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Abstract. The second author has shown that existence of extremal Kähler metrics on semisimple principal toric fibrations is equivalent to a notion of weighted uniform K-stability, read off from the moment polytope. The purpose of this article is to prove various sufficient conditions of weighted uniform K-stability which can be checked effectively and explore the low dimensional new examples of extremal Kähler metrics it provides.

1. Introduction

Calabi's work has been extremely influencial in Kähler geometry, his name being still associated to some of the most fundamental objects of interest. The present article is motivated by two of these, Calabi's extremal Kähler metrics and Calabi's ansatz.

Extremal Kähler metrics provide a natural notion of canonical Kähler metrics in a given Kähler class on a compact Kähler manifold X: they are the metrics that achieve the minimum of the L^2-norm of the scalar curvature. Kähler metrics with constant scalar curvature (cscK metrics for short) are special cases of such metrics, but Calabi showed in [6] that there may exist extremal Kähler metrics when there exists no cscK metrics at all, by exhibiting extremal Kähler metrics on Hirzebruch surfaces. In order to show this, Calabi relied on the simple yet powerful idea that one should search for extremal Kähler metrics among those Kähler metrics that behave well with respect to the geometry of the manifold.

This was not a new idea of course. Matsushima showed for example [26] that cscK metrics must behave well with respect to biholomorphism. More precisely, the automorphism group of X must be the complexification of the isometry group of the cscK metric, if it exists. This is preventing Hirzebruch surfaces from admitting cscK metrics as their automorphism group is non-reductive.

Calabi went further and restricted to metrics that respect the structure of $\mathbb{P}^1$-bundles of Hirzebruch surfaces. He was then able to translate, for such metrics, the extremal property into a simple ODE and to solve it, showing the existence of extremal Kähler metrics. His construction was later referred to as Calabi's ansatz, used in various situations and generalized in various directions. It would be easy to fill pages with a bibliographical review of these, but it is not the purpose of this introduction. We only stress that a common theme is usually the desire to get explicit existence results or criterions. An influential illustration is [1], where a variant of Calabi's ansatz was used to

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show tat on various $\mathbb{P}^1$-bundles, existence of extremal Kähler metrics reduces to checking the positivity of a polynomial on $[-1,1]$, the so-called extremal polynomial. In the series of papers leading to [1], the general idea of Calabi's ansatz was actually pushed way further, allowing for example to consider certain fibrations with toric fiber.

The interest for such fibrations was significantly renewed last year, when the second author proved in [17], using the breakthrough results of Chen and Cheng [7], that a uniform version of the Yau-Tian-Donaldson conjecture holds for semisimple principal toric fibrations, a very large class of toric fibrations. While it allows to translate the question of existence of extremal Kähler metrics on such manifolds into a question of convex geometry on their moment polytopes, it is not yet an explicitly checkable criterion, as the conditions to check still form an infinite dimensional space. Motivated by the more practical philosophy behind Calabi's ansatz, we prove in the present paper various sufficient conditions of existence of extremal Kähler metrics which may be easily checked. Our approach is based on an initial idea by Zhou and Zhu [28], exploited in greater generality by the first author in [8].

Let us now highlight in the remainder of this introduction our main results. For this, a few notations are needed. Semisimple principal toric fibrations are certain holomorphic fiber bundles $\pi : Y \to B$ where the basis $B = \prod_a B_a$ is a product of Hodge manifolds (B_a, ω_a) with constant scalar curvature s_a, and toric fiber X under a torus $\mathbb{T}$. They are constructed from certain types of principal $\mathbb{T}$-bundles, essentially determined by the data of a tuple (p_a) of one-parameter subgroups of $\mathbb{T}$. On such manifolds, a Kähler class is called compatible if it decomposes as the sum of a relative Kähler class induced by a Kähler class $[\omega_X]$ on X, and a sum of real multiples $c_a\pi^*[\omega_a]$ of the pull-backs of the Kähler classes $[\omega_a]$. An admissible Kähler class contains admissible Kähler metrics, that behave well with respect to the fibration structure.

Theorem 1.1. *Assume that Y is a semisimple principal toric fibration, that the toric fiber X is Fano equipped with the Kähler class $[\omega_X] = tc_1(X)$, and let $[\omega_Y]$ be an admissible Kähler class. Assume that with the notations above, for all a, $2\dim(B_a)c_a \geq ts_a$ and that at every vertex x of the moment polytope P of $(X, [\omega_X])$,*

$$2(\dim(Y)+1) + \sum_a \frac{ts_a - 2\dim(B_a)c_a}{p_a(x)+c_a} - tl_{\mathrm{ext}}(x) \geq 0$$

where l_{ext} is the extremal affine function. Then there exists an extremal Kähler metric in $[\omega_Y]$.

Here the extremal function is encoding the extremal vector field, such that the scalar curvature of the extremal Kähler metric, if it exists, is a holomorphy potential of this vector field. We refer to the body of the paper for the precise conventions used. We actually prove a much more general sufficient condition, Theorem 2.6 that does not require the fiber to be Fano. Since we obtain already a wealth of new examples with this particular case, and it is a natural generalization of the $\mathbb{P}^1$-bundle case, we focus on this result for the introduction.

As a simple corollary, we get:

Corollary 1.2. *A Fano semisimple principal toric fibration Y admits an extremal Kähler metric in $c_1(Y)$ if its extremal function l_{ext} satisfies*

$$\sup l_{\text{ext}} \leq 2(\dim(Y) + 1)$$

and the latter obviously needs only be verified at vertices of the moment polytope.

We provide, for the reader's convenience, an elementary Python program implementing the sufficient condition from Theorem 1.1 in the case when there is only one factor in B and the fiber is of dimension one or two. It would be easy to imitate these to allow greater flexibility in the data. It may be used either with all the data given numerically, or some of the data treated as variable. We use this to our advantage to prove the existence of extremal Kähler metrics in a wide range of Kähler classes for some examples of fibrations.

Proposition 1.3. *Let $Y = \mathbb{P}_B(\mathcal{O} \oplus H^{-p_1} \oplus H^{-p_2})$, where B is a Kähler-Einstein Fano threefold, H is the smallest integral divisor of $c_1(X)$ and $1 \leq p_1 \leq p_2$. Then there exists an extremal Kähler metric in the Kähler class $c_1(X) + \lambda c_1(B)$ for $\lambda \geq 7p_2$, where $c_1(X)$ and $c_1(B)$ respectively denote the relative first Chern class and the pull-back of the first Chern class, by an abuse of notations.*

The article is organized as follows. In Section 2, we prove a general sufficient condition for weighted uniform K-stability of labelled polytopes, and consider the special case of monotone polytopes. Section 3 provides the geometric translation of this sufficient condition for weighted cscK metrics on toric manifolds and for extremal Kähler metrics on semisimple principal toric fibrations. There we prove Theorem 1.1 and Corollary 1.2 using the monotone case of Section 2, as well as more general statements. We present various examples of applications of the sufficient condition in Section 4, including Proposition 1.3. Finally, we include in an appendix elementary Python programs computing the sufficient condition for fibrations with only one factor in the basis, and a one or two dimensional Fano fiber.

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2. Weighted K-stability of labelled polytopes: a sufficient condition

2.1. **Weighted K-stability of labelled polytopes.** Let V be an affine space of dimension ℓ, equipped with a fixed Lebesgue measure dx. A *labelled polytope* in V is a pair $(P, \mathbf{L})$ where P is a (compact, convex) polytope in V and $\mathbf{L} = (L_j)_{j=1}^d$ is a set of defining affine functions for P, that is,

$$P = \{x \in V \mid \forall j, \quad L_j(x) \geq 0\}$$

where d is the number of facets (codimension one faces) of P. We denote by $F_j := \{x \in P \mid L_j(x) = 0\}$ the facet of P defined by L_j.

Definition 2.1. *The labelled boundary measure $d\sigma$ is the measure on ∂P whose restriction to the facet F_j is defined by $dL_i \wedge d\sigma = -dx$.*

Following [12, 18, 23], for $\mathrm{v} \in \mathcal{C}^0(P, \mathbb{R}_{\geq 0})$ and $\mathrm{w} \in \mathcal{C}^0(P, \mathbb{R})$, we define the (v, w)-Donaldson–Futaki invariant of the labelled polytope $(P, \mathbf{L})$ as the functional $\mathcal{F}$ on $\mathcal{C}^0(P, \mathbb{R})$ such that

$$\mathcal{F}(f) := 2\int_{\partial P} f(x)\mathrm{v}(x) d\sigma - \int_P f(x)\mathrm{w}(x) dx. \tag{1}$$

Definition 2.2. *A labelled polytope $(P, \boldsymbol{L})$ is (v, w)-uniformly K-stable if there exists a $\lambda > 0$ such that for any continuous convex functions f on P,*

$$\mathcal{F}(f) \geq \lambda \inf_{l \in \mathrm{Aff}(V)} \int_P \left(f + l - \inf_P (f + l) \right) \mathrm{v} dx \tag{2}$$

where $\mathrm{Aff}(V)$ denotes the space of affine functions on V.

Remark 2.3. Note that $\mathcal{F}$ is linear, and the right-hand side of (2) is always non-negative, hence the following is a necessary condition for (2) to hold:

$$\forall f \in \mathrm{Aff}(V), \mathcal{F}(f) = 0. \tag{3}$$

2.2. **The sufficient condition.** We assume from now on that v is given by the restriction of a C^1 function defined on an open subset of V containing P, which is positive on the interior P^0 of P.

We denote by $\mathcal{CV}^0(P)$ the space of continuous convex functions on P, and by $\mathcal{CV}^1(P)$ the space of all convex functions f on P which are the restrictions to P of a C^1 function defined on an open subset of V containing P. Note that by uniform approximation by C^1 functions, it is enough to consider only functions in $\mathcal{CV}^1(P)$ to check condition (2).

In order to deal more efficiently with the right hand side of (2), following [12], we consider the following normalization of functions. We choose a point x_0 in the interior P^0 of the polytope P. It allows to choose a linear complement $\mathcal{CV}^1_*(P)$ to $\mathrm{Aff}(V)$ in $\mathcal{CV}^1(P)$, defined by

$$\mathcal{CV}^1_*(P) := \left\{ f \in \mathcal{CV}^1(P) \mid \forall x, f(x) \geq f(x_0) = 0 \right\}. \tag{4}$$

Then, any $f \in \mathcal{CV}^1(P)$ can be written uniquely as $f = f^* + f_0$, where f_0 is affine and $f^* \in \mathcal{CV}^1_*(P)$, and we will use these notations in the following. By linearity, $\mathcal{F}(f) = \mathcal{F}(f^*)$ if $\mathcal{F}$ vanishes on $\mathrm{Aff}(V)$.

Lemma 2.4. *The labelled polytope $(P, \boldsymbol{L})$ is (v, w)-uniformly K-stable if and only if there exists $\lambda > 0$ such that for all $f \in \mathcal{CV}^1(P)$,*

$$\mathcal{F}(f) \geq \lambda \|\mathrm{v} f^*\|_1 \tag{5}$$

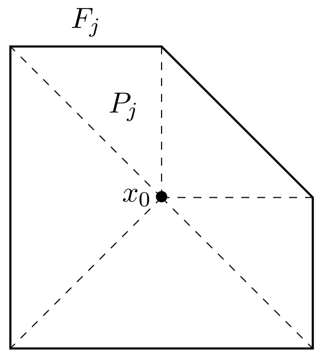
where $\|\cdot\|_1$ denotes the L^1-norm on P with respect to the Lebesgue measure dx.

Proof. One direction is immediate since by definition of $f^* = f - f_0$,

$$\|\mathrm{v} f^*\|_1 \geq \inf_{l \in \mathrm{Aff}(V)} \int_P \left(f + l - \inf_P (f + l) \right) \mathrm{v} dx.$$

Conversely, we can apply the same proof as in [15, Lemma 3.5] (which established this result for $\mathrm{v} = 1$) except one necessary modification: because v can vanish on the

FIGURE 1. The cone decomposition



boundary, we need to use the pre-compactness result of [8, Proposition 7.2] instead of [12, Corollary 5.2.5].

$\square$

Remark 2.5. If $\mathrm{v} > 0$ on P, then the labelled polytope $(P, \mathbf{L})$ is (v, w)-uniformly K-stable if and only if there exists $\lambda > 0$ such that for all $f \in \mathcal{CV}^1(P)$,

$$\mathcal{F}(f) \geq \lambda \|f^*\|_1. \tag{6}$$

Indeed, it suffices to multiply the constant λ by the minimum of v on P to go from one characterization to the other.

Recall that F_j denotes the facet of P defined by L_j. For each j, let P_j be the cone with basis F_j and vertex x_0 as illustrated in Figure 2.2. Given a function $f \in \mathcal{C}^1(P)$, we let $d_x f$ denote its differential at $x \in P$. The following is the main technical result of our paper, it imitates quite closely part of the proof by Zhou and Zhu [28] of a coercivity criterion for the modified Mabuchi functional on toric manifolds.

Theorem 2.6. *Let* $\mathrm{v} \in \mathcal{C}^1(P, \mathbb{R})$ *such that* v *is positive on* P^0 *and let* $\mathrm{w} \in \mathcal{C}^0(P, \mathbb{R})$. *Assume that* $\mathcal{F}$ *vanishes on* $\mathrm{Aff}(V)$ *and that for all* $j = 1, \dots, d$, *for all* $x \in P_j$,

$$\frac{1}{L_j(x_0)} \left(\mathrm{v}(x)(\ell + 1) + d_x \mathrm{v}(x - x_0)\right) - \frac{\mathrm{w}(x)}{2} \geq 0, \tag{7}$$

then $(P, \boldsymbol{L})$ *is* (v, w)*-uniformly K-stable.*

Proof. Since L_j vanishes on x at all points of F_j, we have $L_j(x_0) = d_x L_j(x_0 - x)$ for all $x \in F_j$. In other words,

$$\int_{F_j} f(x)\mathrm{v}(x) d\sigma = \int_{F_j} f(x)\mathrm{v}(x) \frac{-d_x L_j(x - x_0)}{L_j(x_0)} d\sigma$$

For each facet F of ∂P_j different from F_j, and $x \in F$, $x - x_0$ belongs to the vector space direction of F, hence the interior product $\iota_{x - x_0}(dx)$ vanishes on the affine space spanned by F. If we further use that $-dL_j \wedge d\sigma = dx$ on F_j, we obtain

$$\int_{F_j} f(x)\mathrm{v}(x) d\sigma = \frac{1}{L_j(x_0)} \int_{\partial P_j} f(x)\mathrm{v}(x) \iota_{x - x_0}(dx).$$

Hence by Stokes theorem we obtain

$$\int_{F_j} f(x)\mathrm{v}(x)d\sigma = \frac{1}{L_j(x_0)}\int_{P_j}(\mathrm{v}(x)d_xf(x-x_0)+\ell f(x)\mathrm{v}(x)+f(x)d_x\mathrm{v}(x-x_0))\,dx.$$

Summing the previous identities over j we get

$$\begin{aligned}\mathcal{F}(f) =&\sum_{j=1}^{d}\frac{2}{L_j(x_0)}\int_{P_j}(d_xf(x-x_0)-f(x))\,\mathrm{v}(x)dx\\ &+\sum_{j=1}^{d}\int_{P_j}\left(\frac{2}{L_j(x_0)}((\ell+1)\mathrm{v}(x)+d_x\mathrm{v}(x-x_0))-\mathrm{w}(x)\right)f(x)dx.\end{aligned}\tag{8}$$

We argue by contradiction to prove the sufficient condition. Assume condition (7) is satisfied and $(P,\mathbf{L})$ is not (v,w)-uniformly K-stable. Then by Lemma 2.4 and since $\mathcal{F}$ vanishes on $\mathrm{Aff}(V)$, there exists a sequence of $\{f_k\}_{k\in\mathbb{N}}$ in $\mathcal{CV}^1_*(P)$ such that

$$\lim_{k\to\infty}\mathcal{F}(f_k)=0 \qquad\text{and}\qquad \forall k\in\mathbb{N}, \|\mathrm{v}f_k\|_1=1.\tag{9}$$

By [8, Proposition 7.2] (or [12, Corollary 5.2.5] if $\mathrm{v}>0$ on P) and the second equality of (9), $\{f_k\}_{k\in\mathbb{N}}$ converges (up to a sub-sequence, still denoted by f_k) locally uniformly in P^0 to a convex function f_∞. Since all f_k are smooth and convex, $d_xf_k(x-x_0)-f_k(x)\geq 0$ by the normalization condition (4). Then, since condition (7) is assumed to hold, all terms of the sum in (8) are non-negative. Evaluating (8) at f_k and passing to the limit reveals that $\lim_{k\to\infty} d_xf_k(x-x_0)-f_k(x)=0$ almost everywhere in P^0, showing that f_∞ is affine on P^0. Since f_∞ satisfies $f_\infty(x)\geq f_\infty(x_0)=0$ on P^0, f_∞ is identically zero on P^0. That contradicts the second equality of (9) and concludes the proof. ☐

Remark 2.7. If v vanishes on F_{j_0}, then the contribution

$$\int_{F_{j_0}} f(x)\mathrm{v}(x)d\sigma$$

is zero. As a consequence, for $j=j_0$, we may replace $L_{j_0}(x_0)$ in condition (7) by any value! It doesn't mean that condition (7) is equivalent to $\mathrm{w}(x)\leq 0$ on P_{j_0}, since the contribution of the F_{j_0} term was split out in two terms in expression (8), and condition (7) involves only one of these terms.

Remark 2.8. We stress that the choice of $x_0\in P^0$ in the previous section was arbitrary, but condition (7) *depends* on that choice. It is possible and useful in practical uses of the condition to vary this x_0 according to the data of the problem, see 4.4.1 and [8, Section 9 and Section 11].

Remark 2.9. Condition (7) depends continuously on the labelled polytope, the weights, and the choice of x_0.

2.3. **Case of monotone polytopes.** Let us recall the terminology of monotone polytopes, used in [20].

Definition 2.10. *A labelled polytope* $(P,\boldsymbol{L})$ *is* monotone *if there exists an* $x_0\in P^0$ *such that* $L_1(x_0)=L_2(x_0)=\cdots=L_d(x_0)$.

There is thus an obvious choice of x_0 in that case. Our sufficient condition indeed becomes much simpler in that case, since the decomposition of the polytope may essentially be forgotten.

Corollary 2.11. *Let $(P, \boldsymbol{L})$ be a monotone labelled polytope with $L_1(x_0) = L_2(x_0) = \cdots = L_d(x_0) = t$. Let $\mathrm{v} \in \mathcal{C}^1(P, \mathbb{R})$ such that v is positive on P^0 and let $\mathrm{w} \in \mathcal{C}^0(P, \mathbb{R})$. Assume that $\mathcal{F}$ vanishes on $\mathrm{Aff}(V)$ and that for all $x \in P$,*

$$\frac{1}{t}\left(\mathrm{v}(x)(\ell+1) + d_x\mathrm{v}(x - x_0)\right) - \frac{\mathrm{w}(x)}{2} \geq 0, \tag{10}$$

then $(P, \boldsymbol{L})$ is (v, w)-uniformly K-stable.

The conditions involved form a finite set of conditions to check, contrary to the definition of (v, w)-uniform K-stability. It is furthermore easy to implement in a computer program, *via* formal or numerical computations depending on the data $(P, \mathbf{L}, \mathrm{v}, \mathrm{w})$. The same is true for the more general Theorem 2.6, but the decomposition in cones makes it a bit more tedious.

3. Geometric applications

3.1. **Weighted cscK toric manifolds.** The results from Section 2 are motivated by the study of the existence of weighted cscK metrics on toric manifolds, as studied in [17].

Let $\mathbb{T}$ be an ℓ-dimensional compact torus. We denote by $\mathfrak{t}$ its Lie algebra and by $\Lambda \subset \mathfrak{t}$ the lattice of generators of circle subgroups, so that $\mathbb{T} = \mathfrak{t}/2\pi\Lambda$. Let $(X, \omega, \mathbb{T})$ be a compact Kähler toric manifold. Denote by μ the moment map of X with respect to the action of $\mathbb{T}$, and let $P = \mu(X) \subset \mathfrak{t}^*$ be the moment polytope. The polytope P is a Delzant polytope [10], and in particular, there is a natural choice of labelling $\mathbf{L}$ of P such that all the differentials dL_j of the defining affine functions L_j are primitive elements in the lattice Λ.

In the context of toric manifold, the v-weighted scalar curvature was introduced in [23]. To avoid introducing too much notation, we give the definition of [18], which makes sense for general compact Kähler manifold and coincide with the one of [23] in the toric context.

Definition 3.1 (Weighted cscK metrics)**.**

(1) For $\mathrm{v} \in \mathcal{C}^\infty(P, \mathbb{R}_{>0})$, define the v-scalar curvature of ω as the function

$$Scal_{\mathrm{v}}(\omega) := \mathrm{v}(\mu)Scal(\omega) + 2\Delta_\omega\big(\mathrm{v}(\mu)\big) + \mathrm{Tr}\big(G_\varphi \circ (\mathrm{Hess}(\mathrm{v}) \circ \mu)\big),$$

where $Scal(\omega)$ is the usual scalar curvature of the Riemann metric g_ω associated to ω, Δ_ω is the Riemannian Laplacian of g_ω, $\mathrm{Hess}(\mathrm{v})$ is the Hessian of v viewed as bilinear form on $\mathfrak{t}^*$ whereas G_ω is the bilinear form with smooth coefficients on $\mathfrak{t}$, given by the restriction of g_ω on fundamental vector fields.

(2) If furthermore $\mathrm{w} \in \mathcal{C}^\infty(P, \mathbb{R})$, then ω is a (v, w)-cscK metric if

$$\mathrm{Scal}_{\mathrm{v}}(\omega) = \mathrm{w} \circ \mu$$

In general, no YTD correspondence is proved for the existence of weighted cscK metrics on toric manifolds. However by analogy with the unweighted cscK case, the candidate notion of K-stability is known, and translates on the polytope as Definition 2. In fact, the direction from existence of weighted cscK metrics to K-stability was proved in general by Li, Lian, Sheng.

Theorem 3.2 ([24, Theorem 2.1])**.** *If ω is a (v,w)-cscK metric, then $(P,\mathbf{L})$ is (v,w)-uniformly K-stable.*

The converse direction is in general much harder, but is known for special choices of weights.

- If v and w are constants, this is the uniform YTD conjecture for cscK metrics on toric manifolds. Then the converse direction is proved thanks to the combination of [12, 28, 7] (see also [21] for an alternative proof bypassing the argument of [28]). These results hold as well for extremal metrics on toric manifolds, thanks to the adaptation by He [14] of the arguments of Chen and Cheng to the extremal case.
- More generally, the uniform YTD conjecture holds for cscK metrics on spherical manifolds, as Odaka shows by using Chi Li's arguments in the appendix of [8]. In addition, [8] shows that existence of cscK metrics on a *horo*spherical G-manifold (X,L) is equivalent to (v,w)-uniform K-stability of its moment polytope for certain degenerate weights (v,w). Note that for general spherical varieties, the condition is slightly different as one needs to impose slope conditions on the convex functions.
- If only v is constant, the converse of Theorem 3.2 is known for all $\mathrm{w}\in\mathcal{C}^\infty(P,\mathbb{R})$ by [25].
- For v-solitons on Fano toric manifolds, which correspond to choosing an arbitrary weight $\mathrm{v}\in\mathcal{C}^\infty(P,\mathbb{R}_{>0})$ and $\mathrm{w}(x)=2(\ell\mathrm{v}(x)+d_x\mathrm{v}(x))$ (see [3, Proposition 1]), it was proven in [22] that the YTD conjecture holds. Actually, thanks to our Corollary 2.11, we recover the result of Han Lie [22] without the need of special test configuration, see Corollary 3.13.
- Finally, as we shall explain in the next sections, the converse of Theorem 3.2 was proven by the second author for weights corresponding to extremal Kähler metrics on semisimple principal toric fibrations [17].

3.2. **Extremal metrics on semisimple principal toric fibrations.** Let $\mathbb{T}$ be an ℓ-dimensional compact torus. Let X be a toric projective manifold under the action of $\mathbb{T}$. Let (B_a,ω_a), for $1\leq a\leq k$ be a family of cscK Hodge manifolds. Let Q be a principal $\mathbb{T}$-bundle on $B:=\prod_{a=1}^k B_a$, equipped with a principal connection θ whose curvature is of the form

$$d\theta=\sum_{a=1}^k \omega_a\otimes p_a$$

where the $p_a\in\mathfrak{t}$ define one-parameter subgroups of $\mathbb{T}$. The semisimple principal fiber bundle construction associates to the above data a holomorphic fiber bundle $Y=(Q\times X)/\mathbb{T}$ over B, with distinguished Kähler classes called bundle-compatible, associated to the data of a Kähler class on X and certain k-tuples of constants (c_a). More precisely, given a $\mathbb{T}$-invariant Kähler form ω_X on X, with associated moment map $\mu: X\to\mathfrak{t}^*$ with respect to $\mathbb{T}$, and a k-tuple of constants (c_a) such that for all a, $p_a\circ\mu+c_a>0$, a bundle-compatible Kähler metric ω_Y on Y is defined by the following basic form on $Q \times X$:

$$\omega_Y = \omega_X + \sum_{a=1}^{k} c_a \omega_a + d(\mu \circ \theta).$$

We refer to section 4.2 for a practical way to construct such fibrations from a collection of line bundles, and to the remainder of section 4 for more explicit examples.

Theorem 3.3 ([17, Theorem 3]). *Let (Y, ω_Y) be a semisimple principal bundle with toric Kähler fiber (X, ω_X) and denote by P its moment polytope. Then there exists an extremal Kähler metric in $[\omega_Y]$ if and only if P is* (v, w)*-uniformly K-stable, where*

$$\mathrm{v}(x) = \prod_{a=1}^{k} (p_a(x) + c_a)^{n_a} \tag{11}$$

$$\mathrm{w}(x) = \left(l_{\mathrm{ext}}(x) - \sum_{a=1}^{k} \frac{s_a}{p_a(x) + c_a} \right) \mathrm{v}(x), \tag{12}$$

and l_{ext} is the unique affine function such that (3) *holds for* (v, w). *Equivalently, there exists a* (v, w)*-cscK metric in* $[\omega_X]$.

Note that condition (3) corresponds to vanishing of the modified Futaki character, and l_{ext} encodes the extremal vector field. In particular the extremal metric above is cscK if and only if l_{ext} is constant.

Remark 3.4. It is remarkable that the condition depends on the base only through the constants (s_a) and the existence of a principal $\mathbb{T}$-bundle with connection with corresponding data (p_a). In particular, when we obtain an existence result for extremal Kähler metrics, we usually actually obtain the existence of extremal Kähler metrics over a full deformation family of cscK manifolds.

Theorem 3.3 allows us to translate Theorem 2.6 as the following sufficient condition of existence of extremal Kähler metrics on fibrations.

Corollary 3.5 (of Theorem 2.6). *The semisimple principal toric fibration (Y, ω_Y) admits an extremal Kähler metric in $[\omega_Y]$ if there exists an $x_0 \in P^0$ and corresponding cone decomposition $P = \bigcup_j P_j$ such that for all j and for all $x \in P_j$*

$$\frac{1}{L_j(x_0)} \left(\ell + 1 + \sum_{a=1}^{k} \frac{n_a p_a(x - x_0)}{p_a(x) + c_a} \right) - \frac{1}{2} \left(l_{\mathrm{ext}}(x) - \sum_{a=1}^{k} \frac{s_a}{p_a(x) + c_a} \right) \geq 0.$$

Proof. It suffices to note that for the weight v involved, we have

$$d_x \mathrm{v}(y) = \left(\sum_{a=1}^{k} \frac{n_a p_a(y)}{p_a(x) + c_a} \right) \mathrm{v}(x).$$

so that in the condition in Theorem 2.6, we can factor by $\mathrm{v}(x)$ which is positive everywhere. □

3.3. **Fibrations with Fano fiber.** We now turn to the fibrations with Fano fiber, in order to use Corollary 2.11. With the same notations as in Section 3.2, we now assume furthermore that the toric fiber is a Fano manifold, and that the Kähler class $[\omega_X]$ is a multiple of the anticanonical class $c_1(X)$. As a consequence, the moment polytope P is a dilation of a reflexive lattice polytope. This implies that the labelled polytope $(P, \mathbf{L})$ corresponding to the lattice polytope P is monotone, with a preferred point x_0 and $L_1(x_0) = \cdots = L_d(x_0) = t$. Assuming without loss of generality that the (anti-)canonical normalization is used for the moment polytope of the fiber, we may further assume that $x_0 = 0$, and $t = \frac{[\omega]}{c_1(X)}$.

Corollary 3.6. *The semisimple principal toric fibration $(Y, [\omega_Y])$ with Fano toric fiber admits an extremal Kähler metric in $[\omega_Y]$ if $\forall x \in P$,*

$$2(\ell + \sum_a n_a) + 2 + \sum_a \frac{ts_a - 2n_a c_a}{p_a(x) + c_a} - tl_{\mathrm{ext}}(x) \geq 0 \tag{13}$$

Note that $\ell + \sum_a n_a = \dim(Y)$.

Proof. Since all $L_j(x_0)$ are equal to t, the condition from Corollary 3.5 further simplifies to

$$2\ell + 2 + \sum_{a=1}^{k} \frac{2n_a p_a(x) + ts_a}{p_a(x) + c_a} - tl_{\mathrm{ext}}(x) \geq 0 \ \forall x \in P$$

as for Corollary 2.11. Writing $2n_a p_a(x) = 2n_a(p_a(x) + c_a) - 2n_a c_a$ yields the statement. □

While simple enough, and tractable with numerical optimization techniques, the inequality involved is a polynomial inequality in several variables, whose degree can be equal to the dimension of the basis plus one. If is difficult to solve formally. There is a further reduction that allows to get a simpler condition which can be checked by a finite number of evaluations of polynomial functions.

Corollary 3.7. *Assume furthermore that for all a, $c_a \geq \frac{ts_a}{2n_a}$. Then the semisimple principal toric fibration $(Y, [\omega_Y])$ admits an extremal Kähler metric in $[\omega_Y]$ if inequation* (13) *is satisfied at every vertex of P.*

Proof. The inverse of an affine function is convex on the locus where this affine function is positive. Hence under the condition in the statement, the function $\frac{ts_a - 2n_a c_a}{p_a + c_a}$ is concave on P. Condition (13) thus amounts to checking the non-negativity of a concave function on a convex polytope: it is enough to check the non-negativity on vertices. □

Remark 3.8. In the case of a *simple* principal toric fibration, that is, if there is only one factor in the basis, then the condition becomes extremely simple for classes with $c_a \geq \frac{ts_a}{2n_a}$: it is enough to check a degree two polynomial inequation on every vertices of the moment polytope.

Remark 3.9. We can write a similar statement for the general case of toric fibrations, by working on the cone decomposition. In that case the conditions to impose are: for all j, for all a, $L_j(x_0)s_a - 2n_a(p_a(x_0) + c_a) \leq 0$ and condition (3.5) is satisfied at all vertices of P_j, that is, some vertices of P and x_0.

3.4. **Case of Fano fibrations.** An important special case when the toric fiber is Fano is given by the semisimple principal toric fibrations which are themselves Fano.

Lemma 3.10 ([3, Lemma 5.10])**.** *Assume that each B_a is a Fano Kähler-Einstein manifold. Let ω_a denote a Kähler-Einstein metric on B_a so that $I_a[\omega_a] = c_1(B_a)$, where I_a denotes the Fano index of B_a. We fix a principal bundle with connection (Q, θ) as before (with associated data (p_a)). We further assume that (X, ω_X) is a Fano toric manifold with a $\mathbb{T}$-invariant Kähler form $\omega_X \in c_1(X)$, with the natural choice of moment map μ. If for all a, $p_a \circ \mu + I_a > 0$, then the semisimple principal fibration Y associated to the above data is a Fano manifold, and ω_Y is in $c_1(Y)$ for the k-tuple $(c_a) = (I_a)$.*

Note that in the above situation, the scalar curvature of ω_a is indeed constant, equal to $2n_aI_a$ where n_a is the complex dimension of B_a.

By our general sufficient condition, we obtain a very simple condition for the existence of extremal Kähler metrics on Fano toric fibrations.

Corollary 3.11. *A Fano semisimple principal toric fibration Y admits an extremal Kähler metric in $c_1(Y)$ if its extremal function l_{ext} satisfies:*

$$\sup l_{\mathrm{ext}} \leq 2(\dim(Y) + 1) \tag{14}$$

Proof. By Lemma 3.10, for all a, $s_a = 2n_ac_a$ and the condition from Corollary 3.6 becomes

$$2\dim(Y) + 2 - l_{\mathrm{ext}} \geq 0 \text{ on } P$$

□

Of course, as in Corollary 3.7, it is enough to check this condition on vertices of the polytope.

Remark 3.12. If l_{ext} is constant, it is equal to $2\dim(Y)$ since the class is the anticanonical one. As a consequence, the condition is strictly satisfied:

$$2\dim(Y) + 2 - l_{\mathrm{ext}} = 2 > 0.$$

In particular, We recover that a Fano toric fibration with vanishing Futaki invariant admits a Kähler-Einstein metric [3]. Furthermore, by Remark 2.9, it shows that, on a neighborhood of $c_1(X)$ in the subcone of bundle-compatible Kähler classes, there exists extremal Kähler metrics. Of course, this is already known by Lebrun-Simanca [19] and [3]. However, in the present setting, working directly with the condition it is not hard to find an explicit neighborhood which works. More generally, if one focuses on the cscK metrics existence problem, the same remarks as above show that, whether or not the Futaki invariant of $c_1(X)$ does vanish, there is a neighborhood of $c_1(X)$ in the subcone of compatible Kähler classes where existence of a cscK metric is equivalent to vanishing of the Futaki character, a further illustration of a phenomenon observed in [8].

In [22], Han-Li showed that the YTD conjecture holds for v-soliton. By definition a v-soliton is a Kähler metric ω such that

$$Ric(\omega) - \omega = \frac{1}{2}dd^c log(\mathrm{v}),$$

where $Ric(\omega)$ is the Ricci form of ω. On Fano semisimple principal toric fibrations, a Kähler metric $\omega_Y \in 2\pi c_1(Y)$ is a v-soliton iff its corresponding metric $\omega_X \in 2\pi c_1(X)$ is $(\mathrm{v}\mathrm{v}_0, \tilde{\mathrm{v}})$-cscK (see [3, Lemma 2.2, Lemma 5.11]) for the weights $\tilde{\mathrm{v}} := 2(\ell \mathrm{v}_0(x)\mathrm{v}(x) + d_x(\mathrm{v}_0\mathrm{v})(x))$ and v_0 is defined in (11). Since the polytope must be reflexive hence monotone, one has $x_0 = 0$, $t = 1$ and condition (10) becomes $\mathrm{v} \geq 0$ on the polytope, which is obviously satisfied. Moreover, by [17, Proposition 7.8], the $(\mathrm{v}, \tilde{\mathrm{v}})$-uniform K-stability implies the coercivity of the corresponding weighted $(\mathrm{v}, \tilde{\mathrm{v}})$-Mabuchi functional. Involving [22, Theorem 3.5], we obtain the existence of a Ricci-soliton in $2\pi c_1(Y)$ (see also [3, Proof of Theorem 3]). We then recover the result of Han-Li [22], bypassing some of the arguments allowing to reduce to special test configurations:

Corollary 3.13. *Let Y be a Fano semisimple principal toric fibration with associate Delzant polytope P and fix the weights corresponding to* v*-solitons defined above. Then, if the weighted Donaldson-Futaki invariant $\mathcal{F}$ vanishes, there exists a* v*-soliton in $2\pi c_1(Y)$.*

3.5. **Further ways to apply Theorem 2.6.** To end this section, we describe two, somewhat indirect, ways to apply Theorem 2.6, that provide further examples of classes where we may find extremal Kähler metrics.

The first way relies on a by now standard method, often called the adiabatic regime [16, 13, 1, 2, 5, 11]. Given a semisimple principal toric fibration Y, we let the tuples (c_a) parametrizing admissible Kähler classes vary. When, say, c_{a_0} goes to $+\infty$, we observe that in the associated weighted uniform K-stability condition the dominant term is controlled by the analogous weighted uniform K-stability condition for another semisimple principal toric fibration Z, with the same toric fiber, and all data associated to the factor B_{a_0} of the basis removed (for example if Y was simple, then $Z = X$ is simply the toric fiber). If Z is weighted uniformly K-stable for the admissible Kähler class corresponding to the tuple $(c_b)_{c \neq a_0}$, then we deduce that for large enough c_{a_0}, the admissible Kähler class on Y, corresponding to the tuple (c_a) with $c_a = c_b$ for $b \neq a_0$, admits an extremal Kähler metric. We may thus apply our criterions to $Z$and obtain induced extremal Kähler classes on Y. This is explained by the adiabatic picture since Y is actually a fibration over B_{a_0} with fiber Z. One may of course apply the same idea to an arbitrary subset of indices $\{a_0, a_1, \ldots, a_m\}$.

The second way is related to the blowup phenomenon, and provides an easy way to get extremal Kähler classes for polytopes which are not monotone. Consider again a semisimple principal toric fibration Y, and let now some c_a decrease so that, at the limit, some of the corresponding affine functions $p_a + c_a$ vanish on some facets of the polytope P. In particular, the associated weight v vanishes on these facets. If the base point x_0 and labelling $\mathbf{L}$ of P can be chosen so that the $L_j(x_0)$ are independent of j for all j such that v vanishes on the facet F_j, then Remark 2.7 allows to apply Theorem 2.6 in a way that is just as efficient as Corollary 3.6. Although such a data does not strictly speaking correspond to an admissible Kähler class on the fibration, if the polytope P is weighted uniformly K-stable for these weights, then the same limiting argument as before shows that for nearby Kähler classes, there exists extremal Kähler metrics.

4. Examples

4.1. **Examples of bases.** In this section we comment on examples of possibles bases for the semisimple principal toric fibration construction. This allows to determine possible values of s_a to plug into the condition. The easiest way to get a cscK basis is to choose a Kähler-Einstein manifold, equipped with a multiple of its first Chern class when it is definite, and with an arbitrary Kähler class for Calabi-Yau manifolds.

For canonically polarized manifolds, there always exists a Kähler-Einstein metric in $-c_1(X)$, and there exists such manifolds in every dimension. In particular, the value $s_a = -\frac{2n_a}{k_a}$ are always allowed, for $k_a \in \mathbb{Z}_{>0}$. For manifolds with zero first Chern class, there always exist Kähler-Einstein metrics with zero scalar curvature. For the positive curvature case, since the projective space of dimension n is a Kähler-Einstein manifold of index $n+1$, all the values $s_a = 2\frac{n_a(n_a+1)}{k_a}$ are allowed, for $k_a \in \mathbb{Z}_{>0}$. More generally, for a Kähler-Einstein Fano basis of dimension n_a and index I_a, then all the values $s_a = 2\frac{n_a I_a}{k_a}$ are allowed, for $k_a \in \mathbb{Z}_{>0}$. Note that the Fano index of an n-dimensional Fano manifold is always an integer between 1 and $n+1$. Here are a couple known results on existence of Fano Kähler-Einstein manifolds when n is small or I is large:

- if $I = n+1$ then $X = \mathbb{P}^n$ is the n-dimensional projective space, and it is Kähler-Einstein,
- if $I = n$ then $X = Q^n$ is the n-dimensional quadric, and it is Kähler-Einstein,
- if $n = 1$ then $X = \mathbb{P}^1$, $I = 2$ and it is Kähler-Einstein,
- if $n = 2$ then $\mathbb{P}^2$ (index 3), $X = \mathbb{P}^1 \times \mathbb{P}^1$ (index 2) and the blowups of $\mathbb{P}^2$ (index 1) at three or more points are Kähler-Einstein,
- if $n = 3$, then the existence of Kähler-Einstein metrics on a general member of a deformation family of smooth Fano threefolds was recently settled in [4], and the families where the general member is *not* Kähler-Einstein are the following, in the labelling used in [4], 2.23, 2.26, 2.28, 2.30, 2.31, 2.33, 2.35, 2.36, 3.14, 3.16, 3.18, 3.21, 3.22, 3.23, 3.24, 3.26, 3.28, 3.29, 3.30, 3.31, 4.5, 4.8, 4.9, 4.10, 4.11, 4.12, 5.2.

4.2. **Preliminaries: projectivization of sums of line bundles.** Let $(B, \omega_B) := \prod_{a=1}^k (B_a, \omega_a)$ be a product of compact complex manifolds B_a endowed with cscK metrics ω_a with $\frac{1}{2\pi}[\omega_a]$ primitive element of $H^2(B_a, \mathbb{Z})$. We consider holomorphic line bundles $\mathcal{L}_i \longrightarrow B$, $i = 1, \ldots, \ell$, and we suppose that their first Chern classes satisfy

$$2\pi c_1(\mathcal{L}_i) = \sum_{a=1}^k p_{ai}[\omega_a],$$

where by definition p_{ai} is the ω_a-degree of $\mathcal{L}_i$. The natural $\mathbb{C}_i^*$-action on $\mathcal{L}_i$ induces an action of $\mathbb{S}_i^1 \subset \mathbb{C}_i^*$, providing a $\mathbb{T}$-bundle $\pi : Q \longrightarrow B$, where $\mathbb{T} = \prod_i \mathbb{S}_i^1$ is a compact ℓ-torus. We choose a Hermitian metric h_i on $\mathcal{L}_i$ and consider the norm function $r_i(u) := (h_i(u,u))^{\frac{1}{2}}$ for any u in $\mathcal{L}_i$. On $\tilde{\mathcal{L}}_i$, the $\mathbb{C}^*$-bundle obtained from $\mathcal{L}_i$ by removing the zero section, r_i is positive and we let $t_i = \log(r_i)$. We fix a basis $\boldsymbol{\xi} = (\xi_i)_{i=1}^{\ell}$ of the Lie algebra $\mathfrak{t}$ of $\mathbb{T}$ and we denote by $\xi^{\tilde{\mathcal{L}}_i}$ the generator of the $\mathbb{S}_i^1$-action on $\tilde{\mathcal{L}}_i$. We then consider the $\mathfrak{t}$-valued one-form $t := \sum_{i=1}^{\ell} t_i \xi_i$ and we define a connection one-form θ on

Q as the restriction of d^ct to Q, seen as the $\mathbb{T}$-bundle of unit element on each $(\mathcal{L}_i, h_i)$. For all $i = 1, \dots, \ell$, it satisfies

$$\theta(\xi^{\tilde{\mathcal{L}}_i}) = \xi_i.$$

We obtain by construction

$$\begin{aligned} d\theta &= \sum_{i=1}^{\ell} \xi_i \otimes \pi^*(\omega_{h_i}) = \sum_{i=1}^{\ell} \xi_i \otimes \big(\sum_{a=1}^{k} p_{ai}\pi^*(\omega_a)\big) \\ &= \sum_{a=1}^{k} p_a \otimes \pi^*(\omega_a), \end{aligned} \tag{15}$$

where ω_{h_i} is the opposite of the curvature form of the Chern connection of $(\mathcal{L}_i, h_i)$ and $p_a = \sum_{i=1}^{\ell} p_{ai}\xi_i$. We consider the ℓ-projective space $(\mathbb{P}^\ell, \omega_{\mathbb{P}^\ell}, \mathbb{T})$ endowed of an hamiltonian $\mathbb{T}$-action with respect to a fixed Kähler metric $\omega_{\mathbb{P}^\ell}$. We fix the principal $\mathbb{T}$-bundle Q with its connection one form θ, the cscK Kähler manifolds (B_a, ω_a) and the toric Kähler manifold $(\mathbb{P}^\ell, \omega_{\mathbb{P}^\ell}, \mathbb{T})$. From this datas, we define a semisimple principal toric fibration $Y := Q \times_{\mathbb{T}} \mathbb{P}^\ell$. By construction, Y is biholomorphic to the total space of the projectize bundle $\mathbb{P}(E)$, $E := \mathcal{O} \oplus \bigoplus_{i=1}^{\ell} \mathcal{L}_i$.

Suppose $\omega_{\mathbb{P}^\ell}$ belongs the the first Chern class $2\pi c_1(\mathbb{P}^\ell)$ of $\mathbb{P}^\ell$ and denotes by P the canonical ℓ-simplex associate to $(\mathbb{P}^\ell, 2\pi c_1(\mathbb{P}^\ell), \mathbb{T}^\ell)$ via Delzant correspondence [10]. By (15), any compatible Kähler metric on Y is of the form

$$\omega_Y = \sum_{a=1}^{k} \left(\sum_{i=1}^{\ell} p_{ai}x_i + c_a \right)\omega_a + \omega_{\mathbb{P}^\ell}, \quad \boldsymbol{x} = (x)_{i=1}^{\ell} \in P \tag{16}$$

with

$$c_a > \sum_{i=1}^{\ell} p_{ai}, \tag{17}$$

In the above formulas, by abuse of notation, $\omega_{\mathbb{P}^d}$ denotes both the Kähler metric on $\mathbb{P}^\ell$ and its induced metric in $2\pi c_1(O_E(\ell+1))$. The tuples (c_a) satisfying (17), parametrize the compatible Kähler classes.

Furthermore, suppose that B is a product of Kähler-Einstein manifolds $(B, \omega_B) := \prod_{a=1}^{k}(B_a, \omega_a)$. By Lemma 3.10, if we choose c_a equal to the Fano index I_a of B_a, the corresponding compatible Kähler form ω_Y defined in (16) belongs to the first Chern class $2\pi c_1(Y)$. In particular, if

$$I_a > \sum_{i=1}^{\ell} p_{ai}, \tag{18}$$

Y is a Fano manifold with compatible first Chern class.

4.3. $\mathbb{P}^2$**-fiber over Fano threefold.** We consider the 2-dimensional projective space $(\mathbb{P}^2, \mathbb{T}^2, 2\pi c_1(\mathbb{P}^2))$. Identifying the lattice Λ of $\mathbb{T}^2$ with $\mathbb{Z}^2$, we consider its labelled moment polytope $(P, \mathbf{L})$ in $\mathbb{R}^2$

$$P = \left\{(x_1, x_2) =: x \in \mathbb{R}^2 \mid L_1(x) \geq 0,\ L_2(x) \geq 0,\ L_3(x) \geq 0\right\}, \tag{19}$$

where $L_1(x) := x_1+1$, $L_2(x) := x_2+1$, $L_3(x) := -x_1-x_2+1$. Let (B, ω_B) be a KE Fano 3-fold with $\alpha_B := [\omega_B]$ primitive element of $H^2(B, \mathbb{Z})$ proportional to the first Chern class $2\pi c_1(B)$. Let $\mathcal{L}_i \longrightarrow B$ be a holomorphic line bundle of degree p_i proportional to the anticanonical line bundle $-K_B$, i.e. $p_i\alpha_B = 2\pi c_1(\mathcal{L}_i)$. We consider a *simple* principal toric fibration (i.e. the basis has only one factor) $\pi : Y := \mathbb{P}(\mathcal{L}_0 \oplus \mathcal{L}_1 \oplus \mathcal{L}_2) \longrightarrow B$. Since the holomorphic class of Y is invariant by tensoring $\mathcal{L}_0 \oplus \mathcal{L}_1 \oplus \mathcal{L}_2$ with a line bundle, we can suppose without loss of generality that $L_0 = \mathcal{O}$ is the trivial line bundle and $p_i \geq 0$, $i = 1, 2$. When B is a local Kähler product of nonnegative cscK metric and $p_1 = p_2 > 0$ or $p_2 > p_1 = 0$, it is know [1, Proposition 11], that there exists an extremal metric in every compatible Kähler classes. We then suppose $p_2 \geq p_1 > 0$.

The compatible Kähler classes are parametrized by constants c and are of the form

$$\alpha_c := 2\pi c_1(O_E(3)) + c\pi^*(\alpha_B). \tag{20}$$

As introduced in Section 4.1, since B is a Fano threefold, the only possible Fano indices I are 1, 2, 3 or 4. In the case where B is the quadric Q_3 or the projective space $\mathbb{P}^3$ (i.e. if $I = 3$ or $I = 4$ respectively), Leray-Hirch Theorem shows that $H^2(Y, \mathbb{R}) \cong \mathbb{R}^2$. It follows that, up to scaling, all Kähler classes are compatible, i.e. of the form of (20). It is known that [2, Theorem 4] for c sufficiently large, the class α_c is extremal. The following Proposition gives a precise value for c, depending on p_1 and p_2, from which α_c admits an extremal metric.

Proposition 4.1. *Let $Y = \mathbb{P}(\mathcal{O} \oplus \mathcal{L}_1 \oplus \mathcal{L}_2) \longrightarrow B$ be a simple principal toric fibration over a Kähler-Einstein Fano threefold B, where $\mathcal{L}_1$ and $\mathcal{L}_2$ are holomorphic line bundles of degrees $1 \leq p_1 \leq p_2$ proportional to the anti-canonical line bundle $-K_B$ of B. Then there exists an extremal metric in α_c for $c \geq 7p_2$.*

Proof. Since the arguments are identical for each Fano index I, we give the proof only for $I = 4$.

By Corollary 3.7, for $c \geq 4$, it is sufficient to check (13) evaluated in each vertices $v_1 := (-1, 2)$, $v_2 := (-1, -1)$, $v_3 := (2, -1)$ of the polytope P.

Using Programm 2 in Appendix A, we find that the LHS of (13) evaluated in v_1 is a rational fraction in the variables c, p_1, p_2:

$$\text{LHS of (13)} = \frac{P(c, p_1, p_2)}{Q(c, p_1, p_2)}.$$

We give the explicit expression of the polynomials P and Q in Appendix B. Suppose now $c \geq 7p_2$ and $p_2 \geq p_1 \geq 1$. Then we can find two polynomials

$$\begin{aligned} R(c) :=& 12250c^{10} - 73500c^9 - 295470c^8 + 1296540c^7 - 3657150c^6 + 3776220c^5 \\ &- 6537672c^4 + 5624964c^3 - 6193584c^2 + 85920232c - 1889568 \end{aligned}$$

and

$$\begin{aligned}S(c) :=& 6125c^{10} + 18375c^9 + 6615c^8 + 19845c^7 + 127575c^6 \\ &+ 382725c^5 + 17496c^4 + 52488c^3 - 288684c^2 - 866052c\end{aligned}$$

such that

$$0 < R(c) \leq P(c, p_1, p_2)$$

and

$$0 < S(c) \quad \text{and } S(c) \geq Q(c, p_1, p_2).$$

It implies that

$$\text{LHS of (13)} = \frac{P(c, p_1, p_2)}{Q(c, p_1, p_2)} \geq \frac{R(c)}{S(c)} \geq 0.$$

We proceed analogously for the vertex v_2 and v_3. We conclude the proof by involving Corollary 3.7.

□

Remark 4.2. In Proposition 4.1, we obtain a lower bound on c depending only on the degrees p_1 and p_2 of the line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$. For given values of p_1 and p_2 it is possible to obtain a more optimal result. Indeed, suppose p_1 and p_2 are fixed. Then, the LHS of (13) is a rational fraction F depending only on the variable c. We then only need to look for constant α such that F is non-negative for $c \geq \alpha$. For example, if $B = \mathbb{P}^3$, respectively $B = Q_3$, $p_1 = 1$ and $p_2 = 2$, (13) show the existence of an extremal metric in α_c for $c \geq 7.09$, respectively $c \geq 9.08$. We refer to Appendix A for further exemples of application of the sufficient condition on simple principal $\mathbb{P}^2$ fibrations.

4.4. **Comments on the rank one case.**

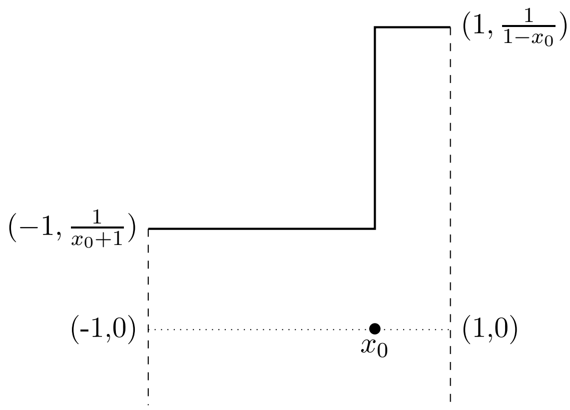
4.4.1. *Varying x_0 and prescribing weighted scalar curvature on $\mathbb{P}^1$.* As noted in Remark 2.8, it can be useful to vary the base point x_0. In this short paragraph, we want to illustrate this phenomenon in the simplest possible case, that is, when working on the one-dimensional polytope $[-1, 1] \subset \mathbb{R}$ with the weights $\mathrm{v} \equiv 1$ and arbitrary w. We further choose the lattice labelling of $[-1, 1]$ induced by the lattice $\mathbb{Z} \subset \mathbb{R}$ (in other word, we work on the anticanonical moment polytope of $\mathbb{P}^1$). More precisely, the labelling (L_1, L_2) is given by $L_1(x) = 1 + x$ and $L_2(x) = 1 - x$. Since $\mathrm{v} \equiv 1$, we have $d_x\mathrm{v} \equiv 0$, hence condition (7) from Theorem 2.6 translates as

$$\frac{1}{4}\mathrm{w}|_{[-1,x_0]} \leq \frac{1}{1 + x_0} \qquad \text{and} \qquad \frac{1}{4}\mathrm{w}|_{[x_0,1]} \leq \frac{1}{1 - x_0}$$

The latter condition is illustrated in Figure 2, and it is obviously less restrictive if one can choose x_0 than the uniform condition corresponding to the obvious choice of $x_0 = 0$ for the monotone lattice polytope $[-1, 1]$.

We end this paragraph by recalling that $(1, \mathrm{w})$-uniform stability of the lattice polytope $[-1, 1]$ translates to existence of certain canonical Kähler metrics on $\mathbb{P}^1$ thanks to [25].

FIGURE 2. Varying x_0



4.4.2. *Extremal metrics on $\mathbb{P}^1$-bundles.* We have focused on applications of our sufficient condition to semisimple principal toric bundles with dimension two toric fiber. This is because in the case of a one-dimensional toric fiber, quite a few strong results have been shown in [1]. For example, it is proved in [1, Proposition 11] that if all factors (B_a, ω_a) of the basis have non-negative constant scalar curvature, and the fiber is one-dimensional, then there exists an extremal Kähler metric in all compatible Kähler classes.

There cannot be such a result if some factors of the basis have negative constant scalar curvature, as shown by examples in [1]. More importantly, some of these examples motivated the initial introduction of the notion of uniform K-stability, as they are likely relatively K-polystable in the sense of [27], but do not admit extremal Kähler metrics.

On the positive side, by [1, Theorem 1], there always exist extremal Kähler metrics on a a semisimple principal $\mathbb{P}^1$-fibration, when all the c_a are large enough, as explained in Section 3.5. However, it is not so easy to derive explicit Kähler classes with extremal Kähler metrics from this asymptotic proof. A possible approach to get explicit classes with extremal Kähler metrics would be to compute the extremal polynomial (in the terminology of [1]) and check when it is positive. This is less practical than our sufficient condition, which involves only checking the positivity of a polynomial at two points. We provide in the appendix an elementary computer program which checks whether our sufficient condition is satisfied for a simple principal $\mathbb{P}^1$-fibration, which could easily be adapted to the case of a semisimple principal $\mathbb{P}^1$-fibration.

4.4.3. *A more explicit example.* Consider B a three-dimensional canonically polarized manifold, equipped with its Kähler-Einstein metric in $-c_1(X)$, whose scalar curvature is thus equal to -6. We consider the sufficient condition for existence of extremal Kähler metrics in admissible Kähler classes on the $\mathbb{P}^1$-bundles $\mathbb{P}(\mathcal{O}_B \oplus K_B^m)$. Up to rescaling and symmetry, this amounts to checking (v, w)-uniform K-stability of the reflexive lattice polytope $[-1, 1] \subset \mathbb{R}$ with respect to the weights

$$\mathrm{v}(x) = (px + c)^3 \qquad \text{and} \qquad \mathrm{w}(x) = \left(l_{\mathrm{ext}}(x) - \frac{-6}{px + c}\right)(px + c)^3$$

where $p \in \mathbb{Q}$, $c \in \mathbb{R}$ and $c > p > 0$. Our sufficient condition allows to obtain the following explicit families of extremal Kähler classes. We only show an example with very rough estimates to illustrate the results, but of course one could get much more classes by using more precise estimates in the proof, and even more classes by using the sufficient condition in Theorem 2.6 in its full generality.

Proposition 4.3. *With the above notations, if* $c \geq 15p$, *then* $[-1, 1]$ *is* (v, w)*-uniformly K-stable. The corresponding Kähler classes on the* $\mathbb{P}^1$*-bundles* $\mathbb{P}(\mathcal{O} \oplus K_B^m)$ *admit extremal Kähler metrics.*

Proof. Using Program 1 in the appendix or straightforward but tedious computations, we obtain up to elementary simplifications that the sufficient condition reads as

$$75c^7 - 300c^6 - 65c^5p^2 + 160c^4p^2 - 15c^3p^4 - 180c^2p^4 - 27cp^6 + 48p^6$$

is greater than

$$|-75c^6p + 5c^4p^3 + 80c^3p^3 - 105c^2p^5 + 15p^7|$$

Without attempting to give an optimal result, we may as well check that it is greater than

$$75c^6p + 5c^4p^3 + 80c^3p^3 + 105c^2p^5 + 15p^7$$

since c and p are positive. Writing $c = \alpha p$ for some $\alpha > 1$ and simplifying by p^6, we get a linear inequation in p

$$pA + B \geq 0 \tag{21}$$

where

$$A = 75\alpha^7 - 75\alpha^6 - 65\alpha^5 - 5\alpha^4 - 15\alpha^3 - 105\alpha^2 - 27\alpha - 15$$
$$B = -300\alpha^6 + 160\alpha^4 - 80\alpha^3 - 180\alpha^2 + 48$$

Since $\alpha > 1$, the coefficient A is larger than $(75\alpha - 307)\alpha^6$ and in particular, it is non-negative for $\alpha \geq \frac{307}{75}$. Using the same lower bound for the leading coefficient, inequation (21) is certainly satisfied at $p = 1$ if

$$(75\alpha - 307)\alpha^6 - 300\alpha^6 - 160\alpha^4 - 80\alpha^3 - 180\alpha^2 - 48 \geq 0$$

Using again $\alpha > 1$ and very rough estimates, this is implied by the inequality

$$(75\alpha - 1075)\alpha^6 \geq 0$$

The latter is satisfied at least for $\alpha \geq 15$, and since $15 \geq \frac{307}{75}$, we obtain that if $\alpha \geq 15$, the sufficient condition is satisfied for all $p \geq 1$. □

Appendix A. An elementary Python program

We provide, as a courtesy to the reader, elementary Python programs using SymPy that checks the sufficient condition from Corollary 3.7 for *simple* principal toric fibrations (that is, the basis has only one factor) with Fano toric fiber X of dimension one or two such that $[\omega_X]$ a multiple of $c_1(X)$.

The only data from the simple principal toric bundle needed to compute the condition is:

- from the basis, the dimension $n \in \mathbb{Z}$ and scalar curvature $s \in \mathbb{Q}$

- from the Fano toric fiber of dimension $\ell \in \{1,2\}$, the reflexive moment polytope $P \subset \mathbb{R}^\ell = \mathbb{Z}^\ell \otimes \mathbb{R}$, and the multiple $t = \frac{[\omega_X]}{c_1(X)} \in \mathbb{R}$
- the one-parameter subgroup p from the principal bundle, identified with an integer $p \in \mathbb{Z}$ if the fiber is one-dimensional, and with an element $p = (p_1, p_2) \in \mathbb{Z}^2$ if the fiber is of dimension two,
- and the constant $c \in \mathbb{R}$ defining the admissible Kähler class.

We wish to compute the expression given by the right-hand side of (13)

$$\texttt{test} = 2(\ell + n + 1) + \frac{ts - 2nc}{p(x) + c} - tl_{\mathrm{ext}}(x)$$

in order to check the condition. For this, it suffices to compute the extremal function l_{ext} by solving the linear system which defines it. Our short programs compute l_{ext}, then `test`, then evaluate `test` at the vertices of P and returns the minimum if all data are explicitly given. It the minimum returned by the program is non-negative, the data correspond to a simple principal toric fibration with an admissible Kähler class and $c > \frac{ts}{2n}$, then there exists an extremal Kähler metric. We may also let some of the data remain unknown and treat them as variables.

```
1  import sympy as sym
2  # variable on the line (here the fiber is one-dimensional)
3  x = sym.symbols('x')
4  # data of the simple principal toric fibration
5  p, c = sym.symbols('p,c')
6  n, s, t = 3, -6, 1
7  # weights
8  l = c+p*x
9  v, w0 = l**n, -s*l**(n-1) # for now, unknown l_ext is replaced with zero
10 # Donaldson-Futaki invariant with weights (v,w0)
11 def DF0(f):
12     interior=sym.integrate(f*w0, (x, -t, t))
13     facets=(f*v).subs(x,-t)+(f*v).subs(x,t)
14     return(interior+facets)
15 # Compute the extremal function lext
16 X=sym.Matrix(2, 1, [1, x])
17 M=sym.Matrix(2, 2, lambda i,j:
18     sym.integrate(X[i,0]*X[j,0]*v, (x, -t, t)))
19 V=sym.Matrix(2, 1, [DF0(1), DF0(x)])
20 Lext=M.LUsolve(V)
21 lext=((Lext.T)*X)[0,0]
22 # Compute expression test at the two vertices and print it
23 test=2*(1+1+n)+(t*s-2*n*c)/l-t*lext
24 print(sym.factor(test.subs(x,-t)))
25 print(sym.factor(test.subs(x,t)))
```

Program 1. Rank one simple principal toric fibrations

Program 1 prints the condition to check when c and p are variables, $n = 3$, $s = -6$ and $t = 1$, as used in Proposition 4.3. By modifying Line 5 and 6, one can obtain the conditions for an arbitrary simple principal $\mathbb{P}^1$-bundle.

```
1 import sympy as sym
2 # variables on the plane
3 x1, x2 = sym.symbols('x1,x2')
4 # data of toric fibration and admissible Kahler class
5 c, p1, p2, n, s, t = 12, 1, 2, 3, 18, 1
6 ## weights associated to the data
7 l=c+p1*x1+p2*x2
8 v=l**n
9 w0=-s*l**(n-1) # for now, unknown l_ext replaced with zero
10 # list of vertices of the polytope
11 vert= [[2*t,-t], [-t,-t], [-t,2*t]]
12 # Donaldson-Futaki invariant with weights (v,w0)
13 def DF0(f):
14   interior=sym.integrate(sym.integrate(f*w0,(x2,-t,t-x1)),(x1,-t,2*t))
15   facet1=sym.integrate((2*f*v).subs(x2,-t),(x1,-t,2*t))
16   facet2=sym.integrate((2*f*v).subs(x2,t-x1),(x1,-t,2*t))
17   facet3=sym.integrate((2*f*v).subs(x1,-t),(x2,-t,2*t))
18   return(interior+facet1+facet2+facet3)
19 # Compute the extremal function l_ext
20 X=sym.Matrix(3, 1, [1, x1, x2])
21 M=sym.Matrix(3, 3, lambda i,j:
22     sym.integrate(sym.integrate(X[i,0]*X[j,0]*v,(x2,-t,t-x1)),(x1,-t,2*t)
    ))
23 V=sym.Matrix(3, 1, [DF0(1), DF0(x1), DF0(x2)])
24 Lext=M.LUsolve(V)
25 lext=((Lext.T)*X)[0,0]
26 # Compute and print the minimum of expression test on vertices
27 test=2*(1+2+n)+(t*s-2*n*c)/l-t*lext
28 test_vertices=test.subs(x1,vert[0][0]).subs(x2,vert[0][1])
29 for i in range(1,len(vert)):
30   test_vertices=sym.Min(test_vertices,
31                         test.subs(x1,vert[i][0]).subs(x2,vert[i][1]))
32 print("The minimum of expression test on vertices is ", test_vertices)
```

PROGRAM 2. Simple principal $\mathbb{P}^2$ toric fibrations

Program 2 computes the condition when all the data are given the fixed values $(c, p1, p2, n, s, t) = (12, 1, 2, 3, 18, 1)$. Changing the values on the right-hand side of Line 5 allows to check the sufficient condition for arbitrary fixed values. If one wants one or several of the above quantities to be treated as variables, for example c, p_1 and p_2, it suffices to remove these and the corresponding values on the right in Line 5 and add the line

```
6 c, p1, p2 = sym.symbols('c,p1,p2')
```

Since the program will now compute values of `test` as symbolic expressions, it will no longer be able to determine the minimum. One should thus replace Lines 28–32 for example by

```
28 print(sym.separatevars(test.subs(x1,vert[2][0]).subs(x2,vert[2][1])))
```

to get the expressions from appendix B, to be used in the proof of Proposition 4.1.

Similarly, it is very easy to modify the program to consider another Fano toric surface as fiber (Recall that there are five smooth Fano toric surfaces: $\mathbb{P}^1 \times \mathbb{P}^1$ and the blowups of $\mathbb{P}^2$ at up to three fixed points under the torus action). It suffices to modify Lines 10–18 according to the desired polytope. For example, if one wants to work with fiber the first Hirzebruch surface (i.e. the blowup of $\mathbb{P}^2$ at one point), then it suffices to replace Lines 10–18 with

```
10 # list of vertices of the polytope
11 vert= [[-t,-t], [t,-t], [t,0], [-t,2t]]
12 # Donaldson-Futaki invariant with weights (v,w0)
13 def DF0(f):
14   interior=sym.integrate(sym.integrate(f, (x2, -t, t-x1)), (x1, -t, t))
15   facet1=sym.integrate(f.subs(x2,-t), (x1, -t, t))
16   facet2=sym.integrate(f.subs(x2,t-x1), (x1, -t, t))
17   facet3=sym.integrate(f.subs(x1,-t), (x2, -t, 2t))
18   facet4=sym.integrate(f.subs(x1,t), (x2, -t, 0))
19   return(interior+facet1+facet2+facet3+facet4)
```

Appendix B. Complement of proof of Proposition 4.1

$$
\begin{aligned}
P(c,p_1,p_2) :={}&12250c^{10} + 24500c^9p_1 - 39690c^8p_1^2 + 18060c^7p_1^3 - 22470c^6p_1^4 - 31752c^5p_1^5\\
&- 53376c^4p_1^6 + 22740c^3p_1^7 - 57024c^2p_1^8 + 1312p_1^{1}0 - 49000c^9p_2\\
&+ 34650c^7p_1^2p_2 + 286860c^6p_1^3p_2 + 152460c^5p_1^4p_2 + 360972c^4p_1^5p_2\\
&- 59520c^3p_1^6p_2 + 230112c^2p_1^7p_2 + 18288cp_1^8p_2 - 464p_1^9p_2 - 127890c^8p_2^2\\
&- 212310c^7p_1p_2^2 - 615510c^6p_1^2p_2^2 - 373212c^5p_1^3p_2^2 - 921924c^4p_1^4p_2^2\\
&- 425376c^2p_1^6p_2^2 - 160632cp_1^7p_2^2 - 19296p_1^8p_2^2 + 141540c^7p_2^3 + 657300c^6p_1p_2^3\\
&+ 603288c^5p_1^2p_2^3 + 1408632c^4p_1^3p_2^3 + 390936c^3p_1^4p_2^3 + 571536c^2p_1^5p_2^3\\
&+ 349440cp_1^6p_2^3 + 41376p_1^7p_2^3 - 328650c^6p_2^4 - 531720c^5p_1p_2^4 - 1421136c^4p_1^2p_2^4\\
&- 806100c^3p_1^3p_2^4 - 829080c^2p_1^4p_2^4 - 497592cp_1^5p_2^4 - 22416p_1^6p_2^4 + 212688c^5p_2^5\\
&- 43812c^3p_1^5p_2^2 + 860184c^4p_1p_2^5 + 849456c^3p_1^2p_2^5 + 906192c^2p_1^3p_2^5\\
&+ 485712cp_1^4p_2^5 - 7488p_1^5p_2^5 - 286728c^4p_2^6 - 527016c^3p_1p_2^6 - 725760c^2p_1^2p_2^6\\
&- 329952cp_1^3p_2^6 + 127890c^8p_1p_2 + 7352cp_1^9 + 22656p_1^4p_2^6 + 150576c^3p_2^7\\
&+ 363168c^2p_1p_2^7 + 156096cp_1^2p_2^7 - 25728p_1^3p_2^7 - 90792c^2p_2^8 - 46368cp_1p_2^8\\
&+ 16992p_1^2p_2^8 + 10304cp_2^9 - 7040p_1p_2^9 + 1408p_2^{1}0 + 132300c^7p_1^2 + 105840c^6p_1^3\\
&- 11340c^5p_1^4 + 125496c^4p_1^5 + 151200c^3p_1^6 - 79056c^2p_1^7 + 60048cp_1^8\\
&- 12096p_1^9 - 396900c^7p_1p_2 - 449820c^6p_1^2p_2 - 260820c^5p_1^3p_2 - 374220c^4p_1^4p_2\\
&- 420336c^3p_1^5p_2 + 358992c^2p_1^6p_2 - 364176cp_1^7p_2 + 17712p_1^8p_2 + 396900c^7p_2^2\\
&+ 714420c^6p_1p_2^2 + 601020c^5p_1^2p_2^2 + 378756c^4p_1^3p_2^2 + 743904c^3p_1^4p_2^2\\
&- 557280c^2p_1^5p_2^2 + 734832cp_1^6p_2^2 + 84240p_1^7p_2^2 - 476280c^6p_2^3 - 680400c^5p_1p_2^3\\
&- 282744c^4p_1^2p_2^3 - 728784c^3p_1^3p_2^3 + 99792c^2p_1^4p_2^3 - 1073520cp_1^5p_2^3\\
&- 287280p_1^6p_2^3 + 340200c^5p_2^4 + 45360c^4p_1p_2^4 + 568512c^3p_1^2p_2^4 + 829440c^2p_1^3p_2^4\\
&+ 1551312cp_1^4p_2^4 - 244944c^3p_1p_2^5 - 241056c^2p_2^7 - 736128cp_1p_2^7 - 18144c^4p_2^5\\
&+ 415152p_1^5p_2^4 - 279072p_1^2p_2^7 + 184032cp_2^8 + 139968p_1p_2^8 - 31104p_2^9\\
&- 1175472c^2p_1^2p_2^5 - 1732752cp_1^3p_2^5 - 358992p_1^4p_2^5 + 81648c^3p_2^6\\
&+ 843696c^2p_1p_2^6 + 1436400cp_1^2p_2^6 + 323568p_1^3p_2^6
\end{aligned}
$$

$$
\begin{aligned}
Q(c,p_1,p_2) :=& 6125c^9 + 2205c^7p_1^2 + 210c^6p_1^3 + 14175c^5p_1^4 - 7812c^4p_1^5 + 24c^3p_1^6 \\
&+ 9072c^2p_1^7 - 5004cp_1^8 + 688p_1^9 - 2205c^7p_1p_2 - 315c^6p_1^2p_2 - 28350c^5p_1^3p_2 \\
&- 31752c^2p_1^6p_2 + 20016cp_1^7p_2 - 3096p_1^8p_2 + 2205c^7p_2^2 - 315c^6p_1p_2^2 + 42525c^5p_1^2p_2^2 \\
&- 7812c^4p_1^3p_2^2 + 4356c^3p_1^4p_2^2 + 40824c^2p_1^5p_2^2 - 40320cp_1^6p_2^2 + 4464p_1^7p_2^2 \\
&+ 210c^6p_2^3 - 28350c^5p_1p_2^3 - 7812c^4p_1^2p_2^3 - 8592c^3p_1^3p_2^3 - 22680c^2p_1^4p_2^3 \\
&+ 50904cp_1^5p_2^3 - 1176p_1^6p_2^3 + 19530c^4p_1^4p_2 - 72c^3p_1^5p_2 \\
&+ 14175c^5p_2^4 + 19530c^4p_1p_2^4 + 4356c^3p_1^2p_2^4 - 22680c^2p_1^3p_2^4 - 56196cp_1^4p_2^4 \\
&- 1224p_1^5p_2^4 - 7812c^4p_2^5 - 72c^3p_1p_2^5 + 40824c^2p_1^2p_2^5 + 50904cp_1^3p_2^5 - 1224p_1^4p_2^5 \\
&+ 24c^3p_2^6 - 31752c^2p_1p_2^6 - 40320cp_1^2p_2^6 - 1176p_1^3p_2^6 + 9072c^2p_2^7 + 20016cp_1p_2^7 \\
&+ 4464p_1^2p_2^7 - 5004cp_2^8 - 3096p_1p_2^8 + 688p_2^9
\end{aligned}
$$

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