_n$. The equilibrium $B^B_{n,W}$ is such that:*

i) $B^B_{n,W}(v)$ is strictly increasing and continuous in v; in particular, $B^B_{n,W}(W) = W$ for all n.

ii) $W/n \leq B^B_{n,W}(0) \leq 1/n$ and $B^B_{n,W}(0) \sim W$ when $W \to 0$, for all n.

iii) The average budget spent is:

$$\overline{W}^B_{n,W} = \overline{W}^T_{n,1} \left[1 - (1-W)^n(1+nW)\right] . \tag{10}$$

iv) The average number of items bought is

$$\overline{N}^B_{n,W} = 1 - (1-W)^n . \tag{11}$$

Remark 3. *We can assess how small the budget W must be for the solution of Theorem 4 to hold. The equation determining W_n is (8):*

$$\frac{W}{2} = \frac{1-(1-W)^n}{n} ,$$

discarding the value $W = 0$ which also solves this equation. The first values are:

n	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*
W_n	*1*	*0.634*	*0.456*	*0.356*	*0.291*	*0.246*	*0.213*	*0.188*	*0.168*

The sequence W_n appears to be decreasing with respect to $n \geq 2$. An asymptotic analysis concludes that:

$$W_n \sim \frac{a}{n}, \quad \textit{where } a \simeq 1.594 \textit{ solves } a = 2(1 - e^{-a}).$$

Using these values in (10) *and* (11)*, we deduce that when n is large,*

$$\overline{W}^B_{n,W_n} \sim \frac{a(a-1)}{n} \simeq \frac{0.946}{n}, \qquad \overline{N}^B_{n,W_n} = \frac{nW_n}{2} \sim \frac{a}{2} \simeq 0.797.$$

We conclude from this that, even if we do not know yet what is an equilibrium bid for budget values $W > W_n$, we can state that, by setting $W = W_n$, the buyer can expect to buy at least 0.8 *items approximately. In addition, the budget needed for that goes to 0 when the number of sellers grows large.*

4.5 The case with $n = 2$ bidders

Assume in this section that there are $n = 2$ bidders. We do not assume *a priori* that the bid function $B(\cdot)$ is strictly increasing, so that ties in bids can occur with non-negative probability. Assuming that the winner is chosen by tossing a fair coin in such a situation, the winning event (6), refined with the consideration of ties, reduces to the conditions:

$$\begin{cases} & b < B(v_1) \text{ and } b \leq W \\ \text{or} & B(v_1) < b \text{ and } B(v_1) + b \leq W \\ \text{or} & B(v_1) = b \text{ and } 2b \leq W \\ \text{or} & B(v_1) = b,\ 2b > W \text{ and } b \leq W \text{ and the distinguished player wins the toss.} \end{cases} \tag{12}$$

The fourth line is the case where bids tie but the total bid is beyond budget.

If $b > W$, none of the conditions in (12) match, and the event is empty. Then, $\pi(b) = 0$ if $b > W$.

For $0 \leq b \leq W$, evaluating the probability of event (12) leads to the general formula below. Lines (13) and (14) are the respective probabilities of the two first lines in (12) and line (15) sums the probabilities of the two last lines in (12).

$$\pi(b) = \int_0^\infty \mathbf{1}_{\{b<B(x)\}} f(x) \mathrm{d}x \tag{13}$$

$$+ \int_0^\infty \mathbf{1}_{\{B(x)<b\}} \mathbf{1}_{\{B(x)+b\leq W\}} f(x) \mathrm{d}x \tag{14}$$

$$+ \int_0^\infty \mathbf{1}_{\{B(x)=b\}} \left(\mathbf{1}_{\{2b\leq W\}} + \frac{1}{2} \mathbf{1}_{\{2b>W\}} \right) f(x) \mathrm{d}x\ . \tag{15}$$

Assume now that values are uniformly distributed over the interval $[0, 1]$.[3] The equilibrium bid function for $n = 2$ has already been determined for $W \leq W_2 = 1$ in Theorem 4. It has also been determined for $W \geq 2$ in Proposition 3. There remains to identify it for $W \in [1, 2]$. The results are summarized in the following proposition, where cases already solved appear for completeness. The proof is given in Appendix C.

Theorem 6. *Assume that there are two bidders, and values are uniformly distributed over* $[0, 1]$*. Then equilibrium bid functions are given by:*

[3] These results can be extended to the case of uniformly distributed values over $[a, b]$, using the formulas: $B_{[a,b]}(v) = a + B_{[0,1]}((v - a)/(b - a))$.

if $W \in [0,1]$*:*

$$B^B_{2,W} = (v) \;=\; \begin{cases} \dfrac{1+v}{2} - \dfrac{(1-W)^2}{2(1-v)} & \textit{if } 0 \le v \le W \\ v & \textit{if } W \le v \le 1. \end{cases} \tag{16}$$

if $W \in [1,2]$*: either*

$$B^B_{2,W}(v) \;=\; \begin{cases} \dfrac{W}{2} & \textit{if } 0 \le v \le \sqrt{W-1} \\ \dfrac{1+v}{2} & \textit{if } \sqrt{W-1} < v \le 1, \end{cases} \tag{17}$$

or:

$$B^B_{2,W}(v) \;=\; \begin{cases} \dfrac{W}{2} & \textit{if } 0 \le v < \sqrt{W-1} \\ \dfrac{1+v}{2} & \textit{if } \sqrt{W-1} \le v \le 1. \end{cases} \tag{18}$$

if $W \ge 2$*:* $B^B_{2,W}(v) = W/2$*.*

The bid functions featuring in Theorem 6 are represented in Figure 1. The values $W \in [0,1]$ are on the left-hand side, values of $W \in [1,2]$ on the right-hand side.

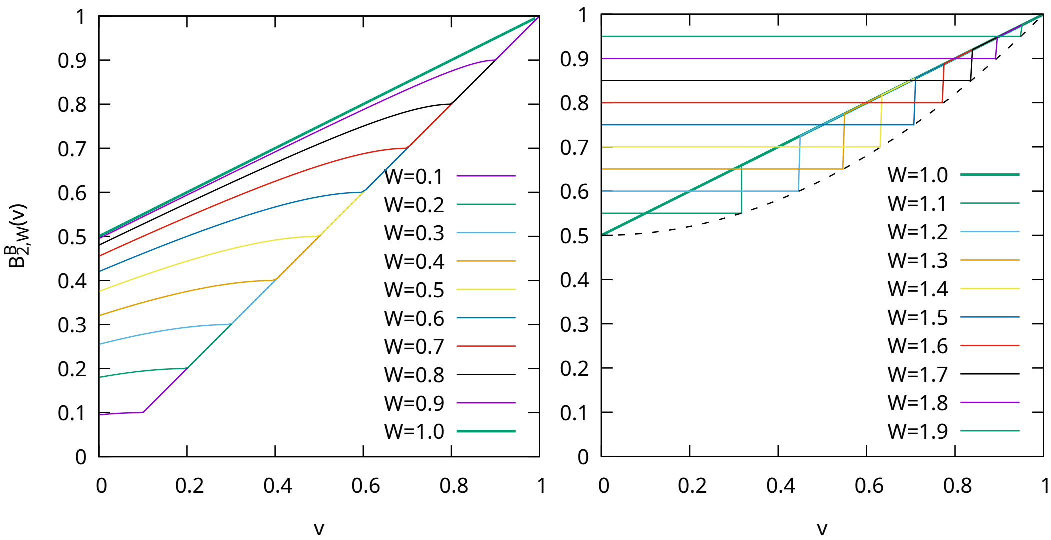

Figure 1: Equilibrium bid functions for $n = 2$ bidders: budget $W \in [0,1]$ (left) and $W \in [1,2]$ (right); the vertical part of the curve materializes the discontinuity but is not part of the function.

Remark 4. *Several remarks are in order. First of all, Theorem 6 proposes two equilibrium bid functions (17) and (18) for* $W \in (1,2)$*. These functions are discontinuous at* $v = \sqrt{W-1}$ *and both of them yield the same expected benefit for sellers: there is indifference between playing* $b = W/2$ *or playing* $b = (1+v)/2$ *for this value.*

In contrast, the solution for $W \le 1$ *is unique in the class of increasing and differentiable functions, see Remark 2. The argument in the proof of Theorem 6 can be used to show that when* $W < 1$*, there is no (discontinuous) equilibrium bidding function of the form* (21) *for some* $v_0 \in (0,1)$*.*

These features illustrate the complexity of finding equilibrium bids in the budget-constrained auctions. Till now, we were not able to find a solution where bids are continuous functions, for values of $W \in (1,2)$*.*

Some properties of the bid functions $B^B_{2,W}(\cdot)$ are as follows. The proof follows from Proposition 5 or simple calculations.

Proposition 7. *The average budget spent by the buyer is:*

$$\overline{W}^B_{2,W} = \begin{cases} \frac{2}{3}\,W^2(3-2W) & \text{if } W \le 1 \\ \frac{4W-1}{3}\,\sqrt{W-1} + \frac{5}{3} - W & \text{if } 1 \le W \le 2 \\ W & \text{if } W \ge 2. \end{cases}$$

The average number of items bought is:

$$\overline{N}^B_{2,W} = \begin{cases} W(2-W) & \text{if } W \le 1 \\ W & \text{if } 1 \le W \le 2 \\ 2 & \text{if } W \ge 2. \end{cases}$$

5 Comparison of target-constrained and budget-constrained auctions

Theorem 4 and Theorem 6 show an equilibrium bid for budget auctions provided that the item are not splittable, for some "small" range of values of W and in the case of two players. We are now able to compare with the equilibrium bid of target-constrained auctions for $n = 2$ and at least for some other values of n, using Lemma 1.

Proposition 8. *If $n = 2$ and $W = 1$ in the budget-constrained auction, and $q = 1$ in the target-constrained auction, the equilibrium bid functions coincide:*

$$B^T_{2,1}(v) = B^B_{2,1}(v) = \frac{1+v}{2}.$$

The average spent budget are the same, equal to 2/3, and the number of items bought is one.

Proof. The equivalence of the bid equilibrium is evident from Lemma 1 and Theorem 6. Moreover from Proposition 5 ii) and iii) the average spent budget is the same. For the number of bought items note that as $W = 1$ for any cost draw, one unit is bought. □

In Proposition 8 we answer the following question in a particular case: which budget must be announced in the budget-constrained auction in order to have the same equilibrium than in a target-constrained auction with $q = 1$.

We now turn to the case of general $n \ge 2$ but “small” budget. First of all, we state the following comparisons, easily derived from the formulas of Theorem 4 and Proposition 5:

Corollary 9. *When $W \le W_n$, $n > 2$, we have:*

$$B^B_{n,W} < B^T_{n,1}, \quad \overline{W}^B_{n,W} < \overline{W}^T_{n,1}, \quad \overline{N}^B_{n,W} < \overline{N}^T_{n,1}.$$

Proof. The comparisons follow from the observation that the values $\overline{W}^T_{n,1}$ and $\overline{N}^T_{n,1}$ are obtained from (10) and (11) by setting $W = 1$, and that these functions $W \mapsto \overline{W}^B_{n,W}$ and $W \mapsto \overline{N}^B_{n,W}$ are strictly increasing for $W < 1$. Observe also that $W_n < 1$ for $n > 2$. □

As a consequence, there is a trade-off between quantities bought and budget spent between the two auction formats: by using the budget-constrained auction, less budget will be spent, but also less items will be bought. Still, we would like to compare, in some sense, both formats. To make a fair and relevant comparison, we need to define *equivalent constraints*. We follow the idea proposed in [Schilizzi, 2017, Boxall et al., 2017, Coiffard et al., 2023], where the authors set one of the constraint exogenously and then define the other constraint endogenously according to the results of the auctions with exogenous constraints. We set here the target constraint $q = 1$ in the target-constrained auction and then we choose as budget in the budget-constrained auction the value $W = \overline{W}^T_{n,1}$, that is, the average of the budget spent in the target-constrained auction. This will be possible only for values $n \leq 6$ because for $n > 6$ we have that $W_n < \overline{W}^T_{n,1}$. The budget-constrained auction will be then characterized by a couple $(\overline{N}^B_{n,\overline{W}^T_{n,1}}, \overline{W}^B_{n,\overline{W}^T_{n,1}})$. However, Corollary 9 still applies to these values so that it is not clear what is the "best" situation.

The objective of the auctioneer is both to maximize the number of items purchased and to minimize the budget spent. As in [Coiffard et al., 2024] we use as a synthetic criterion an average unit cost (UC), defined as the ratio between average spent budget and average of bought items. We assume that the auctioneer prefers to use the auction format that yields the smallest UC.

The results of this procedure are presented in Table 1. The two lines labelled with "Target" and "Budget" show, for each value of n, the couples (average number of units, average of budget). In other terms, these are respectively $(\overline{N}^T_{n,1}, \overline{W}^T_{n,1})$ and $(\overline{N}^B_{n,\overline{W}^T_{n,1}}, \overline{W}^B_{n,\overline{W}^T_{n,1}})$. The two last lines labeled with "UC Target" and "UC Budget" show the ratios $\overline{W}^T_{n,1}/\overline{N}^T_{n,1}$ and and $\overline{W}^B_{n,\overline{W}^T_{n,1}}/\overline{N}^B_{n,\overline{W}^T_{n,1}}$. All values in this table are rounded to three decimal places.

n	2	3	4	5	6
Target	(1, 0.667)	(1, 0.5)	(1, 0.4)	(1, 0.334)	(1, 0.286)
Budget	(0.889, 0.494)	(0.875, 0.344)	(0.870, 0.265)	(0.868, 0.216)	(0.867, 0.183)
UC Target	0.667	0.5	0.4	0.334	0.286
UC Budget	0.556	0.393	0.304	0.249	0.211

Table 1: Comparison of both auction schemes with the equivalent constraints: couples (average number of units, average of budget) when $q = 1$ for Target and $W = \overline{W}^T_{n,1}$ for Budget, and criterion UC.

From this table, we can state the following comparison result:

Proposition 10. *With equivalent constraints and using the defined UC criteria, the budget-constrained auction performs better than the target-constrained auction when* $2 \leq n \leq 6$.

A graphical way to see this comparison is to represent the situation in the space (average number of items, average budget). This is done in Figure 2, where we plot with solid lines, the points $(\overline{N}^B_{n,W}, \overline{W}^B_{n,W})$ for $W \in [0, W_n]$. The curve is prolonged in dashed lines for values of $W \in [W_n, 1]$. Interestingly, as observed in the proof of Corollary 9, this prolongation includes the point $(\overline{N}^T_{n,1}, \overline{W}^T_{n,1})$, represented with squares. The points on these curves where $W = W_n$ are represented with crosses. The points $(\overline{N}^B_{n,\overline{W}^T_{n,1}}, \overline{W}^B_{n,\overline{W}^T_{n,1}})$ are represented with triangles. As observed above, the triangles are located on the solid part of the lines only for $n \leq 6$. According to Remark 3, the abscissas of the crosses would converge to $\simeq 0.797$ when $n \to \infty$. Following the same asymptotic analysis, the abscissas of the triangles would converge to $1 - e^{-2} \simeq 0.865$.

The criterion UC is the slope of the straight line joining the origin and the point considered. The fact that this criterion is smaller for the budget-constrained auction (triangles) than for the target-constrained auction (squares) results from the fact that the curve is increasing and convex, a feature proved in Appendix D.2.

6 Conclusions and extensions

We have shown in this papers the first theoretical results for budget-constrained auctions. These results take advantage of the fact that, for indivisible items, when the budget constraint is strong,

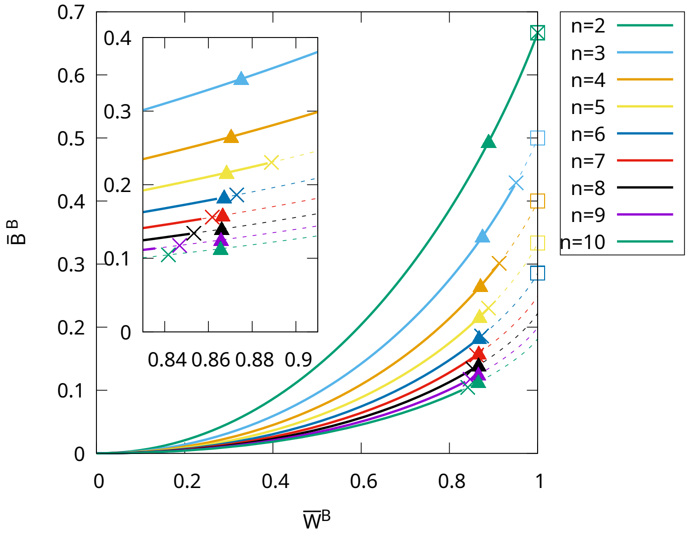

Figure 2: The average budget as a function of the average quantity

only one item is bought and the analysis is similar to the target-constrained auction with $q = 1$ item.

The next steps in this analysis will be to extend the solution to values of the budget larger than the critical value W_n. In that situation, the buyer can possibly afford buying two items, and the probability of a seller winning a bid is more complex. Referring to Section 4 and Appendix A, the event called E_2 has a positive probability, and more lines in the event (6) may have positive probability. This applies also to the case $n = 2$ and $W \in (1, 2)$: a complete solution could provide equilibrium bid functions without the discontinuity of the solutions in Theorem 6.

Another relevant line of research is about the case of *divisible* items. On the one hand, the situation may look more regular since there is a more smooth transition for sellers, between not selling and selling their item. However, the evaluation of the winning probability (or rater in this situation, winning proportion), appears to be even more difficult than for indivisible items.

A Proof of Theorem 4

Proof. The proof consists in showing that the function $B^B_{n,W}(\cdot)$ is effectively an equilibrium under the assumption that $W \le W_n$. For the purpose of this proof, we simplify the notation $B^B_{n,W}$ and use B instead.

We begin with a preliminary analysis of the function $B(v)$. It is easily shown that this function is strictly increasing and piecewise differentiable. Moreover, $B(0) = (1-(1-W)^n)/n \ge W/2$ under the assumption that $W \le W_n$. To see this, consider the function $W \mapsto h(W) := (1-(1-W)^n)/n$. This is an increasing function. We have $h(0) = 0$ and $h(1) = 1/n$. Moreover, $h'(0) = 1$. On the other hand, the function $W \mapsto k(W) := W/2$ is also increasing and such that $k(0) = 0$ but with a smaller slope than $h(\cdot)$. Since $k(1) = 1/2 \ge h(1)$, there exists an intersection W_n of both functions in the interval $[0, 1]$. If $0 \le W \le W_n$ then $h(W) \ge k(W) = W/2$. The fact that $B(0) \ge W/2$ implies that the bid of all other players is $> W/2$ with probability 1, and that the sum of two bids exceeds the budget with probability 1. In the condition (5), this means that the number of items bought Q cannot be larger than 1 if all bidders use $B(\cdot)$ as bid function.

Consider the best response of some distinguished player to the bid function $B(v)$ assumed to be used by the $n-1$ other players. Let v be the private value of the player and b her bid. According to the conditions (6) and taking into account the fact that $B(V^{(1)}) + B(V^{(2)}) > 1$ w.p. 1, the bid is won by this player if and only if:

$$\underbrace{\Big(b \leq B(V^{(1)}) \ \text{ and } \ b < W\Big)}_{E_1} \ \text{or} \ \underbrace{\Big(B(V^{(1)}) \leq b < B(V^{(2)}) \text{ and } B(V^{(1)}) + b \leq W\Big)}_{E_2}.$$

The second event E_2 is empty: if $b \geq B(V^{(1)})$ then $b + B(V^{(1)}) \geq 2B(V^{(1)}) > W$.

The first event E_1 is equivalent to

$$b \leq W \quad \text{and} \quad \min\{V_1, \ldots, V_{n-1}\} > B^{-1}(b).$$

The possibility that $B(V_i) = b$ for some players i is discarded because it is of probability 0, since $B(\cdot)$ is strictly increasing and the random variable V has a density. It follows that, for $b \leq W$:

$$\pi(b) = \mathbb{P}(\text{bid } b \text{ wins}) = \mathbb{P}(E_1) = (1 - B^{-1}(b))^{n-1} \ ,$$

whereas $\mathbb{P}(\text{bid } b \text{ wins}) = 1$ if $b \leq B(0)$ and $\mathbb{P}(\text{bid } b \text{ wins}) = 0$ for $b > W$.

The distinguished player must optimize the function $G(v, b) := (b - v)\pi(b)$ with respect to b. We have just shown that:

$$G(v,b) = \begin{cases} b - v & b \leq B(0) \\ (b-v)(1 - B^{-1}(b))^{n-1} & B(0) \leq b \leq W \\ 0 & b > W. \end{cases} \tag{19}$$

Clearly (see Remark 1), if $v > W$, $G(v, b) \leq 0$ for all values of b, so that the optimum is to bid any $b > W$ so as to secure $G(v, b) = 0$. In particular, choosing $B(v) = v$ is optimal, which agrees with (9).

For $v \leq W$, the problem becomes to maximize (19) for $b \in [v, W]$. Indeed, for such values of b, $G(v, b) \geq 0$, so that any bid $b < v$ or $b > W$ would be sub-optimal. The first-order condition for an optimum bid is:

$$0 = (1 - B^{-1}(b))^{n-1} - (b - v)(n-1)\frac{(1 - B^{-1}(b))^{n-2}}{B'(B^{-1}(b))} \ ,$$

or equivalently:

$$0 = B'(B^{-1}(b))(1 - B^{-1}(b)) - (n-1)(b - v) \ .$$

Requiring that $b = B(v)$ solves this equation leads to:

$$0 = B'(v) \ (1 - v) - (n-1)(B(v) - v) \ . \tag{20}$$

Using the definition (9), we check that for $v \leq W$,

$$\begin{aligned}
&B'(v)(1-v) - (n-1)(B(v) - v) \\
&= \left(\frac{n-1}{n} - \frac{n-1}{n}\frac{(1-W)^n}{(1-v)^n}\right)(1-v) - (n-1)\left(\frac{1+(n-1)v}{n} - \frac{(1-W)^n}{n(1-v)^{n-1}} - v\right) \\
&= \frac{n-1}{n}\left(1 - \frac{(1-W)^n}{(1-v)^n}\right)(1-v) - \frac{n-1}{n}\left(1 - v - \frac{(1-W)^n}{(1-v)^{n-1}}\right) = 0.
\end{aligned}$$

The number $b = B(v)$ therefore solves (20) and the optimization problem if and only if $B(v) \in [v, W]$ for all $v \leq W$. We have:

$$B(v) - v = \frac{1-v}{n} - \frac{(1-W)^n}{n(1-v)^{n-1}} = \frac{(1-v)^n - (1-W)^n}{n(1-v)^{n-1}} \ \geq 0.$$

Then, since B is increasing, $v \leq W$ implies $B(v) \leq B(W) = W$. We conclude that indeed $B(v) \in [v, W]$ and that $B(v)$ solves the fixed-point problem for $v \leq W$ as well. This concludes the proof. $\square$

B Proof of Proposition 5

Proof. The proof of $i)$ follows from simple calculations.

Proof of $ii)$: the value of $B^B_{n,W}(0)$ is $(1-(1-W)^n)/n \leq 1/n$. The lower bound follows from the fact that $(1-W)^n \leq 1-W$. The asymptotic result follows from the fact that $(1-W)^n \sim 1-nW$ when $W \sim 0$.

Proof of $iii)$: the budget B of the buyer and the number of items bought Q can be expressed as follows. Let $M_n := \min\{B^B_{n,W}(V_1), \ldots, B^B_{n,W}(V_n)\}$. Then

$$B = \begin{cases} M_n & \text{if } M_n \leq W \\ 0 & \text{if } M_n > W \end{cases} \qquad Q = \begin{cases} 1 & \text{if } M_n \leq W \\ 0 & \text{if } M_n > W. \end{cases}$$

Accordingly,

$$\begin{aligned}
\overline{W}^B_{n,W} &= \int_{v_1=0}^{W} n! \left(\int_{v_1 \leq v_2 \leq \ldots \leq v_n \leq 1} B^B_{n,W}(v_1) \mathrm{d}v_2 \ldots \mathrm{d}v_n \right) \mathrm{d}v_1 \\
&= n! \int_0^W \frac{1}{n} \left(1 + (n-1)v_1 - \frac{(1-W)^n}{(1-v_1)^{n-1}} \right) \frac{(1-v_1)^{n-1}}{(n-1)!} \,\mathrm{d}v_1 \\
&= \int_0^W \left((1+(n-1)v_1)(1-v_1)^{n-1} - (1-W)^n \right) \mathrm{d}v_1 \\
&= \int_0^W (n + (n-1)(v_1-1))\,(1-v_1)^{n-1} \,\mathrm{d}v_1 - W(1-W)^n \\
&= \frac{n}{n} \left[-(1-v_1)^n \right]_0^W - \frac{n-1}{n+1} \left[-(1-v_1)^{n+1} \right]_0^W - W(1-W)^n \\
&= 1 - (1-W)^n - \frac{n-1}{n+1} \left\{ 1 - (1-W)^{n+1} \right) - W(1-W)^n \\
&= \frac{2}{n+1} - (1-W)^n \left[\frac{n-1}{n+1}(1-W) + 1 + W \right] \\
&= \frac{2}{n+1} \left[1 - (1-W)^n (1+nW) \right]
\end{aligned}$$

which is (10).

Proof of $iv)$: two cases occur: either some bid is smaller than W and one item exactly is bought, or else no transaction occurs. Accordingly, $\overline{N}^B_{n,W} = \mathbb{E}(Q) = 1 - \mathbb{P}(M_n > W) = 1 - (1-W)^n$ which is (11). ☐

C Proof of Theorem 6

Proof. The statement for $W \in [0,1]$ is contained in Theorem 4 since $W_2 = 1$ (see Remark 3). The statement for $W \geq 2$ is Proposition 3, since $n = 2$ and $\bar{v} = 1$. There remains to prove the case $W \in [1,2]$: the proof consists in computing $\pi(b)$, then the best response to the proposed bid function. If this best response coincides with the proposed bid, we have an equilibrium.

Assume that the opponent uses the bid function (17), that is,

$$B(v) \;\; = \begin{cases} W/2 & \text{if } v \leq v_0 \\ (1+v)/2 & \text{if } v > v_0, \end{cases} \tag{21}$$

where $v_0 = \sqrt{W-1} \in [0,1]$. Observe that $(1+v_0)/2 \geq W/2$ (or equivalently, $v_0 \geq W-1$), so that the function $B(v)$ is indeed increasing although not continuous at $v = v_0$.

Assume now that $b \leq W/2$. Then either $b < W/2$ and the bid wins certainly, or $b = W/2$ and the bid may tie with the opponent's with probability v_0. But since the sum of both bids is smaller than W, the item is bought. In summary, $\pi(b) = 1$ in this case.

Assume now that $b \in (W/2, (1+v_0)/2]$. The probability to sell the item is the probability of winning the bid because the total budget is necessarily larger than W. Then we have $\pi(b) = \mathbb{P}(b < B(V)) = \mathbb{P}(V > v_0) = 1 - v_0$.

Next, if $W \geq b \geq (1+v_0)/2$, this probability is $\pi(b) = \mathbb{P}(b < B(V)) = \mathbb{P}(b < (1+V)/2) = \mathbb{P}(V > 2b-1) = (1-(2b-1))^+ = 2(1-b)^+$ (using the information that $b > W/2 \geq 1/2$).

Finally, if $b > W$, $\pi(b) = 0$. In summary, $\pi(b)$ is:

$$\pi(b) = \begin{cases} 1 & \text{if } b \leq W/2 \\ 1 - v_0 & \text{if } W/2 < b \leq (1+v_0)/2 \\ 2(1-b) & \text{if } (1+v_0)/2 \leq b \leq 1 \\ 0 & \text{if } b > 1. \end{cases}$$

If the opponent would use the bid function (18), the probability of winning $\pi(b)$ would be the same: this follows from the fact that $\mathbb{P}(V = v_0) = 0$. This enumeration defines four intervals for b, denoted with: I_1, I_2, I_3 et I_4. We shall study the maximum of $G(v,b) = (b-v)\pi(b)$ over each of them, then globally. Obviously, over I_4 we have $G(v,b) = 0$.

Over intervals I_1 and I_2, since $\pi(b)$ does not depend on b, the maximum is clearly to take b as large as possible. The potential maxima are therefore:

$$G(v,b_1) := \frac{W}{2} - v \qquad G(v,b_2) := \left(\frac{1+v_0}{2} - v\right)(1-v_0)$$

with $b_1 = W/2$ and $b_2 = (1+v_0)/2$. Over interval I_3, the function $G(v,b)$ is quadratic and concave. Its maximum is attained at value $b_3 = (1+v)/2$, provided $b_3 \in I_3$. Since $v \leq 1$, we have $b_3 \leq 1$. Therefore, $b_3 \in I_3$ if and only if $b_3 \geq (1+v_0)/2$, that is, $v \geq v_0$.

The discussion therefore involves a comparison of v with v_0.

Case $v \leq v_0$**.** In that case, $b_3 \notin I_3$, and the maximum of $G(v,b)$ over I_3 is attained at $b_2 = (1+v_0)/2$. Then we must compare $G(v,b_1)$ and $G(v,b_2)$. We have the equivalences:

$$\begin{aligned} G(v,b_1) = \frac{W}{2} - v &\lesseqqgtr \left(\frac{1+v_0}{2} - v\right)(1-v_0) = G(v,b_2) \\ W &\lesseqqgtr (1+v_0-2v)(1-v_0) + 2v = 1 - v_0^2 + 2vv_0 = 2 - W + 2vv_0 \\ 0 &\lesseqqgtr 2vv_0 - 2v_0^2 = 2(v-v_0)v_0 \ . \end{aligned}$$

We conclude that $G(v,b_1)$ is always larger, so that the best response is $b = W/2$.

Case $v \geq v_0$**.** In that case, the maximum over I_3 is indeed attained at b_3, and the value is:

$$G(v,b_3) := \frac{(1-v)^2}{2} \ .$$

Since $b_2 \in I_3$, we have $G(v,b_3) \geq G(v,b_2)$ and we must compare $G(v,b_1)$ and $G(v,b_3)$. We have the equivalences:

$$\begin{aligned} G(v,b_1) = \frac{W}{2} - v &\leq \frac{(1-v)^2}{2} = G(v,b_3) \\ W - 2v &\leq 1 - 2v + v^2 \\ W - 1 &\leq v^2, \end{aligned}$$

that is, $v_0 \leq v$. The global maximum of $G(v,b)$ is indeed attained at b_3 and the best bid is $(1+v)/2$. For the particular value $v = v_0$, we have $b_2 = b_3$ and also $G(v,b_1) = G(v,b_3)$, so that there is indifference between the two bids. Intermediate bids are however not optimal.

Synthesizing the cases, we conclude that the optimal bid of the distinguished player is given by either (17) or (18). □

D Properties of equilibrium bids

D.1 Proof of Lemma 1

Proof. When values are uniformly distributed over the interval $[0,1]$, then $f(x) = 1$ and $F(x) = x$ for $x \in [0,1]$, whereas $f(x) = 0$ otherwise. Then (2) becomes:

$$B^T_{n,1}(v) = \frac{\displaystyle\int_v^1 u(1-u)^{n-2}\mathrm{d}u}{\displaystyle\int_v^1 (1-u)^{n-2}\mathrm{d}u}, \quad \text{where:}$$

$$\int_v^1 (1-u)^{n-2}\mathrm{d}u = \frac{(1-v)^{n-1}}{n-1}$$

$$\begin{aligned}\int_v^1 u(1-u)^{n-2}\mathrm{d}u &= \int_v^1 (u-1+1)(1-u)^{n-2}\mathrm{d}u = -\int_v^1 (1-u)^{n-1}\mathrm{d}u + \int_v^1 (1-u)^{n-2}\mathrm{d}u \\ &= -\frac{(1-v)^n}{n} + \frac{(1-v)^{n-1}}{n-1} = (1-v)^{n-1}\,\frac{n-(1-v)(n-1)}{n(n-1)}\end{aligned}$$

$$B^T_{n,1}(v) = \frac{1+(n-1)v}{n}. \tag{3}$$

The value of average budget spent follows since

$$\begin{aligned}\overline{W}^T_{n,1} &= \mathbb{E}B_1(V^{(1:n)}) \\ &= \int_0^1 B_1(v)n(1-v)^{n-1}\mathrm{d}v \;=\; \int_0^1 (1+(n-1)v)(1-v)^{n-1}\mathrm{d}v \\ &= n\int_0^1 (1-v)^{n-1}\mathrm{d}v - (n-1)\int_0^1 (1-v)^n\mathrm{d}v \;=\; 1-\frac{n-1}{n+1} \;=\; \frac{2}{n+1}\,. \end{aligned} \tag{4}$$

□

D.2 Proof of convexity

The curve under study is a parametric curve with

$$x = 1-(1-W)^n, \qquad y = \frac{2}{n+1}\,[1-(1-W)^n(1+nW)].$$

Eliminating W from the first equation, we have: $W = 1-(1-x)^{1/n}$ and then:

$$\begin{aligned} y = f(x) &= \frac{2}{n+1}\,\left[1-(1-x)(1+n(1-(1-x)^{1/n}))\right] \\ &= \frac{2(1-(1-x)(1+n))}{n+1} + \frac{2n}{n+1}(1-x)^{1/n+1}\end{aligned}$$

We want to prove that f is increasing and convex for $x \in [0,1]$. It is the sum of an affine part and a second part which is convex since the function $(1-x)^{1/n+1}$ is convex. Differentiating,

$$\begin{aligned} f'(x) &= 2 - \frac{2n}{n+1}\,(1/n+1)\,(1-x)^{1/n} \\ &= 2(1-(1-x)^{1/n}) \geq 0.\end{aligned}$$

So indeed, f is increasing and convex.

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