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Topology of the space of locally separable functions

Hassan Boualem[*], Robert Brouzet[†]

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Abstract. Among functions of n variables, the simplest are certainly those which are the sum of n functions, each of them depending only on one variable. Nevertheless, the set consisting of such particular functions is rather small compared to the whole space of general functions. Now, if we relax this condition and only ask for a local separability in some well chosen coordinates, is this request also binding and is the corresponding set of functions, that we will denote by $\mathcal{S}$, small again? In this paper we will show that it is not actually the case and that $\mathcal{S}$ has a large set of interior points that we try to identify as well as possible.

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1 Introduction

The problem of the separability of functions, in various coordinates systems, is a crucial one in many questions concerning mechanics [14, 9, 1, 2, 4]. By the way, the starting point of our questioning is strongly linked to mechanics because it concerns a class of Hamiltonians which appeared many years ago in the problem of the connection between Arnold-Liouville completely integrable systems and bi-Hamiltonian ones [15, 6, 7, 10]. Despite this mechanics origin, we will never mention any mechanics in this paper and we will start directly from the problem that we find mathematically interesting in itself. Our main question is: among all the smooth functions, which of them can be locally written as a sum of functions of only one variable? Moreover could we evaluate the size of the subset of such functions relative to the whole set of functions?

Let us now specify what we mean by the separability of functions. For that, let us consider the space $E = \mathcal{C}^\infty(U)$ of smooth functions, with real values, defined on some open neighborhood U of the origin O in the Euclidean space $\mathbb{R}^n$, for example the open ball $B(O,1)$. If we ask the functions $H : (x_1, \cdots, x_n) \mapsto H(x_1, \cdots, x_n)$ to be separable with respect to the variables x_i, *i.e* to be a sum of the type $H = \sum_{i=1}^n H_i(x_i)$, it is obvious to check if a given function verifies, or not, the property. Moreover, it is rather intuitive that the set of such functions has certainly a small size relative to the whole set of smooth functions. Now if we ask a more relaxed separability it will be rather different. Precisely, let us consider the subset $\mathcal{S}$ of E consisting of functions $H : (x_1, \cdots, x_n) \mapsto H(x_1, \cdots, x_n)$ which can be locally separated in the

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sense that there exists a smooth diffeomorphism φ (depending on H), fixing O, from an open neighborhood V of O onto its image W

$$\varphi : V \to W,\ (x_1, \cdots, x_n) \mapsto (u_1(x_1, \cdots, x_n), \cdots, u_n(x_1, \cdots, x_n)),$$

such that for all $(x_1, \cdots, x_n) \in V$ we have

$$H(x_1, \cdots, x_n) = H_1(u_1(x_1, \cdots, x_n)) + \cdots + H_n(u_n(x_1, \cdots, x_n)).$$

It is generally not very easy to verify if a given function satisfies, or not, this condition.

The question of the size of the subset of functions that interests us in the present work is not obvious. Topology offers a good framework to discuss this question using the concepts of *residual set*, *meagre set* in the context of *Baire categories*.

Despite what may seem like a very strong condition, the property to be locally separable is *generic* in the sense that it is verified by a residual set of functions.

We will endow our space with the *Fréchet topology* which construction we will briefly recall. Then, using classical results of differential calculus (respectively the submersion lemma and the Morse lemma (see section 3.5)), it is readily seen that the property to be locally separable is a generic one. Indeed, it is well known that a function f with a Taylor expansion at the origin which is not too degenerate (by this we mean that $df_O \neq 0$ or $df_O = 0$ but the Hessian matrix of f at O is invertible) then the function f is locally separable; moreover this kind of "non-degenerate" functions constitute an everywhere dense open subset. Because of that, we will search to determine the interior of $\mathcal{S}$. It will be possible in the case where functions have just two variables. But, in the case of a greater number of variables, we will just get a range for this interior. In order to do that we need to introduce the notation $T_k(f)$ which denotes the homogeneous part of degree k in the Taylor's expansion at O of the function f; then, the set O_k will be the set (open) of functions f with $T_k(f) \neq 0$. With these notations, we will state and prove the two next results:

Theorem 19 *For $n = 2$,*

$$\mathring{\mathcal{S}} = O_1 \cup O_2 \cup O_3^2,$$

where O_3^2 is the subset of O_3 containing the functions f with $T_3(f)$ in the orbit of $x^3 + y^3$ for the natural action of the linear group on the set of homogeneous polynomials.

Theorem 22 For any integer n,

1. If r_n is the largest integer such that $r_n < \frac{(n-1)(n-2)}{3(n+1)}$ then

$$\mathring{\mathcal{S}} \subset O_1 \cup O_2(n) \cup O_2(n-1) \cup \cdots \cup O_2(r_n + 1),$$

where $O_2(k)$ denotes the subset of elements of O_2 for which the rank of the Hessian at O is equal to k.

2. For $r \leqslant n-2$, $O_2(r) \not\subset \mathring{\mathcal{S}}$.

2 Fréchet topology on the space of smooth functions

In what follows we will study a particular class of smooth functions from a topological point of view. For that, we need to define some topological structure on the functional space $\mathcal{C}^\infty(U, \mathbb{R})$. In order to get it, we will define a distance d on $\mathcal{C}^\infty(U, \mathbb{R})$, using a family of semi-norms. The obtained metric space $(\mathcal{C}^\infty(U, \mathbb{R}), d)$ will be complete and is called a Fréchet space. Let us briefly recall this classic construction of d [13].

Let $(K_p)_p$ be an exhaustive sequence of compact sets of U. We will denote for $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}^n$, $|\alpha| := \sum_{i=1}^n \alpha_i$ and, if $f = f(x_1, \cdots, x_n)$ is an element of $\mathcal{C}^\infty(U, \mathbb{R})$,

$$\partial^\alpha f := \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}},$$

with the convention of notation: $\partial^{(0,\cdots,0)} f = f$. Let us denote for $m \in \mathbb{N}$,

$$\mathcal{A}_m := \{\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}^n, \ |\alpha| = m\}.$$

Now, for each integer $k \in \mathbb{N}$, we can define a sequence $(\| \ \|_{k,p})_{p \in \mathbb{N}^*}$ of semi-norms on $\mathcal{C}^k(U, \mathbb{R})$, by the formula

$$\forall f \in \mathcal{C}^k(U, \mathbb{R}), \ \|f\|_{k,p} := \sum_{m=0}^{k} \left(\sum_{\alpha \in \mathcal{A}_m} \sup_{x \in K_p} |\partial^\alpha f(x)| \right).$$

In fact, for a given function f, we get a double sequence $(\|f\|_{k,p})_{(k,p) \in \mathbb{N} \times \mathbb{N}^*}$ which will allow us to consider a summable double family. Precisely, for f and g in $\mathcal{C}^\infty(U, \mathbb{R})$, let us define

$$d(f, g) := \sum_{(k,p) \in \mathbb{N} \times \mathbb{N}^*} \frac{1}{2^{k+p}} \frac{\|f - g\|_{k,p}}{1 + \|f - g\|_{k,p}}.$$

Because the family is summable the previous formula defines a non-negative real number and it is easy to verify that we get a distance d on $\mathcal{C}^\infty(U, \mathbb{R})$. Moreover, the metric space $(\mathcal{C}^\infty(U, \mathbb{R}), d)$ is complete. The notion of convergence in the sense of this metric topology is simple: a sequence $(f_n)_n$ has for limit f if all the derivatives of f_n converge uniformly on all the compact sets included in U towards the corresponding derivative of f.

A subset A of such a complete metric space E is meagre (or of first category of Baire) when it is contained in a countable union of nowhere dense closed subsets of E, in other words if it is a subset of a nowhere dense F_σ. On the other hand, A is residual when it is the complement of a meagre set, in other words contains an everywhere dense G_δ. A very particular case of such residual subsets is given in the case where A contains an everywhere dense open set; on the other hand, a very particular case of meagre set is provided by a set contained in a nowhere dense closed set. In what follows, we will be concerned only with these particular situations.

In this work we will have also to deal with some finite dimensional subspaces of the metric space $(\mathcal{C}^\infty(U, \mathbb{R}), d)$, namely spaces of homogeneous polynomial functions of a given degree d. These subspaces are finite dimensional and so are endowed with their natural topology describing closeness of two elements as the closeness of each of their coordinates. It is not hard to convince yourself that, in this finite dimensional case, the Fréchet topology and the natural one are equivalent.

3 Separable functions

3.1 Functions with separate variables

As it is mentioned in the abstract, among all the smooth functions $H(x_1, \cdots, x_n)$ of n variables defined on an open set of $\mathbb{R}^n$, those which are separate in the variables $(x_1, \cdots, x_n)$ constitute a rather "small" set.

Proposition 1 *Let $\mathfrak{S}$ be the subset of $\mathcal{C}^\infty(U,\mathbb{R})$ consisting of functions with separate variables, i.e. satisfying the partial differential equations system:*

$$\forall i \neq j,\ \frac{\partial^2 f}{\partial x_i \partial x_j} = 0.$$

Then $\mathfrak{S}$ is a closed nowhere dense subset of $\mathcal{C}^\infty(U,\mathbb{R})$ and so is meagre. Therefore, the set of smooth functions which have not separate variables is residual or, in other words the property for a function to have separate variables is generic.

Proof The main point is that for all $\varepsilon > 0$ and $f \in \mathfrak{S}$ the function $f + \varepsilon x_1 x_2$ does not belong to $\mathfrak{S}$. This remark implies that $\mathfrak{S}$ is nowhere dense. The fact that it is a closed subset is a rather straightforward sequential reasoning. □

3.2 A relaxed separability

We recall that the set, already mentioned in the introduction, that we want to study in this paper, is the subset $\mathcal{S}$ of the set $E = \mathcal{C}^\infty(B(O,1),\mathbb{R})$ consisting of functions $H : (x_1,\cdots,x_n) \mapsto H(x_1,\cdots,x_n)$ which can be locally separated in the sense that there exists a smooth diffeomorphism φ, fixing O, from an open neighborhood V of O onto its image W

$$\varphi : V \to W,\ (x_1,\cdots,x_n) \mapsto (u_1(x_1,\cdots,x_n),\cdots,u_n(x_1,\cdots,x_n)),$$

such that for all $(x_1,\cdots,x_n) \in V$ we have

$$H(x_1,\cdots,x_n) = H_1(u_1(x_1,\cdots,x_n)) + \cdots + H_n(u_1(x_1,\cdots,x_n)).$$

From now and until the end of this paper, by "*separable function*" we will mean that the considered function is some element of $\mathcal{S}$.

Remark 1 *Of course this notion of separability is actually relaxed since, for example, the function defined by $H(x_1,\cdots,x_n) = x_1 x_2$ obviously does not belong to $\mathfrak{S}$ but belongs to $\mathcal{S}$:*

$$H = \frac{1}{4}u_1^2 - \frac{1}{4}u_2^2 \text{ with } u_1 = x_1 + x_2,\ u_2 = x_1 - x_2,\ u_3 = x_3,\cdots,u_n = x_n.$$

3.3 About the jets of a function

In order to deal with our problem we need to work with the jets of functions at the origin O. Let us introduce some notations.

Around the origin O let us write the Taylor-Young formula

$$H = T_0(H) + T_1(H) + T_2(H) + \cdots,$$

where, for each k, T_k is the homogeneous part of degree k of the Taylor expansion. Let us recall that the sum of the first k polynomials $T_j(H)$ is called the k-jet of H at O:

$$J_O^k(H) = \sum_{j=0}^{k} T_j(H) = H(O) + dH(O).X + \frac{1}{2!}d^2H(O).X^{\otimes 2} + \cdots + \frac{1}{k!}d^kH(O).X^{\otimes k},$$

where $X = (X_1, \cdots, X_n)$.

Now, for $H \in E$, let us define:

$$d(H) := \inf\{k \in \mathbb{N}^*,\ T_k(H) \neq 0\},$$

with the convention that this infimum will be taken equal to $+\infty$ in the case where all the $T_k(H)$ are zero, in other words in the case where H is a flat function at O.

For all $k \in \mathbb{N}$ the function $T_k : H \mapsto T_k(H)$ which maps E to some $\mathbb{R}^{N_k}$, where N_k is the dimension of the space of k-homogeneous polynomials, is clearly linear and so the sets

$$E_k := \ker T_k = \{H \in E,\ T_k(H) = 0\}$$

are vector subspaces of E. We can also define for all integers k the space

$$F_k = E_1 \cap \cdots \cap E_k = \{H \in E,\ J_O^k(H) = 0\}.$$

It is easy to prove the following statement:

Proposition 2 *For all $k \in \mathbb{N}$, the function*

$$T_k :\ H \mapsto T_k(H)$$

is continuous. So the subspaces E_k and F_k are closed.

For $k \in \mathbb{N}$, let us define $O_k := E \backslash E_k$ and $\Omega_k = E \backslash F_k = O_1 \cup \cdots \cup O_k$. Because of the previous proposition, these sets are open subsets of E.

The fact to be in some E_k is not invariant by change of coordinates. So it is not an intrinsic property because, for each $k \geqslant 2$, the differential $d^k H$ of order k is not tensorial. Nevertheless the following proposition, which is essential throughout this paper in order to simplify a lot of proofs, states that we can find some intrinsic object in this landscape:

Proposition 3

1. For all integers k and for all smooth functions H the two following properties are equivalent:

i) $J_O^k(H) = 0$ and $T_{k+1}(H) \neq 0$

ii) for all local diffeomorphism φ around O and fixing O,

$$J_O^k(H \circ \varphi) = 0 \text{ and } T_{k+1}(H \circ \varphi) \neq 0.$$

In other words the property, for a given function, to belong to $F_k \cap O_{k+1}$ is intrinsic in the sense that it does not depend on the chosen coordinates system.

2. An homogeneous polynomial H with degree $k+1$ is separable if and only if there is some ***linear*** *change of coordinates $(x_1, \cdots, x_n) \mapsto (u_1, \cdots, u_n)$ and some real numbers $\alpha_1, \cdots, \alpha_n$ such that*

$$H = \sum_{i=1}^{n} \alpha_i u_i^{k+1}.$$

Proof

1. It is easy to verify that if φ is a change of coordinates around O, fixing O and H a smooth function defined around O then for all integers $k \in \mathbb{N}$, we have

$$\frac{\partial^{k+1}(H \circ \varphi)}{\partial u_{i_1} \cdots \partial u_{i_{k+1}}} = \sum_{j_1, \cdots, j_{k+1}=1}^{n} \frac{\partial^{k+1} H}{\partial x_{j_1} \cdots \partial x_{j_{k+1}}} \frac{\partial x_{j_1}}{\partial u_{i_1}} \cdots \frac{\partial x_{j_{k+1}}}{\partial u_{i_{k+1}}} + \mathcal{L}(T_1(H), \cdots, T_k(H)),$$

where $\mathcal{L}(T_1(H), \cdots, T_k(H))$ is a linear expression in the $T_1(H), \cdots, T_k(H)$. The property is now easy to prove using induction since the "tensorial part" of the right hand side, namely

$$\sum_{j_1, \cdots, j_{k+1}=1}^{n} \frac{\partial^{k+1} H}{\partial x_{j_1} \cdots \partial x_{j_{k+1}}} \frac{\partial x_{j_1}}{\partial u_{i_1}} \cdots \frac{\partial x_{j_{k+1}}}{\partial u_{i_{k+1}}},$$

and $\frac{\partial^{k+1}(H \circ \varphi)}{\partial u_{i_1} \cdots \partial u_{i_{k+1}}}$ are vanishing simultaneously according to the fact that the Jacobian matrix $\left(\frac{\partial x_j}{\partial u_i}\right)_{i,j}$ is invertible. This later property is necessary in order to get the *equivalence* between the vanishing of the partial derivatives of H and that of $H \circ \varphi$.

2. Obviously, if H can be written for a linear change of coordinates as $H = \sum_{i=1}^{n} \alpha_i u_i^{k+1}$ then H is an homogeneous polynomial with degree $k+1$ and H is separable. Conversely, let us assume that H is an homogeneous polynomial with degree $k+1$ and that H is separable. Then it exists a change of coordinates

$$(x_1, \cdots, x_n) \mapsto (u_1 = a_{11}x_1 + \cdots + a_{1n}x_n + h.o.t., \cdots, u_n = a_{n1}x_1 + \cdots + a_{nn}x_n + h.o.t.),$$

where the matrix $(a_{ij})_{i,j}$ is invertible, such that

$$H(x_1, \cdots, x_n) = H_1(u_1) + \cdots + H_n(u_n),$$

condition that we can also write

$$H = \sum_{i=1}^{n} \alpha_i(u_i) u_i^{k+1}$$

for some smooth functions $\alpha_i(u_i)$.

Now, because H is $(k+1)$-homogeneous, we get, replacing a fixed $(x_1, \cdots, x_n)$ by $(tx_1, \cdots, tx_n)$ (where t is describing the real line) and simplifying by t^{k+1}, that

$$\forall t \in \mathbb{R}, \; H(x_1, \cdots, x_n) = \sum_{i=1}^{n} \alpha_i(tu_i)(a_{i1}x_1 + \cdots + a_{in}x_n + o(1))^{k+1}.$$

We outline that here the terms $o(1)$ are relative to t (the x_i are fixed) and so denote terms whith a limit equal to 0 when t goes toward 0. Now considering the limit when t goes toward 0 we get

$$H(x_1, \cdots, x_n) = \sum_{i=1}^{n} \alpha_i(0)(a_{i1}x_1 + \cdots + a_{in}x_n)^{k+1},$$

and so the separability of H by the way of a linear change of coordinates. $\square$

3.4 An example of function which is not separable

As it was already mentioned, the checking that a given function does not belong to $\mathcal{S}$ requires more or less long calculations. In the next proposition, we give an example of a family of such functions.

Proposition 4 *For all integer $m \geqslant 2$, the functions defined by $H(x_1, \cdots, x_n) = x_1^m x_2$ are not separable.*

Proof We will prove it arguing by contradiction. So, let us assume that $H \in \mathcal{S}$. Using proposition 3, because the m-jet of H is zero and $T_{m+1}(H) \neq 0$, we get that we could write by the mean of a linear change of coordinates

$$x_1^m x_2 = \alpha_1 u_1^{m+1} + \cdots + \alpha_n u_n^{m+1}$$

where at least one α_i is not zero. Now, because the change of coordinates is linear, x_1 and x_2 are linear forms φ_1 and φ_2 in the u$_i$ and they are linearly independent. So we are in the situation where two independent linear forms φ_1 and φ_2 verify the condition that the product $\varphi_1^m \varphi_2$ has separated variables. So applying for $i \neq j$ the operator $\frac{\partial^2}{\partial u_i \partial u_j}$ to this product we must get zero. So we get

$$m(m-1)\varphi_1^{m-2}\frac{\partial \varphi_1}{\partial u_i}\frac{\partial \varphi_1}{\partial u_j}\varphi_2 + m\varphi_1^{m-1}\frac{\partial^2 \varphi_1}{\partial u_i \partial u_j}\varphi_2 + m\varphi_1^{m-1}\left(\frac{\partial \varphi_1}{\partial u_i}\frac{\partial \varphi_2}{\partial u_j} + \frac{\partial \varphi_1}{\partial u_j}\frac{\partial \varphi_2}{\partial u_i}\right) + \varphi_1^m \frac{\partial^2 \varphi_2}{\partial u_i \partial u_j} = 0.$$

Now because the φ_i are linear forms, their second derivatives vanish; moreover, because φ_1 is not zero we can simplify and get the relation

$$(m-1)\frac{\partial \varphi_1}{\partial u_i}\frac{\partial \varphi_1}{\partial u_j}\varphi_2 + \varphi_1\left(\frac{\partial \varphi_1}{\partial u_i}\frac{\partial \varphi_2}{\partial u_j} + \frac{\partial \varphi_1}{\partial u_j}\frac{\partial \varphi_2}{\partial u_i}\right) = 0.$$

Now, because the linear forms φ_1 and φ_2 are linearly independent, necessarily we must have that

$$\forall i \neq j, \ \frac{\partial \varphi_1}{\partial u_i}\frac{\partial \varphi_1}{\partial u_j} = 0 \text{ and } \frac{\partial \varphi_1}{\partial u_i}\frac{\partial \varphi_2}{\partial u_j} + \frac{\partial \varphi_1}{\partial u_j}\frac{\partial \varphi_2}{\partial u_i} = 0.$$

Because one of the derivatives $\frac{\partial \varphi_1}{\partial u_i}$ is not zero we can assume, without loss of generality that, for example, $\frac{\partial \varphi_1}{\partial u_1} \neq 0$. Then the first group of relations above gives us that for all $j \geqslant 2$, $\frac{\partial \varphi_1}{\partial u_i} = 0$ and so φ_1 is colinear to $u_1 \mapsto u_1$. Now the second group of relations above gives us that also is φ_2 and so that φ_1 and φ_2 are colinear, so a contradiction because they are two rows of the Jacobian matrix of a diffeomorphism. □

3.5 About some large families of separable functions

As we have already seen functions which are separated in some given system of coordinates (for example cartesian coordinates) constitute a very small subset among all the smooth functions. On the other hand, using classic theorems of calculus, it is easy to prove that the functions which are separable up to some change of coordinates constitute a very large subset.

This first classic result of calculus is the *submersion lemma*. We recall it.

Lemma 5 ***(submersion lemma)*** *Let H be a regular function at the origin. Then it exists a local diffeomorphism $\varphi : V \to W$, $(x_1, \cdots, x_n) \mapsto (u_1, \cdots, u_n)$ with V and W open neighborhoods of O, $\varphi(O) = O$, and such that $u_1 = H$ on V.*

Corollary 6 *The set O_1 of regular functions at the origin is contained in the set $\mathcal{S}$ of separable functions. Moreover, it is a dense open subset of E.*

Proof The fact that O_1 is open subset of $\mathcal{S}$ follows immediately from the submersion lemma.

Now, if $H \in E_1 := E \backslash O_1$, then for all $\varepsilon > 0$ the function $H_\varepsilon := H + \varepsilon x_1$ belongs to O_1 and an easy calculation shows that $d(H, H_\varepsilon) \leqslant 4\varepsilon$. So O_1 is everywhere dense in E. □

Comment 1 *It follows from this last result that the set $\mathcal{S}$ is residual, or, in other words that, for a smooth function the property to belong to $\mathcal{S}$ is generic. Moreover the inclusion $O_1 \subset \mathcal{S}$ and the property for O_1 to be open give the inclusion*

$$O_1 \subset \mathring{\mathcal{S}}.$$

After submersion lemma we can use an other result of calculus, namely the *Morse lemma* [16].

Lemma 7 ***(Morse lemma)*** *If some function H is not regular at the origin but has a non-degenerate Hessian at this point, then we can find a local diffeomorphism $\varphi : (x_1, \cdots, x_n) \mapsto (u_1, \cdots, u_n)$ around O such that*

$$H(\varphi^{-1}(u_1, \cdots, u_n)) = H(0, \cdots, 0) + u_1^2 + \cdots + u_p^2 - u_{p+1}^2 - \cdots u_n^2,$$

where $(p, n-p)$ is the signature of the quadratic form defined by the Hessian of H at O.

It is well known that the so-called *Morse functions*, namely functions of which all critical points are non-degenerate, constitute a very important class of functions. Here we are only interested in what happens at the origin O. So we will introduce a terminology for our purpose and will say that H is a Morse function at the origin if either H is regular at O $(dH(O) \neq 0)$ or O is a non-degenerate critical point of H. We will denote by $\mathcal{M}_O$ the set of such functions.

It results from the definitions and Morse lemma the following statement:

Proposition 8 *We have the following inclusions:*

$$O_1 \subset \mathcal{M}_O \subset \mathcal{S}.$$

Remark 2 *We can notice that the Morse functions belong to $\mathcal{M}_O$ and that this later set, because it is open, is included in $\mathring{\mathcal{S}}$.*

Non-degenerate functions are very particular elements of O_2. Let us introduce some notations. For $r \in \{1, \cdots, n\}$ let us denote

$$O_2(r) := \{H \in \mathcal{C}^\infty(B(O,1), \mathbb{R}),\ \mathrm{rank}(\mathrm{Hess}_O(H)) = r)\},$$

where $\text{Hess}_O(H)$ denotes the Hessian of the function H at the origin O. Then, of course,

$$O_2 = \bigcup_{r=1}^{n} O_2(r).$$

We must pay attention: the notation is somewhat dangerous because one might think that $O_2(r)$ is an open set. It is not the case. Only unions of such sets for all the indices greater than a given one, namely $\bigcup_{r \geqslant k} O_2(r)$, are open sets (because of the semi-continuity of the rank).

Now, with these notations, the set $\mathcal{M}_O$ is nothing else than

$$\mathcal{M}_O = O_1 \cup O_2(n).$$

Actually we can slightly relax the condition on the rank of the Hessian matrix at O by using a generalization of the Morse lemma called the Relative Morse lemma. Namely we have the following statement [8, 12, 13, 11]:

Theorem 9 ***(Relative Morse lemma)***

Let f be a $\mathcal{C}^\infty$ function defined on a neighborhood of O in $\mathbb{R}^n$ with $df(O) = 0$. Let r be the rank of the Hessian form $H_f(O)$ of f at O. Then there exists a local coordinate system $(x_1, \cdots, x_r, y_1, \cdots, y_{n-r})$ on $\mathbb{R}^n$ centered at O and a function g defined on a neighborhood of O in $\mathbb{R}^{n-r}$ such that f can be written

$$f = f(O) + x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots x_r^2 + g(y_1, \cdots, y_{n-r}),$$

where the function g is of order $\geqslant 3$ at O.

This theorem allows us to get an enlargement of functions contained in $\mathcal{S}$ and even in the interior of $\mathcal{S}$, namely:

Proposition 10 *We have for $n \geqslant 2$ the following inclusion*

$$O_1 \cup O_2(n-1) \cup O_2(n) \subset \mathring{\mathcal{S}}.$$

Proof We have already seen that $O_1 \cup O_2(n) \subset \mathcal{S}$ using the submersion and Morse lemmas so we get $O_1 \cup O_2(n) \subset \mathring{\mathcal{S}}$ because the left hand side is an open set. Now, using the Relative Morse lemma, stated above, we get the improvement given in this lemma, namely the inclusion of $O_2(n-1)$ in $\mathcal{S}$. So, finally $O_1 \cup O_2(n-1) \cup O_2(n) \subset \mathcal{S}$ and even better $O_1 \cup O_2(n-1) \cup O_2(n) \subset \mathring{\mathcal{S}}$ because the left hand side is an open subset. □

So, if we define the set of *almost Morse functions at the origin* as the set

$$\mathcal{AM}_O = O_1 \cup O_2(n-1) \cup O_2(n),$$

then we know now that

$$O_1 \subset \mathcal{M}_O \subset \mathcal{AM}_O \subset \mathring{\mathcal{S}}.$$

We can remark that, in the case where $n = 2$, $O_2 = O_2(1) \cup O_2(2)$ and so in this particular case we get

$$\mathcal{AM}_O = O_1 \cup O_2.$$

It follows that in the particular case where $n = 2$ we have the inclusion $O_1 \cup O_2 \subset \mathcal{S}$ and so, because the left hand side is open, $O_1 \cup O_2 \subset \mathring{\mathcal{S}}$.

4 More about the topology of the set of separable functions

In the previous sections we have seen that the set $\mathcal{S}$ is rather large since its interior set contains the set $\mathcal{AM}_0$ of the almost Morse functions at the origin. Our goal in this section is to improve our knowledge of $\mathring{\mathcal{S}}$. For this study it will be necessary to distinguish two cases: $n = 2$ and $n \geqslant 3$. In the first case we will be able to determine exactly the interior of $\mathcal{S}$. But, in the second case, we will only able to give a range for $\mathring{\mathcal{S}}$ in the sense that we will give only a double inclusion with $\mathring{\mathcal{S}}$ as the central term. Before dealing with this topic let us remark that $\mathcal{S}$ is not closed. Indeed, let us define the sequence of functions $(H_k)_{k\in\mathbb{N}}$ where for $k \in \mathbb{N}$,

$$H_k(x_1, \cdots, x_n) = \frac{1}{k+1}(x_1^2 + \cdots + x_n^2) + x_1^2 x_2.$$

Each of these functions is separable because of the Morse lemma but its limit function is $H(x_1, \cdots, x_n) = x_1^2 x_2$ which is not separable. Now, we get that $\mathcal{S}$ is not open either. Indeed,

$$\tilde{H}_k(x_1, \cdots, x_n) = \frac{1}{k+1} x_1^2 x_2$$

is arbitrary close to the separable zero function and is not itself separable. To end with this short introduction to this section, let us also state that our main results (theorems 19 and 22) will imply that the separable flat functions are in $\mathcal{S} \backslash \mathring{\mathcal{S}}$.

4.1 The case where the number of variables is equal to 2

In the case of two variables, the interior of $\mathcal{S}$ contains more than just the almost Morse functions at the origin. Indeed, it also contains a part of the open subset O_3, just a part because $O_3 \not\subset \mathcal{S}$. Indeed, according to the example given in the proposition 4 for the case $n = 2$, the function H defined by $H(x, y) = x^2 y$ belongs to O_3 but is not a separable function.

In order to describe the part of O_3 which is contained in the set $\mathcal{S}$, we must first study the issue of separability when we limit it to homogeneous polynomials of degree 3.

Proposition 3 says that if $H \in \mathbb{R}[x, y]$ is a non zero homogeneous polynomial function of degree 3 that belongs to $\mathcal{S}$ then H can be written as

$$H(x, y) = \alpha(ax + by)^3 + \beta(cx + dy)^3 \ (*),$$

for some real numbers $\alpha, \beta, a, b, c, d$ such that $(\alpha, \beta) \neq (0, 0)$ and $ad - bc \neq 0$.

A geometric way to describe what we need is the following. Let us consider the group action of the linear group $GL_2(\mathbb{R})$ on the set of homogeneous polynomial with degree 3: one simply replaces x and y with respectively $ax + by$ and $cx + dy$ where a, b, c, d are real numbers such that $ad - bc \neq 0$. In other words if H is an homogeneous polynomial with degree 3, we define for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{R})$, $A.H := H(AX)$ where AX is the product of A with the column vector $X = \begin{pmatrix} x \\ y \end{pmatrix}$. Among the polynomials H satisfying $(*)$ we can consider three cases:

- ⋄ If $\alpha = \beta = 0$ then $H = 0$ which corresponds to the orbit of 0.
- ⋄ If $\alpha \neq 0$ and $\beta = 0$ or $\alpha = 0$ and $\beta \neq 0$ then H is in the orbit of x^3
- ⋄ If $\alpha \neq 0$ and $\beta \neq 0$ then H is in the orbit of $x^3 + y^3$.

Now let us write O_3 as the disjoint union of three subsets, $O_3 = O_3^1 \cup O_3^2 \cup O_3^3$, where O_3^1 is the set of functions H such that $T_3(H)$ belongs to the orbit of x^3, O_3^2 is the set of functions H such that $T_3(H)$ belongs to the orbit of x^3+y^3 and O_3^3 is what remains. If we denote by $\mathcal{H}_3$ the vector space of homogeneous polynomial functions of degree 3 then we can state:

Proposition 11

1. $O_3^1 \cap \mathcal{H}_3$ is not open.

2. $O_3^2 \cap \mathcal{H}_3$ is open.

3. The polynomial $x^3 + xy^2$ belongs to O_3^2, so is in the orbit of $x^3 + y^3$.

Proof

For all three points, we need to perform similar calculations. Indeed, for points 1. and 2. we have to determine the isotropy subgroup of the action respectively at x^3 and x^3+y^3 and so to calculate real coefficients a, b, c, d with $ad - bc \neq 0$ such that

$$(ax+by)^3 = x^3 \quad \text{(resp. } (ax+by)^3 + (cx+dy)^3 = x^3 + y^3\text{)},$$

and for the point 3. we have to find real coefficients a, b, c, d such that

$$(ax+by)^3 + (cx+dy)^3 = x^3 + xy^2.$$

Precisely, we have to respectively solve the following systems of equations:

$$\left\{\begin{array}{l} a^3 = 1 \\ 3a^2b = 0 \\ 3ab^2 = 0 \\ b^3 = 0 \end{array}\right., \quad \left\{\begin{array}{l} a^3 + c^3 = 1 \\ 3a^2b + 3c^2d = 0 \\ 3ab^2 + 3cd^2 = 0 \\ b^3 + d^3 = 1 \end{array}\right., \quad \left\{\begin{array}{l} a^3 + c^3 = 1 \\ 3a^2b + 3c^2d = 0 \\ 3ab^2 + 3cd^2 = 1 \\ b^3 + d^3 = 0 \end{array}\right.$$

For the first system we get $a = 1$ and $b = 0$ so the isotropy subgroup of x^3 is

$$\left\{ \begin{pmatrix} 1 & 0 \\ c & d \end{pmatrix},\ (c,d) \in \mathbb{R} \times \mathbb{R}^* \right\}.$$

For the second system, $c = 0$ or $d = 0$ and so $a = 0$ or $b = 0$ because if none of them was zero we would get by division member to member from $a^2b = -c^2d$ and $ab^2 = -cd^2$ that $ad - bc = 0$. It results that the isotropy subgroup of $x^3 + y^3$ is

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}.$$

Finally the third system leads to

$$a = c = \frac{1}{\sqrt[3]{2}} \text{ and } b = -d = \frac{\pm 1}{\sqrt[3]{2}\sqrt{3}}.$$

This last calculation gives 3. For the points 1. and 2. we recall that for a group G acting on a set E, there is a one to one correspondence between the orbit of a point $m \in E$ and the set G/G_m of the left (or right) classes of G modulo the isotropy subgroup G_m of m. Moreover, because here G is a Lie group acting smoothly on a manifold E this one to one correspondence is a smooth immersion. In our case, because $G = GL_2(\mathbb{R})$ is a 4-dimensional Lie group and the set E is the space $\mathcal{H}_3$ of homogeneous polynomial functions of degree 3 a 4-dimensional vector space, we get in the first case that the orbit $O_3^1 \cap \mathcal{H}_3$ is an immersed submanifold of dimension 2 in a space of dimension 4 and so cannot be open. On the other hand, for the orbit $O_3^2 \cap \mathcal{H}_3$, we obtain that its dimension is 4 in the 4-dimensional vector space $\mathcal{H}_3$ and so it is an open subset of $\mathcal{H}_3$. $\square$

Corollary 12 *O_3^2 is an open subset of $(\mathcal{C}^\infty(U,\mathbb{R}), d)$.*

Proof Indeed we have seen in the proposition 2 that the application

$$T_3 : \mathcal{C}^\infty(U,\mathbb{R}) \to \mathcal{H}_3, \ H \mapsto T_3(H)$$

is continuous. So, because by its very definition $O_3^2 = T_3^{-1}(O_3^2 \cap \mathcal{H}_3)$, it is open since $O_3^2 \cap \mathcal{H}_3$ is itself open. $\square$

Let us define the polynomial functions $P, Q \in \mathbb{R}[t]$ by the formulas $P(t) = H(1,t)$ and $Q(t) = H(t,1)$. We can notice that $Q(t) = t^3 P(1/t)$. Then, as the following lemma says, the separability of H expressed by the special form $(*)$ of H, can be translated using the type of roots belonging to the polynomial functions P and Q.

The next result will be very useful in what follows but its proof is just elementary calculations based upon a discussion of different cases and left to the reader.

Lemma 13
Let H be a non zero homogeneous polynomial with degree 3. If H is separable then at least one of the two polynomial functions P or Q defined above has its degree equal to 3 and has either a triple real root or only one real root and two (non real) conjugated complex roots.

Remark 3 *Using the previous result we can get another proof that the function H defined by $H(x,y) = x^2 y$ does not belong to $\mathcal{S}$. Indeed in this case, with the notations introduced above, neither $P(t) = t$ nor $Q(t) = t^3 P(1/t) = t^2$ have their degree equal to 3.*

The statement of the previous lemma can be easily generalized to any function H such that $d(H) = 3$. Indeed,

Lemma 14 *Let H be a function with a 2-jet zero and belonging to O_3. If H is separable then $T_3(H)$ can be written as*

$$T_3(H) = \alpha(ax+by)^3 + \beta(cx+dy)^3.$$

Proof Indeed the terms of degree $\geqslant 4$ do not contribute in the discussion according to the proposition 3. $\square$

Many functions belonging to O_3 and not to $\mathcal{S}$ can be obtained also using the next result:

Proposition 15 *If a, b, c are three distinct real numbers then the homogeneous polynomial function $H(x,y) = (y-ax)(y-bx)(y-cx)$ does not belong to $\mathcal{S}$.*

Proof Indeed in this case P and Q have three distinct real roots a, b, c so it cannot belong to $\mathcal{S}$. ▫

Proposition 16

1. *Let be H a function with a vanishing 2-jet at O. Then,*

$$H \in O_3^1 \Rightarrow H \notin \mathring{\mathcal{S}}.$$

2. $O_3^2 \subset \mathring{\mathcal{S}}$.

Proof

1. Let H be a function with a vanishing 2-jet at O and such that $H \in O_3^1 \cap \mathcal{S}$. Then $T_3(H) = (ax + by)^3$ for some real numbers a, b such that $(a, b) \neq (0, 0)$. Let us assume for example that $a \neq 0$. For $\varepsilon > 0$ let us denote

$$H_\varepsilon := H + \varepsilon(ax + by)(ax + by - \varepsilon y)(ax + by + \varepsilon y).$$

Then H_ε is ε-close to H and

$$T_3(H_\varepsilon) = (ax + by)^3 + \varepsilon(ax + by)(ax + by - \varepsilon y)(ax + by + \varepsilon y).$$

If we denote $X = ax + by$ then

$$T_3(H_\varepsilon) = X^3 + \varepsilon X(X - \varepsilon y)(X + \varepsilon y) = X((1 + \varepsilon)X^2 - \varepsilon^3 y^2).$$

If we consider the polynomial $Q_\varepsilon(t) := T_3(H_\varepsilon)(t, 1)$, then it has three distinct real roots, corresponding to the roots of linear equations

$$at + b = 0,\ at + b = \frac{\varepsilon^{3/2}}{\sqrt{1+\varepsilon}},\ at + b = -\frac{\varepsilon^{3/2}}{\sqrt{1+\varepsilon}},$$

namely

$$-\frac{b}{a} - \frac{\varepsilon^{3/2}}{a\sqrt{1+\varepsilon}},\ -\frac{b}{a},\ -\frac{b}{a} + \frac{\varepsilon^{3/2}}{a\sqrt{1+\varepsilon}}.$$

It results from the proposition 15 that H_ε does not belong to $\mathcal{S}$, so $H \notin \mathring{\mathcal{S}}$.

2. By the very definition of O_3^2, a function H belonging to O_3^2 is such that $T_3(H)$ is in the orbit of $x^3 + y^3$ and so, according to the theorem 17 below, we get that up to a change of coordinates H can be considered as $H = x^3 + y^3$ and so belongs to $\mathcal{S}$. So $O_3^2 \subset \mathcal{S}$ and because O_3^2 is open we get $O_3^2 \subset \mathring{\mathcal{S}}$. ▫

In the previous proof we used the next theorem that we can find in the references [8, 3, 11, 5]. More precisely, for example in [5], one can find it in chapter 15, theorem 15.1 or, in the paper [11], stated in the theorem 3.1 p 364. Let us outline that the singularity which appears here is called a D_4-singularity or hyperbolic umbilic. Recall that this singularity can be viewed as well as $x^3 + y^3$ or $x^3 + xy^2$ (see the point 3 of Proposition 11 above).

Theorem 17 *Let H be a function of two variables with a vanishing 2-jet at O. If $T_3(H) = x^3 + xy^2$, then it exists a change of variables φ in a neighborhood of O such that $H \circ \varphi(u, v) = u^3 + uv^2$.*

Corollary 18 *If H has its 3-jet at O equal to zero then H does not belong to $\mathring{\mathcal{S}}$.*

Proof Indeed, if the 3-jet of H at O is zero then for all $\varepsilon > 0$ the function $H_\varepsilon := \varepsilon x^2 y + H$ is ε-close of H and cannot belong to $\mathcal{S}$. ▫

The following theorem summarizes the previous results:

Theorem 19 *For $n = 2$ we have $\mathring{\mathcal{S}} = \mathcal{AM}_O \cup O_3^2 = O_1 \cup O_2 \cup O_3^2$.*

4.2 The case where the number of variables is greater than 3

The case of a sufficiently great number of variables ($n \geqslant 3$) is based on a simple inequality, true in this case and false for $n = 2$, as we will see in the lines below.

In order to deal with this case we need to introduce some objects and notations. Let us denote by $\mathcal{H}_k$ the vector space of real homogeneous polynomials of degree k with n indeterminates (we consider the polynomial zero as homogeneous of degree k). For $r \in \mathbb{N}$ and $I \subset [\![1, n]\!]$ let us denote

$$\mathcal{A}_{r,I} = \left\{ x_1 P_1 + \cdots + x_r P_r + \sum_{i \in I} x_i^3 \ \ / \ \ \text{where } P_i \in \mathcal{H}_2 \right\}.$$

The set $\mathcal{A}_{r,I}$ is an affine subspace of $\mathcal{H}_3$ of dimension $\frac{rn(n+1)}{2}$.

Now let us denote by $\varphi_{r,I}$ the function defined on $GL(\mathbb{R}^n) \times \mathcal{A}_{r,I}$ by $\varphi_{r,I}(A, P) = P(AX)$ and

$$\Omega_r := \bigcup_{I \subset [\![1,n]\!]} \mathrm{Im}(\varphi_{r,I}).$$

Lemma 20 *If H is a function singular at the origin, belonging to $O_2(r) \cap \mathcal{S}$, then $T_3(H) \in \Omega_r$.*

This lemma remains true for $r = 0$ and, in this case, the notation $O_2(0)$ will denote by extension the space E_2 of functions H with $T_2(H) = 0$.

Proof By the very definition of the set $\mathcal{S}$, if $H \in \mathcal{S}$ then there is a change of variables $(x_1, \cdots, x_n) \mapsto (u_1, \cdots, u_n)$ and n smooth functions $H_1, \cdots, H_n$, each of them depending only on one variable, such that H can be written in a neighborhood of the origin as

$$H(x_1, \cdots, x_n) = H_1(u_1) + \cdots + H_n(u_n).$$

Because the 1-jet of H is zero at the origin we get, using proposition 3, that the first derivative at 0 of the functions H_i vanish.

We also know that the rank of the Hessian of H at the origin is the same as those of the Hessian at the origin of the function $H_1(u_1) + \cdots + H_n(u_n)$ in the variables $(u_1, \cdots, u_n)$. By reindexing the variables u_i if necessary, we can assume that the Taylor expansions of the functions H_i at the origin have the form:

$$\forall i \in [\![1, r]\!] \ \ H_i(u_i) = \alpha_i u_i^2 + \alpha_i' u_i^3 + \cdots \ \text{with } \alpha_i \neq 0 \text{ and } \forall i \in [\![r+1, n]\!] \ \ H_i(u_i) = \alpha_i' u_i^3 + \cdots$$

For each $1 \leqslant i \leqslant n$ the Taylor expansion at the order 1 of u_i can be written

$$u_i = \sum_{j=1}^{n} a_{ij} x_j + \sum_{1 \leqslant p,q \leqslant n} b_{ipq} x_p x_q + \cdots$$

The equality

$$H(x_1, \cdots, x_n) = \sum_{i=1}^{i=r} \alpha_i u_i^2 + \alpha_i' u_i^3 + \sum_{i=r+1}^{n} \alpha_i' u_i^3 + \cdots$$

implies that the polynomials $T_2(H)$ and $T_3(H)$ are given by

$$T_2(H) = \sum_{i=1}^{r} \alpha_i \left(\sum_{j=1}^{n} a_{ij} x_j \right)^2$$

and

$$T_3(H)(x_1, \cdots, x_n) = 2 \sum_{i=1}^{i=r} \alpha_i \left(\sum_{j=1}^{n} a_{ij} x_j \right) \left(\sum_{1 \leqslant p,q \leqslant n} b_{ipq} x_p x_q \right) + \sum_{i=1}^{n} \alpha_i' \left(\sum_{j=1}^{n} a_{ij} x_j \right)^3 .$$

Let us denote $I = \{ i \in [\![1, n]\!] \ / \ \alpha_i' \neq 0 \}$ and $a_{ij}' = (\alpha_i')^{1/3} a_{ij}$ if $i \in I$ and $a_{ij}' = a_{ij}$ if $i \notin I$. The matrix $A = (a_{ij}')_{1 \leqslant i,j \leqslant n}$ is invertible because the matrix $(a_{ij})_{1 \leqslant i,j \leqslant n}$ is itself invertible.

Let us denote $b_{ipq}' = \frac{2\alpha_i}{(\alpha_i')^{1/3}} b_{ipq}$ if $i \in I$ and $b_{ipq}' = 2\alpha_i b_{ipq}$ if $i \notin I$ and let us define the matrices $B_i' = (b_{ipq}')_{1 \leqslant p,q \leqslant}$ and $B_i =^t A^{-1} B_i' A^{-1}$. Finally if we denote by P_i the polynomials of $\mathcal{H}_2$ defined by

$$P_i(x_1, \cdots, x_n) =^t X B_i X \text{ where } X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \text{ and } P = \sum_{i=1}^{r} x_i P i + \sum_{i \in I} x_i^3,$$

we get the equality $T_3(H) = \varphi_{r,I}(A, P)$ proving that $T_3(H)$ belongs to Ω_r. □

Lemma 21 *If $r < \frac{(n-1)(n-2)}{3(n+1)}$ then the set Ω_r has no interior point.*

Proof

If $r < \frac{(n-1)(n-2)}{3(n+1)}$ then the dimension $n^2 + \frac{rn(n+1)}{2}$ of the affine space $\mathcal{M}_n(\mathbb{R}) \times \mathcal{A}_{r,I}$ is strictly less than $\frac{n(n+1)(n+2)}{6}$ which is the dimension of $\mathcal{H}_3$. As $\varphi_{r,I}$ is a differentiable map defined on the open set $GL(\mathbb{R}^n) \times \mathcal{A}_{r,I}$ of $\mathcal{M}_n(\mathbb{R}) \times \mathcal{A}_{r,I}$ with values in $\mathcal{H}_3$, then $\mathrm{Im}(\varphi_{r,I})$ has measure zero [8, 13].

It follows that the set Ω_r has measure zero because it is a finite union of the sets $\mathrm{Im}(\varphi_{r,I})$, so it has no interior point. □

Theorem 22

1. *If r_n is the largest integer such that $r_n < \frac{(n-1)(n-2)}{3(n+1)}$ then*

$$\mathring{\mathcal{S}} \subset O_1 \cup O_2(n) \cup O_2(n-1) \cup \cdots \cup O_2(r_n + 1).$$

2. *For $r \leqslant n-2$, $O_2(r) \not\subset \mathring{\mathcal{S}}$.*

Proof

1. Let us prove that if $H \notin O_1 \cup O_2(n) \cup O_2(n-1) \cup \cdots \cup O_2(r_n+1)$ then $H \notin \mathring{\mathcal{S}}$. So we suppose that $H \in E_1 \cap O_2(r)$ where $0 \leqslant r \leqslant r_n$ (we recall that E_1 denotes the set of singular function at the origin so O_1^c). Because the set Ω_r has no interior point in $\mathcal{H}_3$ we can find some polynomial Q, close enough to $T_3(H)$, such that $Q \notin \Omega_r$. In this case the function $\tilde{H} := H - T_3(H) + Q$ is close enough to H, belongs to $E_1 \cap O_2(r)$ and $T_3(\tilde{H}) = Q \notin \Omega_r$. It results from the lemma 20 that $\tilde{H}$ does not belong to $\mathcal{S}$ and so $H \notin \mathring{\mathcal{S}}$.

2. Let be $H \in O_2(r)$ defined by $H = x_1^2 + \cdots + x_r^2$. Now, because $r \leqslant n-2$ we can define for all $\varepsilon > 0$ the function $H_\varepsilon := H + \varepsilon x_{r+1}^2 x_{r+2}$. Let us verify that H_ε is not separable. First of all if $r = 0$, it results from the Proposition 4. So let us assume $r \geqslant 1$ and write around O

$$x_1^2 + \cdots + x_r^2 + \varepsilon x_{r+1}^2 x_{r+2} = H_1(u_1) + \cdots + H_n(u_n).$$

Because the 1-jet at O of H_ε is zero and $T_2(H_\varepsilon) \neq 0$, we know from proposition 3 that the 1-jets of all the H_i are vanishing. Moreover the type of the quadratic form defined by the 2-jet is invariant by change of coordinates so, without loss of generality we can suppose that for $1 \leqslant i \leqslant r$ we have $H_i(u_i) = u_i^2 + o(u_i^2)$ and that for $i \geqslant r+1$ we have $H_i(u_i) = O(u_i^3)$. Now, according Morse lemma we can even assume that for $1 \leqslant i \leqslant r$, $H_i(u_i) = u_i^2$ and so

$$x_1^2 + \cdots + x_r^2 + \varepsilon x_{r+1}^2 x_{r+2} = u_1^2 + \cdots + u_r^2 + H_{r+1}(u_{r+1}) + \cdots + H_n(u_n),$$

with $H_i(u_i) = \alpha_i u_i^3 + \cdots$ for $i \geqslant r+1$. Let us go back to the notations used in the lemma 20 for all $1 \leqslant i \leqslant n$, so

$$u_i = \sum_{j=1}^{n} a_{ij} x_j + \sum_{1 \leqslant p,q \leqslant n} b_{ipq} x_p x_q + \cdots$$

By identification we get

$$x_1^2 + \cdots + x_r^2 = \sum_{i=1}^{i=r} \left(\sum_{j=1}^{n} a_{ij} x_j \right)^2 \quad (*)$$

and

$$\varepsilon x_{r+1}^2 x_{r+2} = 2 \sum_{i=1}^{i=r} \left(\sum_{j=1}^{n} a_{ij} x_j \right) \left(\sum_{1 \leqslant p,q \leqslant n} b_{ipq} x_p x_q \right) + \sum_{i=r+1}^{n} \alpha_i \left(\sum_{j=1}^{n} a_{ij} x_j \right)^3 \quad (**).$$

Equality $(*)$ implies that $a_{ij} = 0$ for $1 \leqslant i \leqslant r$ and $j \geqslant r+1$. Then the identity $(**)$ implies that

$$\varepsilon x_{r+1}^2 x_{r+2} = \sum_{i=r+1}^{n} \alpha_i \left(\sum_{j=r+1}^{n} a_{ij} x_j \right)^3.$$

Because the matrix $(a_{ij})_{r+1 \leqslant i,j \leqslant n}$ is invertible, the function $\varepsilon x_{r+1}^2 x_{r+2}$ becomes separable leading to a contradiction with the proposition 4.

Finally, using this result for small enough ε we deduce that H does not belong to $\mathring{\mathcal{S}}$.

□

Remark 4 *As we have already stated in the introduction of this section 4, a straightforward consequence of the theorems 19 and 22 is that the separable flat functions are in* $\mathcal{S} \backslash \mathring{\mathcal{S}}$.

References

[1] Abraham, R., Marsden, J.E.,—Foundations of mechanics, *Addison-Wesley, New-York* 1985.

[2] Arnol'd, V. I., —Mathematical Methods of Classical Mechanics, *Springer-Verlag, New York, 2nd edn* 1989.

[3] Arnol'd, V.I., Gussein-Zade, S.M., Varchenko, A.N., —Singularities of differentiable Maps, Vol.1 *Birkhäuser, Basel* 1985.

[4] Benenti, S., —Orthogonal separation of variables on manifolds with constant curvature, *Diff. Geom. Appl. 2, 351-367.* 1992.

[5] Bröcker, Th., Lander, L. —Differentiable germs and catastrophes, *London Mathematical Society, Cambridge University Press* 1975.

[6] Brouzet, R., —About the existence of recursion operators for completely integrable Hamiltonian systems near a Liouville torus, *J. Math. Phys. 34, 1309-1313* 1993.

[7] Brouzet, R., Molino, P., Turiel, F.J.—Géométrie des systèmes bihamiltoniens, *Indag. Math. NS 4, 269-296* 1993.

[8] Demazure, M., —Bifurcations and Catastrophes *Springer* 2000.

[9] Eisenhart, L. P., —Separable systems of Stäckel, *Ann. of Math. 35, 284-305* 1934.

[10] Fernandes, L., —Completely integrable bi-Hamiltonian systems, *J. Dynamics Eq. Diff. 6, 53-69* 1994.

[11] Golubitsky, M., —An introduction to catastrophe theory and its applications, *SIAM review, Vol 20, No 2., pp. 352-387* Apr., 1978.

[12] Gromoll, D., Meyer, W., —On differentiable functions with isolated critical points *Topology, 8, pp. 361-369* 1969.

[13] Hirsch, M.W., —Differential topology *Springer* 1976.

[14] Levi-Civita, T., —Sulla integrazione della equazione di Hamilton-Jacobi per separazione di variabili, *Math. Ann. 59, 3383-3397* 1904.

[15] Magri, F., Morosi, C., —A geometrical characterization of integrable Hamiltonian systems through the theory of Poisson-Nijenhuis manifolds, *Publication of the departement of Mathematics, University of Milan* 1984.

[16] Milnor, J., —Morse theory *Princeton University Press* 1963.

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