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# THE HYPERELLIPTIC THETA MAP AND OSCULATING PROJECTIONS

MICHELE BOLOGNESI\*  AND NÉSTOR FERNÁNDEZ VARGAS

**Abstract.** Let  $C$  be a hyperelliptic curve of genus  $g \geq 3$ . In this paper, we give a new geometric description of the theta map for moduli spaces of rank 2 semistable vector bundles on  $C$  with trivial determinant. In order to do this, we describe a fibration of (a birational model of) the moduli space, whose fibers are GIT quotients  $(\mathbb{P}^1)^{2g} // \mathrm{PGL}(2)$ . Then, we identify the restriction of the theta map to these GIT quotients with some explicit degree 2 osculating projection. As a corollary of this construction, we obtain a birational inclusion of a fibration in Kummer  $(g-1)$ -varieties over  $\mathbb{P}^g$  inside the ramification locus of the theta map.

## §1. Introduction

Let  $C$  be a complex smooth curve of genus  $g \geq 3$  and  $\mathcal{S}\mathcal{U}_C(r)$  the (coarse) moduli space of semistable vector bundles of rank  $r$  with trivial determinant on  $C$ . It is well known that this moduli space is a normal, projective, unirational variety of dimension  $(r^2 - 1)(g - 1)$ . The study of the projective geometry of moduli spaces of vector bundles in low rank and genus has produced some beautiful descriptions, frequently mingling constructions issued in the context of classical algebraic geometry and the geometry of Jacobians and theta functions [11, 26, 28].

Let  $\mathcal{L}$  be the determinant line bundle on  $\mathcal{S}\mathcal{U}_C(r)$  and  $\varphi_{\mathcal{L}} : \mathcal{S}\mathcal{U}_C(r) \dashrightarrow |\mathcal{L}|^*$  the map induced by global sections of  $\mathcal{L}$ . The linear system  $|\mathcal{L}|^*$  is isomorphic to the  $|r\Theta|$  linear series on the Jacobian variety  $\mathrm{Jac}(C)$ , by the first declination of *strange duality* [3]. This way, we obtain a (in general) rational map

$$\theta : \mathcal{S}\mathcal{U}_C(r) \dashrightarrow |r\Theta|,$$

the celebrated theta map, which is canonically identified to  $\varphi_{\mathcal{L}}$  [3] (see also [10, 16] for more details on the injectivity of  $\theta$ ).

Let us now fix  $r = 2$ . In this setting, the map  $\theta$  is a finite morphism [29]. When  $g = 2$ , the map  $\theta$  is an isomorphism onto  $\mathbb{P}^3$  [25]. For  $g \geq 3$ , the map  $\theta$  is an embedding if  $C$  is nonhyperelliptic, and it is a 2:1 map if  $C$  is hyperelliptic [9, 11, 30] (see Section 2.1 for more details).

The goal of this paper is to describe the geometry associated to the map  $\theta$  in the case  $r = 2$  when  $C$  hyperelliptic. In the nonhyperelliptic case, the papers [6] and [1] outline a connection between the moduli space  $\mathcal{S}\mathcal{U}_C(2)$  and the moduli space  $\mathcal{M}_{0,n}$  of rational curves with  $n$  ordered marked points. A generalization of [1] for higher rank vector bundles has

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been given in [8]. In the present work, we develop once again the interplay with the moduli space of pointed rational curves (more precisely with its GIT compactification  $\mathcal{M}_{0,n}^{GIT}$ ). Thanks to some clever description of the GIT compactification in terms of linear systems on the projective space due to Kumar [21], this also offers a new geometric description of the  $\theta$ -map if  $C$  is hyperelliptic.

Let  $C$  be a hyperelliptic curve of genus  $g \geq 3$  and  $D$  an effective divisor of degree  $g$  on  $C$ .

Let us set  $\mathbb{P}_D^{3g-2} := \mathbb{P}\text{Ext}^1(\mathcal{O}(D), \mathcal{O}(-D)) = |K + 2D|^*$ , where  $K$  is the canonical divisor on  $C$ . The projective space  $\mathbb{P}_D^{3g-2}$  parametrizes equivalence classes of extensions

$$0 \rightarrow \mathcal{O}(-D) \rightarrow E \rightarrow \mathcal{O}(D) \rightarrow 0,$$

and we denote by  $f_D : \mathbb{P}_D^{3g-2} \dashrightarrow \mathcal{SU}_C(2)$  the natural tautological classifying map sending a semi-stable extension class onto its corresponding rank 2 vector bundle. Since the divisor  $K + 2D$  is very ample, the linear system  $|K + 2D|$  embeds the curve  $C$  in  $\mathbb{P}_D^{3g-2}$ . Let  $N := p_1 + \dots + p_{2g}$  be a general reduced effective divisor in the linear system  $|2D|$ , and let  $\mathbb{P}_N^{2g-2}$  be the  $(2g - 2)$ -dimensional linear span in  $\mathbb{P}_D^{3g-2}$  of the  $2g$  marked points  $p_1, \dots, p_{2g}$  on  $C$  ( $\mathbb{P}_N^{2g-2}$  has also a precise description in terms of extensions, see Section 3).

THEOREM A.

1. *There exists a fibration  $p_D : \mathcal{SU}_C(2) \dashrightarrow |2D| \cong \mathbb{P}^g$  whose general fiber is birational to  $\mathcal{M}_{0,2g}^{GIT}$ .*
2. *For every generic divisor  $N \in |2D|$ , the restriction of  $f_D$  to the projective space  $\mathbb{P}_N^{2g-2} \subset \mathbb{P}_D^{3g-2}$  dominates  $p_D^{-1}(N) \stackrel{\text{birat}}{\cong} \mathcal{M}_{0,2g}^{GIT}$ . The generic fiber of  $f_D|_{\mathbb{P}_N^{2g-2}}$  is a rational normal curve passing through the  $2g$  marked points.*
3. *The family of rational normal curves contracted by  $f_D|_{\mathbb{P}_N^{2g-2}}$  is the universal family of pointed rational curves over (an open subset of) the generic fiber  $p_D^{-1}(N) \stackrel{\text{birat}}{\cong} \mathcal{M}_{0,2g}^{GIT}$ .*

In fact, the restriction of  $f_D$  to  $\mathbb{P}_N^{2g-2}$  coincides on an open subset with a dominant rational map  $h_N : \mathbb{P}_N^{2g-2} \dashrightarrow \mathcal{M}_{0,2g}^{GIT}$  (see Section 4.3 for more details) that contracts all the rational normal curves passing through the  $2g$  points of  $N$ . Via the Kapranov blow-up construction of  $\overline{\mathcal{M}}_{0,2g}$  (see Section 4.2.3) one sees that these rational normal curves, and the rational map  $h_N$ , make up the universal family of pointed rational curves over an open subset of  $\mathcal{M}_{0,2g}^{GIT}$ .

Besides its independent interest, Thm. A is important since the map  $\theta$  has a very explicit geometric description, once it is restricted to the generic fibers of the fibration  $p_D$ . To this end, the following construction is crucial.

Let  $p, i(p)$  be two hyperelliptic involution-conjugate points in  $C$ ; and consider the line  $l \subset \mathbb{P}_D^{3g-2}$  secant to  $C$  and passing through  $p$  and  $i(p)$ . We show that this line intersects the subspace  $\mathbb{P}_N^{2g-2}$  in a point. Moreover, the locus  $\Gamma \subset \mathbb{P}_D^{3g-2}$  of these intersections as we vary  $p \in C$  is a rational normal curve passing by the points  $p_1, \dots, p_{2g}$ . One of the consequences of Theorem A, is that the map  $f_D$  contracts the curve  $\Gamma$  onto a point  $w \in p_D^{-1}(N)$  (recall that  $p_D^{-1}(N)$  is birational to  $\mathcal{M}_{0,2g}$ ).

By using Kumar's description [21] of certain rational involutions on  $\mathcal{M}_{0,2g}^{GIT}$ , we can prove the following Theorem.

**THEOREM B.** *The map  $\theta$  restricted to the fibers  $p_D^{-1}(N)$  is a 2-to-1 osculating projection  $\pi_N$  of multiplicity  $g-2$  centered at the point  $w = f_D(\Gamma)$ , up to composition with a birational map.*

In this theorem, and in the rest of the text, multiplicity one will mean the simple linear projection. Furthermore, the image of  $\pi_N$  can be identified with a connected component of the moduli space  $\mathcal{SU}_{C_w}(2)^{inv}$  of hyperelliptic invariant semistable vector bundles with trivial determinant on  $C_w$ , where  $C_w$  is the hyperelliptic 2-to-1 cover of  $\mathbb{P}^1$  ramifying over the  $2g$  points defined by  $w$ . Thanks to [21], we know that the ramification locus of the map  $\pi_N$  is the Kummer variety  $\text{Kum}(C_w) \subset \mathcal{SU}_{C_w}(2)^{inv}$ . These results, combined with Theorem B, allow us to give a quite accurate description of the ramification locus of the map  $\theta$ :

**THEOREM C.** *The ramification locus of the map  $\theta$  has an irreducible component birational to a fibration in Kummer varieties of dimension  $g-1$  over  $|2D| \cong \mathbb{P}^g$ .*

In low genus, we are able to give a more precise description of the Theta map and its interplay with maps classifying extensions. Let us define  $\varphi_D$  as  $\theta \circ f_D$ . The map  $\varphi_D$  is defined by the linear system  $|\mathcal{I}_{\text{Sec}^{g-2}(C)}(g)|$  (see Proposition 2.2).

**THEOREM D.** *Let  $C$  be a hyperelliptic curve of genus 3. Then, for generic  $N$ , the restriction of  $\varphi_D$  to the subspace  $\mathbb{P}_N^{2g-2}$  is exactly the composition  $\pi_N \circ h_N$ . If  $g = 4$  or 5, then  $\text{Sec}^{g-2}(C) \cap \mathbb{P}_N^{2g-2}$  is set-theoretically equal to the base locus of  $\pi_N \circ h_N$ .*

**NOTATION.**  $\mathbb{P}^n = \mathbb{P}(\mathbb{C}^{n+1})$  will denote the  $n$ -dimensional complex projective space of  $\dim 1$  subspaces. Throughout this paper, a form  $F$  of degree  $r$  on  $\mathbb{P}^n$  will denote an element of the vector space  $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(r)) = \text{Sym}^r(\mathbb{C}^{n+1})^*$ . If we fix a basis  $x_0, \dots, x_n$  of  $(\mathbb{C}^{n+1})^*$ ,  $F$  is simply a homogeneous polynomial of degree  $r$  in  $x_0, \dots, x_n$ . Most of the maps in this paper will be rational maps, hence we will often offend good taste by just dropping the adjective rational. We apologize for that.

*Plan of the paper:* In Section 2, we introduce some general results about moduli of vector bundles and the maps classifying extension classes. Sections 3 and 4 are devoted to the study of the restriction of the classifying maps to  $\mathbb{P}_N^{2g-2} \subset \mathbb{P}_D^{3g-2}$  (with  $N$  moving in  $|2D|$ ) and to the relation of such maps with forgetful linear systems contracting rational normal curves. In Section 5, thanks to the projective geometry of the curve  $C$  embedded in  $\mathbb{P}_D^{3g-2}$  the hyperelliptic involution comes into play and we show Theorems A, B, and C. Finally, Sections 6 and 7 are devoted to a more detailed study of the low genus cases, up to genus 6, which are resumed in Theorem D.

## §2. Moduli of vector bundles

We briefly recall here some results about moduli of vector bundles.

### 2.1 Moduli of vector bundles and the map $\theta$

Let  $C$  be a smooth genus  $g$  algebraic curve with genus  $g \geq 2$ . Let us denote by  $\text{Pic}^d(C)$  the Picard variety of degree  $d$  line bundles on  $C$ . The Jacobian of  $C$  is  $\text{Jac}(C) = \text{Pic}^0(C)$ . As customary, we will denote by  $\Theta \subset \text{Pic}^{g-1}(C)$  the Riemann theta divisor.

The Picard group  $\text{Pic}(\mathcal{SU}_C(2))$  is isomorphic to  $\mathbb{Z}$ , and it is generated by the *determinant line bundle*  $\mathcal{L}$  [15]. For every  $E \in \mathcal{SU}_C(2)$ , let us define the *theta divisor*

$$\theta(E) := \{L \in \text{Pic}^{g-1}(C) \mid h^0(C, E \otimes L) \neq 0\}.$$

In the rank 2 case,  $\theta(E)$  is a divisor in the linear system  $|2\Theta| \cong \mathbb{P}^{2^g-1}$  and  $|2\Theta|$  is isomorphic to the linear system  $|\mathcal{L}|^*$  [3]. It is well known that we can identify the map  $SU_C(2) \rightarrow |\mathcal{L}|^*$  with the Theta map

$$\begin{aligned} \theta : SU_C(2) &\rightarrow |2\Theta|; \\ E &\mapsto \theta(E). \end{aligned}$$

In rank 2, the map  $\theta$  is a finite morphism. If  $C$  is not hyperelliptic,  $\theta$  is known to be an embedding [9, 30]. This is also the case in genus 2, where  $\theta$  is an isomorphism onto  $\mathbb{P}^3$  [25]. If  $C$  is hyperelliptic of genus  $g \geq 3$ , we have that  $\theta$  factors through the involution

$$E \mapsto i^*E$$

induced by the hyperelliptic involution  $i$ , embedding the quotient  $SU_C(2)/i^*$  into  $|2\Theta|$  [11]. An interesting explicit description of the image of the hyperelliptic theta map is given in [11].

## 2.2 The classifying maps

Let  $D$  be a general degree  $g$  effective divisor on  $C$ . Let us consider isomorphism classes of extensions

$$(e) \quad 0 \rightarrow \mathcal{O}(-D) \rightarrow E_e \rightarrow \mathcal{O}(D) \rightarrow 0.$$

These extensions are parametrized by the  $(3g-2)$ -dimensional projective space  $\mathbb{P}_D^{3g-2} := \mathbb{P}\text{Ext}^1(\mathcal{O}(D), \mathcal{O}(-D)) = |K+2D|^*$ , where  $K$  is the canonical divisor of  $C$ . The divisor  $K+2D$  is very ample and embeds  $C$  as a degree  $4g-2$  curve in  $\mathbb{P}_D^{3g-2}$ . Let us define the rational surjective classifying map  $f_D : \mathbb{P}_D^{3g-2} \dashrightarrow SU_C(2)$  which sends a semi-stable extension class  $(e)$  to the vector bundle  $E_e$ . The composed map

$$\varphi_D := \theta \circ f_D : \mathbb{P}_D^{3g-2} \dashrightarrow |2\Theta|$$

can be described in terms of polynomial maps. From [4, Theorem 2] we have an isomorphism

$$H^0(SU_C(2), \mathcal{L}) \cong H^0(\mathbb{P}_D^{3g-2}, \mathcal{I}_C^{g-1}(g)), \quad (1)$$

where  $\mathcal{I}_C$  is the ideal sheaf of  $C$ . In particular, we have

**THEOREM 2.1.** *The map  $\varphi_D$  is given by the linear system  $|\mathcal{I}_C^{g-1}(g)|$  of forms of degree  $g$  vanishing with multiplicity at least  $g-1$  on  $C$ .*

Let us denote by  $\text{Sec}^n(C)$  the variety of  $(n+1)$ -secant  $n$ -planes on  $C$ . We have that the singular locus of  $\text{Sec}^{n+1}(C)$  is the secant variety  $\text{Sec}^n(C)$  for every  $n$ . The linear system  $|\mathcal{I}_C^{g-1}(g)|$  can be alternatively characterized as follows (see e.g., [1, Lemma 2.5]):

**PROPOSITION 2.2.** *The linear system  $|\mathcal{I}_C^{g-1}(g)|$  and  $|\mathcal{I}_{\text{Sec}^{g-2}(C)}(g)|$  on  $\mathbb{P}_D^{3g-2}$  are the same.*

## 2.3 The exceptional fibers of the classifying map $f_D$

Since  $\dim SU_C(2) = 3g-3$ , the generic fiber of  $f_D$  has dimension one. The set of stable bundles for which  $\dim(f_D^{-1}(E)) > 1$  is a proper subset of  $SU_C(2)$ . For simplicity, let us define the ‘‘Serre dual’’ divisor  $B := K - D$  with  $\deg(B) = g-2$ . As in the previous paragraphs,

the isomorphism classes of extensions

$$0 \rightarrow \mathcal{O}(-B) \rightarrow E \rightarrow \mathcal{O}(B) \rightarrow 0$$

are classified by the projective space

$$\mathbb{P}_B^{3g-6} := \mathbb{P}\text{Ext}^1(\mathcal{O}(B), \mathcal{O}(-B)) = |K + 2B|^*,$$

which is endowed with the rational classifying map  $f_B : \mathbb{P}_B^{3g-6} \dashrightarrow \mathcal{SU}_C(2)$  defined the same way as  $f_D$ .

PROPOSITION 2.3. *Let  $E \in \mathcal{SU}_C(2)$  be a stable bundle. Then*

$$\dim(f_D^{-1}(E)) \geq 2 \quad \text{if and only if} \quad E \in \overline{f_B(\mathbb{P}_B^{3g-6})}.$$

*Proof.* Let  $E$  be a stable bundle. Then, by Riemann-Roch and Serre duality theorems, the dimension of  $f_D^{-1}(E)$  is given by

$$\begin{aligned} h^0(C, E \otimes \mathcal{O}(D)) &= h^0(C, E \otimes \mathcal{O}(B)) + 2g - 2(g - 1) \\ &= h^0(C, E \otimes \mathcal{O}(B)) + 1 \end{aligned}$$

Thus,  $\dim(f_D^{-1}(E)) > 2$  if and only if there exists a nonzero sheaf morphism  $\mathcal{O}(-B) \rightarrow E$ . This is equivalent to  $E \in \overline{f_B(\mathbb{P}_B^{3g-6})}$ .  $\square$

If  $g > 2$ , the divisor  $|K + 2B|$  embeds  $C$  as a degree  $4g - 6$  curve in  $\mathbb{P}_B^{3g-6}$  (recall that  $\mathbb{P}\text{Ext}^1(\mathcal{O}(B), \mathcal{O}(-B)) = |K + 2B|^*$ ). Again by Theorem 2.1, the map  $\varphi_B$  is given by the linear system  $|\mathcal{I}_C^{g-3}(g-2)|$ . Moreover, by [27, Theorem 4.1] this linear system has projective dimension  $\left(\sum_{i=0}^{g-2} \binom{g}{i}\right) - 1$ .

Let us denote by  $\mathbb{P}_c$  the linear span of  $\overline{\theta(f_B(\mathbb{P}_B^{3g-6}))}$  in  $|2\Theta|$ . Hence  $\mathbb{P}_c$  has dimension  $\left(\sum_{i=0}^{g-2} \binom{g}{i}\right) - 1$ , and Proposition 2.3 also applies to  $\varphi_D$ : the fibers of  $\varphi_D$  with dimension  $\geq 2$  are those over  $\mathbb{P}_c$ .

### §3. A linear projection in $|2\Theta|$

The goal of this Section is to describe the map  $\mathcal{SU}_C(2) \rightarrow \mathbb{P}^g$  whose fibers will be birational—and in some cases even biregular—to the GIT compactification of the moduli space of  $2g$ -pointed rational curves. In order to do this, we describe the projection with center  $\mathbb{P}_c$ , seen as a linear subspace of  $|2\Theta|$ . So far we are not restricting to the case where  $C$  is hyperelliptic.

Let  $p_{\mathbb{P}_c}$  be the linear projection in  $|2\Theta|$  with center  $\mathbb{P}_c$ . Recall that  $\dim \mathbb{P}_c = \left[\sum_{i=0}^{g-2} \binom{g}{i}\right] - 1$ . A straightforward calculation shows that the supplementary linear subspaces of  $\mathbb{P}_c$  in  $|2\Theta|$  are of projective dimension  $g$ . Thus, the image of  $p_{\mathbb{P}_c}$  is a  $g$ -dimensional projective space. Let us set

$$\widehat{\mathcal{SU}}_C(2) := \mathcal{SU}_C(2) \setminus (\text{Kum}(C) \cup \overline{\varphi_D(\mathbb{P}_B^{3g-6})}).$$

This is the open subset of  $\mathcal{SU}_C(2)$  we will be mostly concerned by. Recall that the space  $H^0(C, E \otimes \mathcal{O}(D))$  has dimension 2 for  $E \in \widehat{\mathcal{SU}}_C(2)$ . Consequently, we can pick two sections  $s_1$  and  $s_2$  that constitute a basis for this space.

**THEOREM 3.1.** *The image of the projection  $p_{\mathbb{P}_c}$  can be identified with the linear system  $|2D|$  on  $C$ , in a way such that the restriction of the projection  $p_{\mathbb{P}_c}$  to  $\theta(\widehat{\mathcal{S}\mathcal{U}}_C(2))$  coincides with the map*

$$\begin{aligned} \theta(\widehat{\mathcal{S}\mathcal{U}}_C(2)) &\rightarrow |2D| \\ \theta(E) &\mapsto \text{Zeroes}(s_1 \wedge s_2) \end{aligned}$$

*Proof.* A similar statement was proved in [1] for  $C$  non hyperelliptic, but the proof extends to the hyperelliptic case with no big problem. We recall briefly the lines of [1, Theorem 4.1] and will mention explicitly where the proof for  $C$  hyperelliptic differs. The Picard variety  $\text{Pic}^{g-1}(C)$  contains a model  $\tilde{C}$  of  $C$ , made up by line bundles of type  $\mathcal{O}(B+p)$ , with  $p \in C$ . The span of  $\tilde{C}$  inside  $|2\Theta|^*$  corresponds to the complete linear system  $|2D|^*$ . Moreover, the linear span of  $\tilde{C}$  is the annihilator of  $\mathbb{P}_c$ . In particular, the projection  $p_{\mathbb{P}_c|_{\theta(\widehat{\mathcal{S}\mathcal{U}}_C(2))}}$  determines a hyperplane in the annihilator of  $\mathbb{P}_c$ , which is a point in  $|2D|$ . This projection can be identified with the map

$$\begin{aligned} p_{\mathbb{P}_c|_{\theta(\widehat{\mathcal{S}\mathcal{U}}_C(2))}} : \theta(\widehat{\mathcal{S}\mathcal{U}}_C(2)) &\rightarrow |2D|, \\ \theta(E) &\mapsto \Delta(E), \end{aligned}$$

where  $\Delta(E)$  is the divisor defined by

$$\Delta(E) := \{p \in C \mid h^0(C, E \otimes \mathcal{O}(B+p)) \neq 0\}. \quad (2)$$

Equivalently, we have that  $\Delta(E) = \theta(E) \cap \tilde{C}$ . Now, in order to adapt to the hyperelliptic case, it is enough to observe that since  $\theta(E) = \theta(i^*E)$ , we directly obtain that  $\Delta(E) = \Delta(i^*E)$ . Finally, an easy Riemann-Roch argument shows that that  $\Delta(E)$  is the divisor of zeroes of  $s_1 \wedge s_2$ .  $\square$

Recall that the linear system  $|K+2D|$  embeds the curve  $C$  in the projective space  $\mathbb{P}_D^{3g-2}$ . Let  $N \in |2D|$  be a generic effective reduced divisor and consider the linear span  $\langle N \rangle \subset \mathbb{P}_D^{3g-2}$ . The annihilator of  $\langle N \rangle$  is the vector space  $H^0(C, 2D+K-N)$ , which has dimension  $g$ . In particular, the linear span  $\langle N \rangle$  has dimension  $(3g-2) - g = 2g-2$ . Let us write

$$\mathbb{P}_N^{2g-2} := \langle N \rangle \subset \mathbb{P}_D^{3g-2}.$$

We will study the classifying map  $\varphi_D$  in relation with a fibration  $\mathcal{S}\mathcal{U}_C(2) \rightarrow \mathbb{P}^g$  by considering the restrictions of  $\varphi_D$  to  $\mathbb{P}_N^{2g-2}$ , as  $N$  varies in the linear system  $|2D|$ . The spaces  $\mathbb{P}_N^{2g-2}$  have a very explicit description in terms of extension classes (see [23]).

**NOTATION.** For simplicity, let us write  $\varphi_{D,N}$  for the restricted map  $\varphi_{D|_{\mathbb{P}_N^{2g-2}}}$ .

Before proceeding further, let us recall from [1], the following proposition, which is a consequence of [23, Proposition 1.1].

**PROPOSITION 3.2.** *Let  $N$  in  $|2D|$  be a general divisor on  $C \subset \mathbb{P}_D^{3g-2}$ . Then, the image of*

$$\varphi_{D,N} : \mathbb{P}_N^{2g-2} \dashrightarrow \theta(\mathcal{S}\mathcal{U}_C(2))$$

*is the closure in  $\theta(\mathcal{S}\mathcal{U}_C(2))$  of the fiber over  $N \in |2D|$  of the projection  $p_{\mathbb{P}_c}$ .*

**§4. The modular fibration  $SU_C(2) \rightarrow \mathbb{P}^g$**

Let  $C$  be a smooth genus  $g \geq 3$  curve (not necessarily hyperelliptic). Let  $D$  be a general degree  $g$  effective divisor on  $C$ . Let  $N = p_1 + \dots + p_{2g}$  be a general divisor in the linear system  $|2D|$ . Consider the span  $\mathbb{P}_N^{2g-2}$  in  $\mathbb{P}_D^{3g-2}$  of the  $2g$  marked points  $p_1, \dots, p_{2g}$ . In this section, building on [7] and [1], we will give more information about the restricted map  $\varphi_{D,N}$ . In particular, we will explain the interplay between these maps, rational normal curves in  $\mathbb{P}_N^{2g-2}$  and moduli spaces of rational pointed curves.

**4.1 Linear systems in  $\mathbb{P}_N^{2g-2}$  contracting rational normal curves.**

Recall that the secant variety  $\text{Sec}^{g-2}(C)$  is the base locus for  $\varphi_D$  (see Theorem 2.1 and Proposition 2.2). Hence we will distinguish the following two secant varieties in  $\mathbb{P}_N^{2g-2}$ :

$$\begin{aligned} \text{Sec}^N &:= \text{Sec}^{g-2}(C) \cap \mathbb{P}_N^{2g-2} \text{ ,} \\ \text{Sec}^{g-2}(N) &:= \bigcup_{\substack{M \subset N \\ \#M=g-1}} \text{span}\{M\} \text{ .} \end{aligned}$$

Note that, since the points of  $N$  are already in  $\mathbb{P}_N^{2g-2}$ , we have the inclusion  $\text{Sec}^{g-2}(N) \subset \text{Sec}^N$ . We will also need to consider the linear systems on  $\mathbb{P}_N^{2g-2}$

$$\mathcal{I}_{\text{Sec}^N}(g), \quad \text{and} \quad \mathcal{I}_{\text{Sec}^{g-2}(N)}(g)$$

of forms of degree  $g$  vanishing on the corresponding secant varieties. The previous inclusion of secant varieties implies that  $\mathcal{I}_{\text{Sec}^N}(g)$  is in general a linear subsystem of  $\mathcal{I}_{\text{Sec}^{g-2}(N)}(g)$ .

LEMMA 4.1. *The restricted map  $\varphi_{D,N}$  is given by a linear subsystem  $\mathcal{R}$  of  $|\mathcal{I}_{\text{Sec}^N}(g)|$ .*

*Proof.* This is a direct consequence of Theorem 2.1 and Proposition 2.2. □

**4.2 Moduli spaces of pointed rational curves**

In this section, we will outline the relation between the restricted map  $\varphi_{D,N}$  and the moduli spaces of pointed rational curves.

4.2.1. Two compactifications of  $\mathcal{M}_{0,n}$

The moduli space  $\mathcal{M}_{0,n}$  of ordered configurations of  $n$  distinct points on the projective line is not compact. We will consider two compactifications of  $\mathcal{M}_{0,n}$ . The first one is the GIT quotient

$$\mathcal{M}_{0,n}^{\text{GIT}} := (\mathbb{P}^1)^n // \text{PGL}(2, \mathbb{C})$$

of  $(\mathbb{P}^1)^n$  by the diagonal action of  $G = \text{PGL}(2, \mathbb{C})$  for the natural  $G$ -linearization of the line bundle  $L = \boxtimes_{i=1}^n \mathcal{O}_{\mathbb{P}^1}(1)$  (see [12, 13]). The quotient  $\mathcal{M}_{0,n}^{\text{GIT}}$  is naturally embedded in the projective space  $\mathbb{P}(H^0((\mathbb{P}^1)^n, L)^G)$  of invariant sections.

The second one is the Mumford–Knudsen compactification  $\overline{\mathcal{M}}_{0,n}$  [20]. The points of  $\overline{\mathcal{M}}_{0,n}$  represent isomorphism classes of stable curves. More details on these constructions can be found in [18] and [20].

Both  $\mathcal{M}_{0,2g}^{\text{GIT}}$  and  $\overline{\mathcal{M}}_{0,n}$  contain  $\mathcal{M}_{0,n}$  as an open set. However, the Mumford–Knudsen compactification is finer on the boundary: there exists a contraction dominant morphism

$$c_n : \overline{\mathcal{M}}_{0,n} \rightarrow \mathcal{M}_{0,n}^{\text{GIT}}$$



contracting some components of the boundary of  $\overline{\mathcal{M}}_{0,n}$ , that restricts to the identity on  $\mathcal{M}_{0,n}$  [7]. Moreover, we will denote by

$$\lambda_k : \overline{\mathcal{M}}_{0,n} \rightarrow \overline{\mathcal{M}}_{0,n-1},$$

for  $k = 1, \dots, n$ , the forgetful morphism that forgets the labeling of the  $k$ th point.

#### 4.2.2. A variety of rational normal curves

Let  $e_1, \dots, e_n \in \mathbb{P}^{n-2}$  be  $n$  points in general position. Let  $\mathcal{H}$  be the Hilbert scheme of subschemes of  $\mathbb{P}^{n-2}$ . Let  $V_0(e_1, \dots, e_n) \subset \mathcal{H}$  be the subvariety of rational normal curves in  $\mathbb{P}^{n-2}$  passing through the points  $e_1, \dots, e_n$ , and let  $V(e_1, \dots, e_n)$  be the closure of  $V_0(e_1, \dots, e_n)$  inside  $\mathcal{H}$ . The boundary  $V(e_1, \dots, e_n) \setminus V_0(e_1, \dots, e_n)$  consists on reducible rational normal curves, that is reducible nondegenerate curves of degree  $n$  such that each component is a rational normal curve in its projective span.

There exists an isomorphism  $V_0(e_1, \dots, e_n) \cong \mathcal{M}_{0,n}$  (see [12]) associating to a rational normal curve passing by  $e_1, \dots, e_n$  the corresponding ordered configuration of  $n$  points in  $\mathbb{P}^1$ . This can be extended [18] to an isomorphism between  $V(e_1, \dots, e_n)$  and  $\overline{\mathcal{M}}_{0,n}$ .

#### 4.2.3. The blow-up construction

The following construction is due to Kapranov [17]: let  $e_1, \dots, e_{n-1} \in \mathbb{P}^{n-3}$  be  $(n-1)$  points in general position. Consider the following sequence of blow-ups:

1. Blow-up the points  $e_1, \dots, e_{n-1}$ .
2. Blow-up the proper transforms of lines spanned by pairs of points from  $\{e_1, \dots, e_{n-1}\}$ .
3. Blow-up the proper transforms of planes spanned by triples of points from  $\{e_1, \dots, e_{n-1}\}$ .
- ⋮
- ( $n-4$ ). Blow-up the proper transforms of linear subspaces spanned by  $(n-4)$ -ples of points from  $\{e_1, \dots, e_{n-1}\}$ .

Let  $\text{Bl}(\mathbb{P}^{n-3})$  be the  $(n-3)$ -variety obtained in this way, and  $b : \text{Bl}(\mathbb{P}^{n-3}) \rightarrow \mathbb{P}^{n-3}$  the composition of this sequence of blow-ups. We will call this map the *Kapranov blow-up map centered in the points*  $e_1, \dots, e_{n-1}$ .

**THEOREM 4.2.** (Kapranov [17]) *Let  $n \geq 4$ . Then, the moduli space  $\overline{\mathcal{M}}_{0,n}$  is isomorphic to  $\text{Bl}(\mathbb{P}^{n-3})$ .*

Moreover, the images by  $b$  of the fibres of the map  $\lambda_k$  over the points in the open set  $\mathcal{M}_{0,n-1} \subset \overline{\mathcal{M}}_{0,n-1}$  are the rational normal curves in  $\mathbb{P}^{n-3}$  passing through the  $n-1$  points  $e_1, \dots, e_{n-1}$  (see [19, Proposition 3.1]).

#### 4.2.4. The Cremona inversion

Let  $e_1, \dots, e_{n-1} \in \mathbb{P}^{n-3}$  in general position. Without loss of generality, we may assume  $e_k = [0 : \dots : 1 : \dots : 0]$  for  $k = 1, \dots, n-2$ ; and  $e_{n-1} = [1 : \dots : 1]$ . The *Cremona inversion with respect to*  $e_{n-1}$  is the birational map

$$\begin{aligned} \text{Cr}_{n-1} : \mathbb{P}^{n-3} &\dashrightarrow \mathbb{P}^{n-3} \\ [x_0 : \dots : x_{n-3}] &\mapsto [1/x_0 : \dots : 1/x_{n-3}]. \end{aligned}$$

On  $\mathbb{P}^2$  the Cremona inversion is given by the linear system of quadrics passing through  $e_1, e_2$ , and  $e_3$ , on  $\mathbb{P}^3$  by the cubics that vanish on the six secant lines of  $e_1, e_2, e_3$ , and  $e_4$ , and so on. The Cremona inversion has the following property: any nondegenerate rational normal curve passing through the points  $e_1, \dots, e_{n-1}$  is transformed into a line passing by the point  $\text{Cr}_{n-1}(e_{n-1})$ . Let  $\tau_{n-1} : \mathbb{P}^{n-3} \dashrightarrow \mathbb{P}^{n-4}$  be the linear projection with center  $\text{Cr}_{n-1}(e_{n-1})$ . From the previous property, we obtain that the composition  $\tau_{n-1} \circ \text{Cr}_{n-1}$  contracts nondegenerate rational normal curves passing through  $e_1, \dots, e_{n-1}$ .

Let  $k \in \{1, \dots, n-1\}$ . It is straightforward to see that one can let  $e_k$  play the role of  $e_{n-1}$  in the definition of  $\text{Cr}_{n-1}$ , and define similarly the Cremona inversion  $\text{Cr}_k$ . Let  $\tau_k : \mathbb{P}^{n-3} \dashrightarrow \mathbb{P}^{n-4}$  be the linear projection with center  $\text{Cr}_k(e_k)$ .

LEMMA 4.3. *Let  $e_1, \dots, e_{n-1} \in \mathbb{P}^{n-3}$  in general position. Then, the composition  $\tau_k \circ \text{Cr}_k$  contracts the nondegenerate rational normal curves passing through  $e_1, \dots, e_{n-1}$ .*

Let us denote  $H_t$ , for  $t \neq k$ , the hyperplane in  $\mathbb{P}^{n-3}$  spanned the points  $e_i$ , with  $i \neq k, t$ . There are  $n-2$  such hyperplanes and each one is contracted to a point by  $\text{Cr}_k$ .

PROPOSITION 4.4. [18] *Let  $e_1, \dots, e_{n-1} \in \mathbb{P}^{n-3}$  in general position. Let  $k \in \{1, \dots, n-1\}$ . Then, the following diagram is commutative:*

$$\begin{array}{ccc} \overline{\mathcal{M}}_{0,n} & \xrightarrow{\lambda_k} & \overline{\mathcal{M}}_{0,n-1} \\ b \downarrow & & \downarrow b_k \\ \mathbb{P}^{n-3} & \xrightarrow{\tau_k \circ \text{Cr}_k} & \mathbb{P}^{n-4} \end{array}$$

Here, the map  $b_k$  is the Kapranov blow-up map centered in the images of the hyperplanes  $H_t$ , for  $1 \leq t \leq n-1$  and  $t \neq k$ , by  $\tau_k \circ \text{Cr}_k$ .

REMARK. We observe here a little subtlety. We only get here  $n-1$  forgetful maps through Cremona transformations, because we are tacitly assuming that Kapranov’s blow-up construction of  $\overline{\mathcal{M}}_{0,n}$  labels with integers from 1 to  $n-1$  the points  $e_1, \dots, e_{n-1}$  of the projective base of  $\mathbb{P}^{n-3}$ , and labels as  $n$  the last point (which is free to move inside the  $\mathbb{P}^{n-3}$ , which is a birational model of  $\overline{\mathcal{M}}_{0,n}$ ). This is due to this small asymmetric aspect of Kapranov’s construction, but it is clear that one could assume that the last, freely moving point is labeled with any  $k \in \{1, \dots, n-1\}$ , and obtain other forgetful maps.

#### 4.2.5. Rationalizations of $\mathcal{M}_{0,2g}^{\text{GIT}}$

Let  $g \geq 3$ , and let  $e_1, \dots, e_{2g-1} \in \mathbb{P}^{2g-3}$  in general position. Let  $\Omega$  be the linear system of  $(g-1)$ -forms in  $\mathbb{P}^{2g-3}$  vanishing with multiplicity  $g-2$  in  $e_1, \dots, e_{2g-1} \in \mathbb{P}^{2g-3}$ .

THEOREM 4.5. ([21]) *Let  $g \geq 3$ , and let  $e_1, \dots, e_{2g-1} \in \mathbb{P}^{2g-3}$  be in general position. Then, the rational map*

$$i_\Omega : \mathbb{P}^{2g-3} \dashrightarrow \Omega^*$$

*induced by the linear system  $\Omega$  maps  $\mathbb{P}^{2g-3}$  birationally onto  $\mathcal{M}_{0,2g}^{\text{GIT}}$ .*

We also observe that the contraction map  $c_{2n}$  can also be described in terms of Kumar’s linear system  $\Omega$ :

LEMMA 4.6. *Let  $g \geq 3$ , and let  $e_1, \dots, e_{2g-1} \in \mathbb{P}^{2g-3}$  in general position. Then, the following diagram is commutative:*

$$\begin{array}{ccc} \overline{\mathcal{M}}_{0,2g} & & \\ \downarrow b & \searrow c_{2g} & \\ \mathbb{P}^{2g-3} & \xrightarrow{i_\Omega} & \mathcal{M}_{0,2g}^{\text{GIT}} \end{array}$$

Here, the map  $b$  is the Kapranov blow-up map centered in  $p_1, \dots, p_{2g-1}$ .

Let  $e_0 \in \mathbb{P}^{2g-3}$  such that  $w = i_\Omega(e_0)$  lies in the open set  $\mathcal{M}_{0,2g} \subset \mathcal{M}_{0,2g}^{\text{GIT}}$ . The point  $w$  can represent a hyperelliptic genus  $(g-1)$  curve  $C_w$  (namely the double cover of  $\mathbb{P}^1$  ramifying in the  $2g$  marked points) together with an ordering of the Weierstrass points. Let  $SU_{C_w}(2)^{\text{inv}}$  be the moduli space of rank 2 semistable vector bundles with trivial determinant over the curve  $C_w$ , that are invariant with respect to the hyperelliptic involution.

Consider the partial linear subsystem  $\Lambda$  of  $\Omega$  consisting of the  $(g-1)$ -forms in  $\mathbb{P}^{2g-3}$  vanishing with multiplicity  $g-2$  in all the points  $e_0, e_1, \dots, e_{2g-1}$ . Let  $\kappa : \mathcal{M}_{0,2g}^{\text{GIT}} \dashrightarrow \Lambda^*$  be the rational projection induced by the linear system  $\Lambda$ .

THEOREM 4.7. ([21]) *Let  $g \geq 3$ , and let  $e_1, \dots, e_{2g-1} \in \mathbb{P}^{2g-3}$  be  $2g-1$  points in general position. Let  $e_0 \in \mathbb{P}^{2g-3}$  such that  $w = i_\Omega(e_0)$  lies in the open set  $\mathcal{M}_{0,2g} \subset \mathcal{M}_{0,2g}^{\text{GIT}}$ . Then, the map  $\kappa$  induced by the linear system  $\Lambda$  is a degree 2 osculating projection onto a connected component of the moduli space  $SU_{C_w}(2)^{\text{inv}}$ . Furthermore, the map  $\kappa$  ramifies along the Kummer variety  $\text{Kum}(C_w) \subset SU_{C_w}(2)^{\text{inv}}$ .*

$$\begin{array}{ccc} \mathbb{P}^{2g-3} & \xrightarrow{i_\Omega} & \mathcal{M}_{0,2g}^{\text{GIT}} \\ & \searrow i_\Lambda & \downarrow \kappa \\ & & SU_{C_w}(2)^{\text{inv}} \end{array}$$

In fact, one can talk of osculating projection (and this means  $\mathcal{M}_{0,2g}^{\text{GIT}}$  is embedded in some known projective space) since  $i_\Omega$  embeds  $\mathcal{M}_{0,2g}^{\text{GIT}}$  in its natural projective space of invariants, defined by the democratic polarization with weights vector  $(1, 1, \dots, 1)$  (see also Section 5.2.3).

### 4.3 Forgetful linear systems and $\mathcal{M}_{0,2g}^{\text{GIT}}$

Let  $C$  be a smooth genus  $g \geq 3$  curve (not necessarily hyperelliptic). Let  $D$  be a general degree  $g$  effective divisor on  $C$ . Let  $N = p_1 + \dots + p_{2g}$  be a general reduced divisor in the linear system  $|2D|$ . Consider the span  $\mathbb{P}_N^{2g-2}$  in  $\mathbb{P}_D^{3g-2}$  of the  $2g$  marked points  $p_1, \dots, p_{2g}$ .

We will now apply the discussion of Section 4.2 to the general points  $p_1, \dots, p_{2g}$  in the projective space  $\mathbb{P}_N^{2g-2}$ , taking  $n = 2g + 1$  in the notation therein. For every  $k = 1, \dots, 2g$ , we can compose Proposition 4.4 and Lemma 4.6 and get a commutative diagram

$$\begin{array}{ccccc} \overline{\mathcal{M}}_{0,2g+1} & \xrightarrow{\lambda_k} & \overline{\mathcal{M}}_{0,2g} & & \\ \downarrow b & & \downarrow b_k & \searrow c_{2g} & \\ \mathbb{P}_N^{2g-2} & \xrightarrow{\tau_k \circ \text{Cr}_k} & \mathbb{P}^{2g-3} & \xrightarrow{i_\Omega} & \mathcal{M}_{0,2g}^{\text{GIT}} \end{array} \tag{3}$$

where  $\Omega$  is the linear system of  $(g - 1)$ -forms in  $\mathbb{P}^{2g-3}$  vanishing with multiplicity  $g - 2$  at the  $2g - 1$  points  $\tau_k \circ \text{Cr}_k(H_i)$ , with  $i \neq k$  and  $1 \leq i \leq 2g$ . Let us define the rational map

$$h_N : \mathbb{P}_N^{2g-2} \dashrightarrow |\mathcal{I}_{\text{Sec}^{g-2}(N)}(g)|^*.$$

PROPOSITION 4.8. ([7]) *Let  $N = p_1 + \dots + p_{2g}$  be a general reduced divisor in the linear system  $|2D|$ . Then, the map  $h_N$  coincides with the composition  $i_\Omega \circ \tau_k \circ \text{Cr}_k$  for every  $k = 1, \dots, 2g$ . In particular, the composition  $i_\Omega \circ \tau_k \circ \text{Cr}_k$  does not depend on  $k$ .*

This is due to the fact that the linear system  $|\mathcal{I}_{\text{Sec}^{g-2}(N)}(g)|$  is invariant with respect to the action of the symmetric group  $\Sigma_{2g}$  that operates on  $\mathbb{P}_N^{2g-2}$  by linear automorphisms. Let us put together the results of Lemma 4.3, Theorem 4.5 and Proposition 4.8 in the following Proposition:

PROPOSITION 4.9. *The image of  $h_N$  is isomorphic to the GIT moduli space  $\mathcal{M}_{0,2g}^{\text{GIT}}$  of ordered configurations of  $2g$  points in  $\mathbb{P}^1$ . The map  $h_N$  is dominant and its general fiber is of dimension 1. More precisely,  $h_N$  contracts every rational normal curve  $Z$  passing through the  $2g$  points  $N$  to a point  $z$  in  $\mathcal{M}_{0,2g}^{\text{GIT}}$ . This point represents an ordered configuration of the  $2g$  points  $N$  on the rational curve  $Z$ .*

This is why these maps were dubbed “forgetful linear systems.” In fact the rational normal curves passing through the  $2g$  points make up the universal curve over an open subset of  $\mathcal{M}_{0,2g}^{\text{GIT}}$ .

REMARK. Since  $\mathcal{R}$  is a linear subsystem of  $|\mathcal{I}_{\text{Sec}^{g-2}(N)}(g)|$  by Lemma 4.1, we have that  $\varphi_{D,N}$  factors through  $h_N$ :

$$\begin{array}{ccc}
 \mathbb{P}_N^{2g-2} & \xrightarrow{\quad h_N \quad} & \mathcal{M}_{0,2g}^{\text{GIT}} \\
 \searrow \varphi_{D,N} & & \downarrow \\
 & & \theta(\text{SU}_C(2))
 \end{array} \tag{4}$$

### 4.3.1. A comparison of base loci.

For future use, we need to compare the locus  $\text{Sec}^{g-2}(N)$  and the more intricate locus  $\text{Sec}^N$  obtained by intersecting the base locus of  $\varphi_D$  with  $\mathbb{P}_N^{2g-2}$ . This section is devoted to this comparison.

By definition, the points in  $\text{Sec}^N$  are given by the intersections  $\langle L_{g-1} \rangle \cap \mathbb{P}_N^{2g-2}$ , where  $L_{g-1}$  is an effective divisor of degree  $g - 1$  and  $\langle L_{g-1} \rangle$  is its linear span in  $\mathbb{P}_D^{3g-2}$ . If  $L_{g-1}$  is contained in  $N$ , it is clear that  $\langle L_{g-1} \rangle \subset \text{Sec}^{g-2}(N) \subset \mathbb{P}_N^{2g-2}$ .

LEMMA 4.10. *Let  $L_{g-1}$  be an effective divisor on  $C$  of degree  $g - 1$ , not contained in  $N$ . Then,*

$$\langle L_{g-1} \rangle \cap \mathbb{P}_N^{2g-2} \neq \emptyset \quad \text{if and only if} \quad \dim |L_{g-1}| \geq 1.$$

Moreover, if the intersection is nonempty, we have that

$$\dim(\langle L_{g-1} \rangle \cap \mathbb{P}_N^{2g-2}) = \dim |L_{g-1}| - 1.$$

*Proof.* First, let us suppose that  $L_{g-1}$  and  $N$  have no points in common. The vector space  $V := H^0(C, 2D + K - L_{g-1})$  is the annihilator of the span  $\langle L_{g-1} \rangle$  in  $\mathbb{P}_D^{3g-2}$ . By the Riemann-Roch theorem, we see that  $V$  has dimension  $2g$ , hence

$$\dim \langle L_{g-1} \rangle = (3g - 2) - 2g = g - 2.$$

Let  $d$  be the dimension of the span  $\langle L_{g-1}, N \rangle$  of the points of  $L_{g-1}$  and  $N$ . Since the dimension of  $\mathbb{P}_N^{2g-2} = \langle N \rangle$  is  $2g - 2$ , we have that  $d \leq (g - 2) + (2g - 2) + 1 = 3g - 3$ , where the equality holds iff  $\langle L_{g-1} \rangle \cap \mathbb{P}_N^{2g-2}$  is empty.

In particular, this intersection is nonempty iff  $d \leq 3g - 4$ . Since  $\dim |K + 2D|^* = \dim \mathbb{P}_D^{3g-2} = 3g - 2$ , this is equivalent to the annihilator space

$$W := H^0(C, 2D + K - L_{g-1} - N) = H^0(C, K - L_{g-1})$$

being of dimension  $\geq 2$ . By Riemann-Roch and Serre duality, we obtain that this condition is equivalent to  $\dim |L_{g-1}| \geq 1$ .

More precisely, let us suppose that  $\langle L_{g-1} \rangle \cap \mathbb{P}_N^{2g-2}$  is nonempty and let  $e := \dim(\langle L_{g-1} \rangle \cap \mathbb{P}_N^{2g-2})$ . Then, we have that

$$d = 3g - 3 - (e + 1),$$

and the annihilator space  $W$  is of dimension  $2 + e$ . Again by a Riemann-Roch computation, we conclude that  $e = \dim |L_{g-1}| - 1$ .

Finally, if  $L_{g-1}$  and  $N$  have some points in common, we have to count them only once when defining the vector space  $W$  to avoid requiring higher vanishing multiplicity to the sections. □

From this Lemma, we conclude that  $\text{Sec}^{g-2}(N)$  is a proper subset of  $\text{Sec}^N$  if and only if there exists a divisor  $L_{g-1}$  not contained in  $N$  with  $\dim |L_{g-1}| \geq 1$ . By the Existence Theorem of Brill–Noether theory (see [2, Theorem 1.1, page 206]), this is possible only if  $g \geq 4$  in the nonhyperelliptic case, whereas such a linear system may exist also when  $g = 3$  when  $C$  is hyperelliptic. We will discuss the first low genera cases in Section 7.

### §5. The hyperelliptic involution and rational normal curves

From now on,  $C$  will be a *hyperelliptic* curve of genus  $g \geq 3$ .

As we have seen in Lemma 4.1, the base locus of the map  $\varphi_{D,N}$  contains  $\text{Sec}^N$ . We have seen that the secant variety  $\text{Sec}^{g-2}(N)$  is contained in  $\text{Sec}^N$  and that this inclusion is strict for  $g \geq 4$  in the nonhyperelliptic case.

#### 5.1 A rational normal curve coming from involution invariant secant lines.

In the hyperelliptic case, we have an additional base locus for every  $g \geq 3$ , which appears due to the hyperelliptic nature of the curve. This locus arises as follows. For each pair  $\{p, i(p)\}$  of involution-conjugate points in  $C$ , consider the secant line  $l$  in  $\mathbb{P}_D^{3g-2}$  passing through the points  $p$  and  $i(p)$ . Let  $Q_p$  be the intersection of the line  $l$  with  $\mathbb{P}_N^{2g-2}$ . Let us define  $\Gamma \subset \mathbb{P}_N^{2g-2}$  as the locus of intersection points  $Q_p$  when  $p$  moves inside  $C$ .

LEMMA 5.1. *The locus  $\Gamma \subset \mathbb{P}_N^{2g-2}$  is a nondegenerate rational normal curve in  $\mathbb{P}_N^{2g-2}$ . Moreover,  $\Gamma$  passes through the  $2g$  points of  $N \subset C$ .*

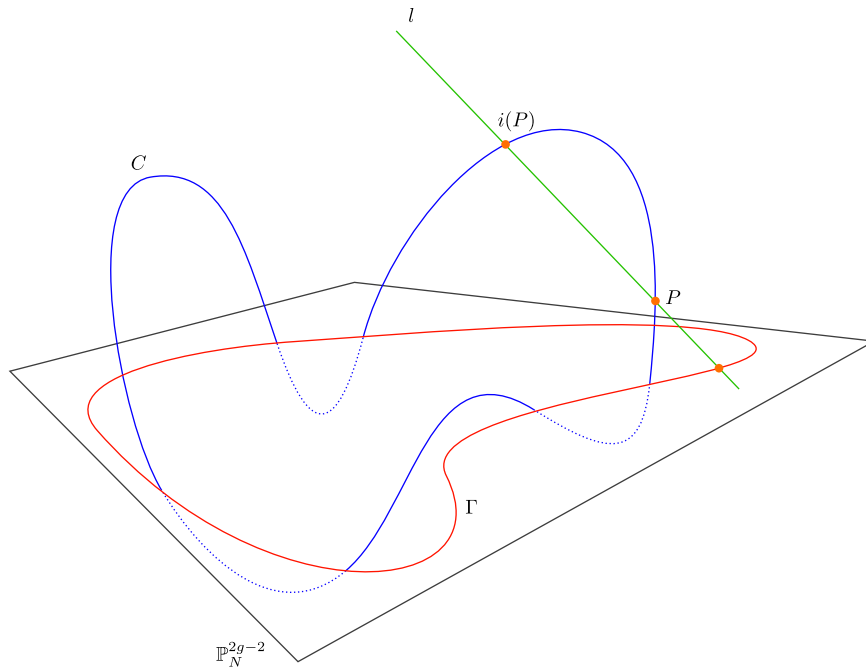


Figure 1.

The situation in genus 3. The curves  $\Gamma$  and  $C$  intersect along the divisor  $D$ , of degree 6. The secant lines  $l$  cutting out the hyperelliptic pencil define the curve  $\Gamma$ .

*Proof.* Let us start by showing that the intersection  $Q_p$  is nonempty for every line  $l = \overline{p, i(p)}$ , with  $p \in C$ . Since  $\dim |p + i(p)| = \dim |h| = 1$ , the intersection  $l \cap \mathbb{P}_N^{2g-2}$  is nonempty by Lemma 4.10.

Let us show that this intersection is a point, that is that the line  $l$  is not contained in  $\mathbb{P}_N^{2g-2}$ . Recall that  $\mathbb{P}_D^{3g-2} = |2D + K|^*$ ; we will discuss three cases.

i) If the points  $p$  and  $i(p)$  are both not contained in the divisor  $N$ , the vector space

$$V := H^0(C, 2D + K - N - (p + i(p))) = H^0(C, 2D + K - N - h)$$

is exactly the annihilator of the span  $\langle l, \mathbb{P}_N^{2g-2} \rangle$  in  $\mathbb{P}_D^{3g-2}$ . In particular, the codimension of  $\langle l, \mathbb{P}_N^{2g-2} \rangle$  in  $\mathbb{P}_D^{3g-2}$  is the dimension of  $V$ . By Riemann-Roch and Serre duality, we get that  $\dim V = g - 2$ , thus  $\dim \langle l, \mathbb{P}_N^{2g-2} \rangle = 3g - 2 - (g - 2) = 2g$ . This means that the intersection  $l \cap \mathbb{P}_N^{2g-2}$  is a point.

ii) For the case  $p \in N$  and  $i(p) \notin N$ , let us remark that the the annihilator of the span  $\langle l, \mathbb{P}_N^{2g-2} \rangle$  is now the vector space  $H^0(C, 2D + K - N - i(p))$ . Since

$$h^0(C, 2D + K - N - i(p)) < h^0(C, 2D + K - N),$$

we conclude that the line  $l$  is not contained in  $\mathbb{P}_N^{2g-2}$ .

iii) The case  $\{p, i(p)\} \subset N$  is excluded by our genericity hypotheses on  $N$ . Hence we deduce that the locus  $\Gamma$  is a curve in  $\mathbb{P}_N^{2g-2}$ .

Let  $q$  be a point of  $N \subset \mathbb{P}_N^{2g-2}$ . The line  $\overline{q, i(q)}$  intersects the plane  $\mathbb{P}_N^{2g-2}$  at  $q$ . Thus  $\Gamma$  passes through all the points of  $N$ . Moreover, it is clear that  $N$  is the only intersection of  $\Gamma$  and  $C$ , that is  $\Gamma \cap C = N$ .

Let us prove now that  $\Gamma$  is a rational normal curve. Since  $\Gamma$  is defined by the hyperelliptic pencil, it is straightforward to see that  $\Gamma$  is rational. Moreover, since the divisor  $D$  is general,

the span of any subset of  $2g - 1$  points of  $D$  is  $\mathbb{P}_N^{2g-2}$ . Thus, it suffices to show that the degree of  $\Gamma \subset \mathbb{P}_N^{2g-2}$  is precisely  $2g - 2$ .

Let us set  $N = q_1 + \dots + q_{2g}$  with  $q_1, \dots, q_{2g} \in C$ . By the previous paragraph,  $\Gamma$  passes through these  $2g$  points. Let us consider a hyperplane  $H$  of  $\mathbb{P}_N^{2g-2}$  spanned by  $2g - 2$  points of  $N$ . Without loss of generality, we can suppose that these points are the first  $2g - 2$  points  $q_1, \dots, q_{2g-2}$ . To show that the degree of  $\Gamma$  is  $2g - 2$ , we have to show that the intersection of  $\Gamma$  with  $H$  consists exactly only of these points.

Let  $l$  be the secant line  $\overline{q, i(q)}$ ,  $q \in C$ . The intersection  $l \cap H$  is empty if and only if the linear span  $\langle l, H \rangle$  of  $l$  and  $H$  in  $\mathbb{P}_D^{3g-2}$  is of maximal dimension  $2g - 1$ , that is of codimension  $g - 1$  in  $\mathbb{P}_D^{3g-2}$ . Consider the divisors

$$D_H = q_1 + \dots + q_{2g-2} \quad \text{and} \quad D_l = q + i(q).$$

As before, if  $\{q, i(q)\} \cap \{q_1, \dots, q_{2g-2}\}$  is empty, the vector space  $W = H^0(C, 2D + K - D_H - D_l)$  is the annihilator of the span  $\langle l, H \rangle$  in  $\mathbb{P}_D^{3g-2}$ . In particular, the codimension of  $\langle l, H \rangle$  in  $\mathbb{P}_D^{3g-2}$  is given by the dimension of  $W$ . Again by Riemann-Roch and Serre duality theorems, we can check that

$$\dim W = h^0(C, -2D + D_H + D_l) + g - 1.$$

Thus, the codimension of  $\langle l, H \rangle$  in  $\mathbb{P}_D^{3g-2}$  is greater than  $g - 1$  if and only if  $h^0(C, -2D + D_H + D_l) > 0$ . Since  $\deg(-2D + D_H + D_l) = 0$ , this is equivalent to  $-2D + D_H + D_l \sim 0$ . Since  $N = q_1 + \dots + q_{2g} \sim 2D$ , we have that

$$\begin{aligned} -2D + D_H + D_l \sim 0 &\iff q + i(q) \sim q_{2g-1} + q_{2g} \\ &\iff h \sim q_{2g-1} + q_{2g} \\ &\iff i(q_{2g-1}) = q_{2g}. \end{aligned}$$

By our genericity hypothesis on  $N$ , the last condition is not satisfied. Consequently, we conclude that the line  $l$  intersects the hyperplane  $H$  iff  $\{q, i(q)\} \cap \{q_1, \dots, q_{2g-2}\}$  is nonempty, that is iff  $q$  or  $i(q)$  is one of the  $q_k$  for  $k = 1, \dots, 2g - 2$ . In particular,

$$\Gamma \cap H = \{q_1, \dots, q_{2g-2}\}$$

as we wanted to show. □

Hence, the curve  $\Gamma$  is contracted by the map  $h_N$  to a point  $w \in \mathcal{M}_{0,2g}^{\text{GIT}}$  by Proposition 4.9. The point  $w$  represents a hyperelliptic curve  $C_w$  of genus  $g - 1$  together with an ordering of the Weierstrass points that correspond to the points of  $N$  on the rational curve  $\Gamma$ .

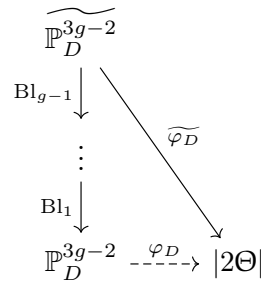
### 5.2 The restriction of the theta map to $\mathcal{M}_{0,2g}^{\text{GIT}}$

Let us set once again  $N = p_1 + \dots + p_{2g}$ , a general divisor in the linear system  $|2D|$ , and consider the span  $\mathbb{P}_N^{2g-2}$  in  $\mathbb{P}_D^{3g-2}$  of the  $2g$  marked points  $p_1, \dots, p_{2g}$ .

In this section, we describe the interplay between  $\theta$  and the linear systems presented in Section 4.

#### 5.2.1. The factorization of the map $\varphi_D$ .

Recall that the base locus of the map  $\varphi_D$  is the secant variety  $\text{Sec}^{g-2}(C)$  by Proposition 2.2. As in [4], one can construct a resolution  $\widetilde{\varphi}_D$  of the map  $\varphi_D$  via a sequence of blow-ups



along the secant varieties

$$C = \text{Sec}^0(C) \subset \text{Sec}^1(C) \subset \dots \subset \text{Sec}^{g-1}(C) \subset \mathbb{P}_D^{3g-2}.$$

This chain of morphisms is defined inductively as follows: the center of the first blow-up  $\text{Bl}_1$  is the curve  $C = \text{Sec}^0(C)$ . For  $k = 2, \dots, g - 1$ , the center of the blow-up  $\text{Bl}_k$  is the strict transform of the secant variety  $\text{Sec}^{k-1}(C)$  under the blow-up  $\text{Bl}_{k-1}$ .

The map  $\varphi_D$  is, by definition, the composition of the classifying map  $f_D$  and the degree 2 map  $\theta$ . Thus, the map  $f_D$  lifts to a morphism  $\widetilde{f}_D$  which makes the following diagram commute:

$$\begin{array}{ccc}
 \widetilde{\mathbb{P}}_D^{3g-2} & \xrightarrow{\widetilde{f}_D} & \mathcal{SU}_C(2) \\
 \searrow \widetilde{\varphi}_D & & \downarrow \theta \\
 & & |2\Theta|
 \end{array} \tag{5}$$

### 5.2.2. Osculating projections

We recall here a generalization of linear projections that will allow us to describe the map  $p$  in higher genus. For a more complete reference, see for example [24]. Let  $X \subset \mathbb{P}^N$  be an integral projective variety of dimension  $n$ , and  $p \in X$  a smooth point. Let

$$\begin{aligned}
 \phi : \mathcal{U} \subset \mathbb{C}^n &\longrightarrow \mathbb{C}^N \\
 (t_1, \dots, t_n) &\longmapsto \phi(t_1, \dots, t_n)
 \end{aligned}$$

be a local parametrization of  $X$  in a neighborhood of  $p = \phi(0) \in X$ . For  $m \geq 0$ , let  $O_p^m$  be the affine subspace of  $\mathbb{C}^N$  passing through  $p \in X$  and generated by the vectors  $\phi_I(0)$ , where  $\phi_I$  is a partial derivative of  $\phi$  of order  $\leq m$ .

By definition, the  $m$ -osculating space  $T_p^m X$  of  $X$  at  $p$  is the projective closure in  $\mathbb{P}^N$  of  $O_p^m$ . The  $m$ -osculating projection

$$\Pi_p^m : X \subset \mathbb{P}^N \dashrightarrow \mathbb{P}^{N-m}$$

is the corresponding linear projection with center  $T_p^m$ .

### 5.2.3. Osculating projections of $\mathcal{M}_{0,2g}^{\text{GIT}}$

In this section, we show how the map  $\varphi_{D,N}$  induces an osculating projection on the copies of  $\mathcal{M}_{0,2g}^{\text{GIT}}$  that appear as factors of the map in Diagram 4.



LEMMA 5.2. *Let  $Q$  be a  $r$ -form in  $\mathbb{P}^n$  vanishing at the points  $P_1$  and  $P_2$  with multiplicity  $l_1$  and  $l_2$  respectively. Then,  $Q$  vanishes on the line passing through  $P_1$  and  $P_2$  with multiplicity at least  $l_1 + l_2 - r$ .*

*Proof.* See, for example, [22, page 2]. □

Let us now consider the linear system  $|\mathcal{I}_{\text{Sec}^N}(g)|$  on  $\mathbb{P}_N^{2g-2}$  (see Section 4). The forms in  $|\mathcal{I}_{\text{Sec}^N}(g)|$  vanish with multiplicity  $g - 1$  along the points of  $C$  (see Lemma 2.2). By Lemma 5.2, these forms vanish then with multiplicity  $(g - 1) + (g - 1) - g = g - 2$  along the secant lines  $l$  cutting out the hyperelliptic pencil. Thus, these forms vanish with multiplicity  $g - 2$  on the curve  $\Gamma$ .

Let us consider the linear system  $|\mathcal{I}_{\text{Sec}^{g-2}(N)}(g)|$ . Let  $\mathcal{I}(\Gamma) \subset |\mathcal{I}_{\text{Sec}^{g-2}(N)}(g)|$  be the partial linear system of forms vanishing (with multiplicity 1) along  $\text{Sec}^{g-2}(N)$ , and with multiplicity  $g - 2$  along  $\Gamma$ . By our previous observation and Lemma 4.1, we have the following inclusions of linear systems:

$$\mathcal{R} \subset |\mathcal{I}_{\text{Sec}^N}(g)| \subset \mathcal{I}(\Gamma) \subset |\mathcal{I}_{\text{Sec}^{g-2}(N)}(g)|.$$

Recall that the map  $\varphi_{D,N}$  is induced by the linear system  $\mathcal{R}$ . The above sequence of inclusions yields the following factorization of maps

$$\begin{array}{ccc} \mathbb{P}_N^{2g-2} & \xrightarrow{h_N} & \mathcal{M}_{0,2g}^{\text{GIT}} \subset |\mathcal{I}_{\text{Sec}^{g-2}(N)}(g)|^* & \xrightarrow{\pi_N} & |\mathcal{I}(\Gamma)|^* \\ & \searrow \varphi_{D,N} & & & \downarrow l_N \\ & & & & \theta(SU_C(2)) \end{array} \quad (6)$$

The first map  $h_N$  is the one defined in Section 4.3, its image is the GIT quotient  $\mathcal{M}_{0,2g}^{\text{GIT}}$ . According to Proposition 4.9, this map contracts the curve  $\Gamma$  to a point  $h_N(\Gamma)$ .

PROPOSITION 5.3. *The map  $\pi_N$  is the  $(g - 3)$ -osculating projection  $\Pi_P^{g-3}$  with center the point  $w = h_N(\Gamma)$ .*

*Proof.* From the geometric description of the linear systems  $\mathcal{I}(\Gamma)$  and  $|\mathcal{I}_{\text{Sec}^{g-2}(N)}(g)|$  (Propositions 4.8 and 4.9), the base locus of the map  $\pi_N$  is the point  $w = h_N(\Gamma)$ , with multiplicity  $g - 2$ . In particular, since the forms in  $\mathcal{I}(\Gamma)$  vanish with multiplicity  $g - 2$  along  $\Gamma$ , the order the projection  $\pi_N$  is  $g - 3$ . □

According to Proposition 4.9, the map  $h_N$  contracts the curve  $\Gamma$  to a point  $w$  in  $\mathcal{M}_{0,2g}^{\text{GIT}}$  representing an ordered configuration of the  $2g$  marked points  $N$ . This point in turn corresponds to a hyperelliptic genus  $(g - 1)$  curve  $C_w$  together with an ordering of the Weierstrass points. Now recall from Section 4.3 that the lower level composed map of Diagram 3 is the map  $h_N$ . The rational normal curve  $\Gamma \subset \mathbb{P}_N^{2g-2}$  is contracted to a point  $e_0 \in \mathbb{P}^{2g-3}$  s.t.  $w = i_\Omega(e_0)$ . Recall, once again from Section 4.3 that  $\mathbb{P}^{2g-3}$  also contains the  $2g - 1$  points  $\tau_k \circ \text{Cr}_k(H_i)$ , images of the hyperplanes  $H_i \subset \mathbb{P}_N^{2g-2}$ , with  $i \neq k$  and  $1 \leq i \leq 2g$ . Let us label them  $e_1, \dots, e_{2g-1}$ . Let now  $\Lambda$  be the partial linear system of  $\Omega$  consisting of the  $(g - 1)$ -forms in  $\mathbb{P}^{2g-3}$  vanishing with multiplicity  $g - 2$  in all the points  $e_0, e_1, \dots, e_{2g-1}$ . As it is explained in Theorem 4.7, the rational map induced by the linear subsystem  $\Lambda$  factors through  $\mathcal{M}_{0,2g}^{\text{GIT}}$ , and the second map  $\kappa : \mathcal{M}_{0,2g}^{\text{GIT}} \dashrightarrow \Lambda^*$  is the osculating projection from the image of  $e_0$  inside  $\mathcal{M}_{0,2g}^{\text{GIT}}$ .

**THEOREM 5.4.** *The map  $\pi_N$  coincides with the map  $\kappa$ . In particular, the map  $\pi_N$  is of degree 2.*

*Proof.* Consider the GIT quotient  $\mathcal{M}_{0,2g}^{\text{GIT}}$  embedded in  $|\Omega|^*$  as we have seen in Theorem 4.5. The osculating projection  $\pi_N$  is given by the linear system  $|H - (g - 2)w|$  of hyperplanes vanishing in  $w$  with multiplicity  $g - 2$ . By definition of  $\Omega$ , this linear system pulls back via  $i_\Omega$  to the linear system of  $(g - 1)$ -forms in  $\mathbb{P}^{2g-3}$  vanishing with multiplicity  $g - 2$  in  $e_1, \dots, e_{2g-1}$ , and also with multiplicity  $g - 2$  in  $e_0$ , which is precisely  $\Lambda$ . Hence, the map  $\pi_N$  is the map induced by the same linear system as  $\kappa$  (see Theorem 4.7).  $\square$

We will show in the next Section that the map  $l_N$  from Diagram 6 is actually birational, and that the map  $\pi_N$  (birationally) coincides with the restriction of the map  $\theta$ .

### 5.3 The hyperelliptic theta map and rational involutions on $\mathcal{M}_{0,2g}^{\text{GIT}}$ and $SU_C(2)$

The resolution  $\widetilde{\varphi}_D$  of  $\varphi_D$  factors through the degree 2 map  $\theta$  as shown in Diagram 5. In the preceding section, we have shown that, when we restrict  $\varphi_{D,N}$  to  $\mathbb{P}_N^{2g-2}$ , it factors through the degree 2 map  $\pi_N$ . Now, we link these two factorizations. The identification of maps in the following claim must be intended as rational maps, since for example  $\pi_N$  is not everywhere defined.

**THEOREM 5.5.** *Let  $N \in |2D|$  be a general effective divisor. Then, the restricted map  $\theta|_{f_D(\mathbb{P}_N^{2g-2})}$  is the map  $\pi_N$  up to composition with a birational map.*

*Proof.* Let us place ourselves on the open set  $\widehat{SU}_C(2) \subset SU_C(2)$  of general stable bundles. First we remark that the factorization  $\widetilde{\varphi}_D = \theta \circ \widetilde{f}_D$  of Diagram 5 is the Stein factorization of the map  $\widetilde{\varphi}_D$  along  $\widetilde{\mathbb{P}}_D^{3g-2}$ . Indeed, the map  $\theta$  is of degree 2 as explained in Section 1. Moreover, the preimage of a general stable bundle  $E$  by the map  $f_D$  is the  $\mathbb{P}^1$  arising as the projectivization of the space of extensions of the form

$$0 \rightarrow \mathcal{O}(-D) \rightarrow E \rightarrow \mathcal{O}(D) \rightarrow 0.$$

In particular, the fibers of  $\widetilde{f}_D$  over  $\widehat{SU}_C(2)$  are connected.

The restriction of  $\varphi_D$  to  $\mathbb{P}_N^{2g-2}$  factors through the maps  $h_N$  and  $\pi_N$  (see Diagram 6), followed by the map  $l_N$ . According to Proposition 4.9, the fibers of  $h_N$  are rational normal curves, thus connected. Moreover, the map  $\pi_N$  is degree 2 by Theorem 5.4. By unicity of the Stein factorization, we have our result.

Comparing with the factorization  $\widetilde{\varphi}_D = \theta \circ \widetilde{f}_D$ , we see that  $l_N$  cannot have relative dimension  $> 0$ . Hence,  $l_N$  is a finite map. Since the degree of the map  $\theta$  in the Stein factorization is 2, which is equal to the degree of  $\pi_N$ , we have that  $l_N$  cannot have degree  $> 1$ . In particular, we have that the map  $l_N$  is a birational map.  $\square$

We are now in the position to show the following.

**PROPOSITION 5.6.**

1. *There exists a fibration  $p_D : SU_C(2) \dashrightarrow |2D| \cong \mathbb{P}^g$  whose general fiber is birational to  $\mathcal{M}_{0,2g}^{\text{GIT}}$ .*

2. For every generic divisor  $N \in |2D|$ , the restriction of  $f_D$  to the  $2g$ -pointed projective space  $\mathbb{P}_N^{2g-2} \subset \mathbb{P}_D^{3g-2}$  dominates  $p_D^{-1}(N) \stackrel{\text{birat}}{\cong} \mathcal{M}_{0,2g}^{\text{GIT}}$ . The generic fiber of  $f_D|_{\mathbb{P}_N^{2g-2}}$  is a rational normal curve passing through the  $2g$  marked points.
3. The family of rational normal curves contracted by  $f_D|_{\mathbb{P}_N^{2g-2}}$  is the universal family of pointed rational curves over (an open subset of) the generic fiber  $p_D^{-1}(N) \stackrel{\text{birat}}{\cong} \mathcal{M}_{0,2g}^{\text{GIT}}$ .

*Proof.* The fibration  $p_D$  is defined by the composed map  $p_D := p_{\mathbb{P}_c} \circ \theta : \mathcal{S}U_C(2) \dashrightarrow |2D|$ . The birationality  $p_D^{-1}(N) \stackrel{\text{birat}}{\cong} \mathcal{M}_{0,2g}^{\text{GIT}}$  for a general  $N \in |2D|$  is given by the classifying map  $f_D$ . In fact, if we fix  $N \in |2D|$ , then by Proposition 3.2 we know that the (closure of the) image of the restricted map  $\varphi_{D,N}$  is the fiber of  $p_{\mathbb{P}_c}$ . Thanks to the discussion in Section 5.2.3 (see in particular Diagram 6 and Theorem 5.4) we know that birationally  $\varphi_{D,N}$  factors via the map  $h_N$  that contracts rational normal curves through the  $2g$  points of  $N$  (see Proposition 4.9), and the degree 2 osculating projection  $\pi_N$ . Finally, from Theorem 5.5 and its proof we see that birationally  $h_N$  is equal to the restriction of  $f_D$  to  $\mathbb{P}_N^{2g-2}$ , the rational normal curves contracted by  $h_N$  are the fibers of the classifying map  $f_D$ , and  $\pi_N$  is the restriction of  $\theta$  to  $f_D(\mathbb{P}_N^{2g-2})$ . This implies claims (1) and (2). Claim (3) is a consequence of the properties of  $h_N$  described in Proposition 4.9 and the fact that over an open set  $f_D|_{\mathbb{P}_N^{2g-2}}$  coincides with  $h_N$ . Remark that the existence of this universal family also induces, in a more intrinsic way, the birational map  $p_D^{-1}(N) \stackrel{\text{birat}}{\cong} \mathcal{M}_{0,2g}^{\text{GIT}}$  by means of the universal property of the categorical quotient  $\mathcal{M}_{0,2g}^{\text{GIT}}$ .  $\square$

REMARK. With a little effort, it is also possible to reconstruct a semistable, rank 2 vector bundle on  $C$  with trivial determinant starting from a general configuration of points in  $\mathcal{M}_{0,2g}^{\text{GIT}} \stackrel{\text{birat}}{\cong} p_D^{-1}(N)$ , thus producing an inverse to the birational map  $p_D^{-1}(N) \dashrightarrow \mathcal{M}_{0,2g}^{\text{GIT}}$  induced by  $f_D$ , but we will refrain to develop the details.

The results proved and collected in the preceding sections put us now in the position to claim the following theorem.

**THEOREM 5.7.** *The restriction of  $\theta$  to the general fiber of the fibration  $p_D : \mathcal{S}U_C(2) \dashrightarrow |2D| \cong \mathbb{P}^g$  has a ramification locus birational to the Kummer variety of dimension  $g - 1$ , obtained from the Jacobian of the hyperelliptic curve that is the double cover of  $\mathbb{P}^1$  ramified along the  $2g$  points represented by  $w = h_N(\Gamma)$ .*

*Proof.* The proof of this theorem is a collection of the results that we have developed so far. We will refer to Figure 2 as a guiding line. As we have observed (see Proposition 5.6 (2)—top right corner of Figure 2), the generic fiber of the projection  $p_D$  is birational to  $\mathcal{M}_{0,2g}^{\text{GIT}}$ , and on an open subset of the fiber the map  $\theta$  coincides with the degree 2 map  $\pi_N$  (Theorem 5.5—vertical map on the RHS of the figure). In turn, the map  $\pi_N$  coincides with the osculating projection  $\kappa$  (Theorem 5.4) defined on  $\mathcal{M}_{0,2g}^{\text{GIT}}$ . Hence, by Kumar’s beautiful description (see [21] and Theorem 4.7—bottom right corner of the figure) of the Kummer variety as the ramification locus of  $\kappa$ , we have the claim. Let us also underline that we are also tacitly showing that the map  $p_D$  factors through the map  $\theta$  (this is the full RHS of the diagram in Figure 2).  $\square$

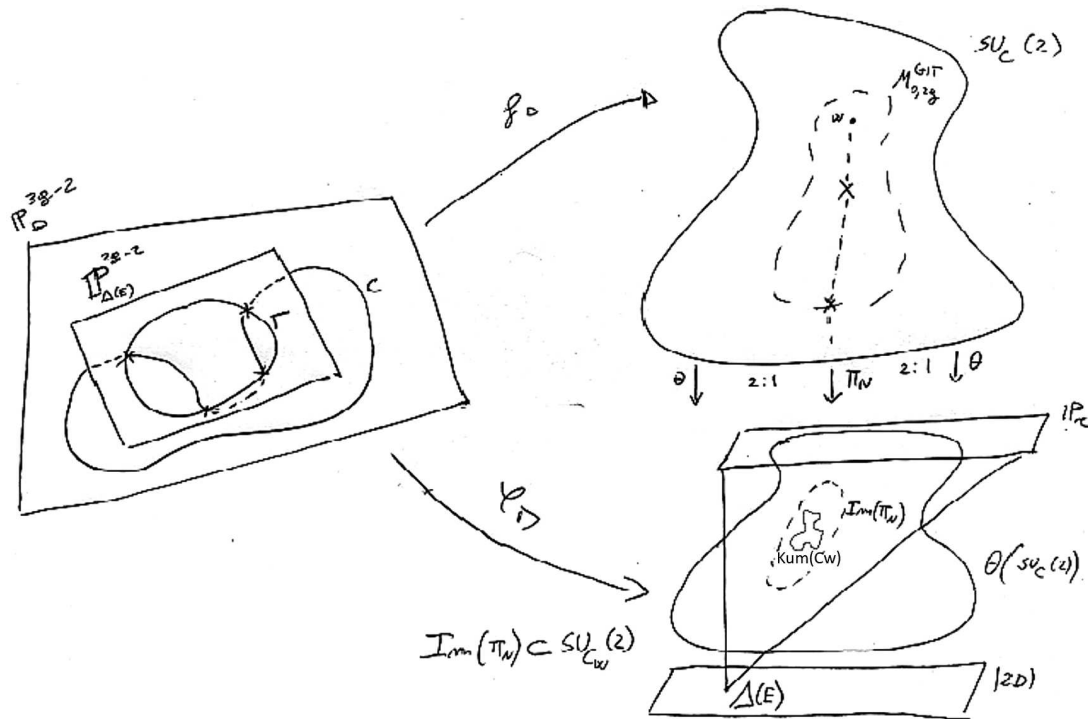


Figure 2. An intuitive view of our construction.

COROLLARY 5.8. One of the irreducible components of the ramification locus of the theta map is birational to a fibration in Kummer  $(g - 1)$ -folds over  $\mathbb{P}^g$ .

Results from [31, App. E] imply that the ramification locus is in fact nonirreducible.

### §6. The case $g = 3$

Let us now illustrate the geometric situation by explaining in detail the first case in low genus. Let  $C$  be a hyperelliptic curve of genus 3. In this setting, we have that the map  $\theta$  factors through the involution  $i^*$ , and embeds the quotient  $SU_C(2)/\langle i^* \rangle$  in  $\mathbb{P}^7 = |2\Theta|$  as a quadric hypersurface (see [11]). Let  $D$  be a general effective divisor of degree 3. The projective space  $\mathbb{P}_D^7$ , as defined in Section 1, parametrizes the extension classes in  $\text{Ext}^1(\mathcal{O}(D), \mathcal{O}(-D))$ . The classifying map  $\varphi_D$  is given in this case by the complete linear system  $|\mathcal{I}_C^2(3)|$  of cubics vanishing on  $C$  with multiplicity 2. The forms from this linear system vanish along the secant lines of  $C$ , and in particular along the secant lines passing through involution-conjugate points. These form a pencil parametrized by the linear system  $|h|$ .

The image of the projection of  $\theta(SU_C(2))$  with center  $\mathbb{P}_c = \mathbb{P}^3 \subset |2\Theta|$  is also a  $\mathbb{P}^3$ , that is identified with  $|2D|$  by Theorem 3.1. Let  $N \in |2D|$  be a general reduced divisor. By Proposition 3.2, the closure of the fiber  $p_{\mathbb{P}_c}^{-1}(N)$  is the image via  $\varphi_D$  of the  $\mathbb{P}_N^4$  spanned by the six points of  $N$ .

### 6.1 The restriction to $\mathbb{P}_N^4$

The base locus of the restricted map  $\varphi_{D,N} = \varphi_D|_{\mathbb{P}_N^4}$  contains  $\text{Sec}^N = \text{Sec}^1(C) \cap \mathbb{P}_N^4$  by Lemma 4.1. The secant variety  $\text{Sec}^1(N) \subset \text{Sec}^N$  is the union of the 15 lines passing through pairs of the 6 points of  $N$ . According to Lemma 4.10, the further base locus  $\text{Sec}^N \setminus \text{Sec}^1(N)$  is given by the intersections of  $\mathbb{P}_N^4$  with the lines spanned by degree 2 divisors  $L_2$  on  $C$  not contained in  $N$  satisfying  $\dim|L_2| \geq 1$ . By Brill–Noether theory, there is only one linear system of such divisors on a genus 3 curve, namely the hyperelliptic linear system  $|h|$  (see, e.g., [2], Chapter V). We will review these ideas in Section 7. This linear system traces the curve  $\Gamma$  that we introduced in Section 5 as the set of intersection points of  $\mathbb{P}_N^{2g-2}$  with the lines spanned by the divisors in the hyperelliptic pencil. Hence, we have that  $\text{Sec}^N = \{15 \text{ lines}\} \cup \Gamma$ , and the restricted map  $\varphi_{D,N}$  factors as

$$\begin{array}{ccc}
 \mathbb{P}_N^4 & \xrightarrow{h_N} & \mathcal{M}_{0,6}^{\text{GIT}} \subset \mathbb{P}^4 \\
 \searrow \varphi_{D,N} & & \downarrow \pi_N \\
 & & \mathbb{P}^3
 \end{array}$$

where  $h_N$  is the map defined by the complete linear system  $|\mathcal{I}_{\text{Sec}^1(N)}(3)|$  of cubics vanishing along the 15 lines defined by the points of  $N$ , and  $\pi_N$  is the projection with center the image via  $h_N$  of the rational normal curve  $\Gamma$ .

The image of  $\varphi_{D,N}$  is a  $\mathbb{P}^3$ . Indeed, this image cannot have higher dimension, since the map factors through the projection from a point of  $\mathcal{M}_{0,6}^{\text{GIT}} \subset \mathbb{P}^4$ . Also, it cannot have dimension strictly smaller than 3 since otherwise the relative dimension of  $\varphi_{D,N}$  would be bigger than 1, or equivalently the global map  $\varphi_D$  would not surject onto  $SU_C(2)$ . Hence, in this case the map  $\varphi_{D,N}$  is defined exactly by the system of cubics in  $\mathbb{P}_N^4$  vanishing on  $\text{Sec}^N$ .

According to Proposition 4.9, the image of  $h_N$  is the GIT moduli space  $\mathcal{M}_{0,6}^{\text{GIT}}$  if  $N$  is general and reduced. It is a classical result that this GIT quotient is embedded in  $\mathbb{P}^4$  as the Segre cubic  $S_3$  (see for instance [12] or [14]). This three-fold arises by considering the linear system of quadrics in  $\mathbb{P}^3$  that pass through five points in general position, thus it is isomorphic to the blow-up of  $\mathbb{P}^3$  at these points, followed by the blow-down of all lines joining any two points. The composition of this map with the projection off a smooth point of  $S_3$  gives a  $2 : 1$  rational map  $\mathbb{P}^3 \dashrightarrow \mathbb{P}^3$  whose ramification locus is a Weddle surface [21, 5]. The curve  $\Gamma \subset \mathbb{P}_N^4$  is a rational normal curve by Lemma 5.1, hence  $\Gamma$  is contracted to a point  $w$  by  $h_N$  again by Proposition 4.9.

By [4] and Lemma 4.1, the linear system  $|\mathcal{O}_{S_3}(1)|$  of hyperplanes in  $S_3$  is pulled back by  $h_N$  to  $|\mathcal{I}_{\text{Sec}^1(N)}(3)|$  on  $\mathbb{P}_N^4$ . The linear system  $|\mathcal{O}_{S_3}(1) - w|$  of hyperplanes in  $S_3$  passing through  $w$  is pulled back to the complete linear system  $|\mathcal{I}_{\text{Sec}^N}(3)|$  defining  $\varphi_{D,N}$ . Hence, the map  $\pi_N$  is the linear projection with center  $w$ . Since  $S_3$  is a cubic, the projection  $\pi_N$  is a degree 2 map. We will see in the next Section that this will be also the case for higher genus. The following proposition resumes what we have seen so far in this section.

**PROPOSITION 6.1.** *Let  $C$  be a hyperelliptic curve of genus 3. Then, for generic  $N$ , the restriction of  $\varphi_D$  to the subspace  $\mathbb{P}_N^{2g-2}$  is exactly the composition  $\pi_N \circ h_N$ .*

The point  $w$  in  $\mathcal{M}_{0,6}^{\text{GIT}}$  represents a rational curve with six marked points. Let  $C'$  be the hyperelliptic genus 2 curve constructed as the double cover of this rational curve ramified in these six points. According to Theorem 4.2 of [21], the Kummer variety  $\text{Kum}(C')$  is

contained in the image of  $\pi_N$ , and it is precisely its ramification locus. Recall that, when  $g = 3$ , the linear system  $|2D|$  is a  $\mathbb{P}^3$ . By Proposition 3.2, the image of  $\mathbb{P}_N^4$  by  $\varphi_D$  is the closure of the fiber  $p_{\mathbb{P}_c}^{-1}(N)$ . For each point  $N$  in  $|2D|$ , this image is  $\mathbb{P}^3 = |\mathcal{I}_{\text{Sec}^N}^2(3)|^*$ , which is the image of the Segre variety  $\mathcal{M}_{0,6}^{\text{GIT}}$  under the projection with center  $P$ . Thus, the image of the global map  $\varphi_D$  is birational to a  $\mathbb{P}^3$ -bundle over  $|2D| = \mathbb{P}^3$ . Of course this is also the case since the image of the theta map is a quadric hypersurface in  $\mathbb{P}^7$  [11].

**§7. Explicit descriptions in low genera**

In this section, we will go through an explicit description of the classifying maps and how they factor through forgetful linear systems and osculating projections, for low values of the genus  $g(C)$  of the hyperelliptic curve. In these cases the map remains fairly simple. These computations seem completely out of reach without the help of a computer for higher genus.

Recall from Section 4 that the intersection  $\text{Sec}^N = \text{Sec}^{g-2}(C) \cap \mathbb{P}_N^{2g-2}$  arises naturally as part of the base locus of the restricted map  $\varphi_{D,N}$ . The subvarieties  $\text{Sec}^{g-2}(N)$  and  $\Gamma$  of  $\text{Sec}^N$  yield the factorization of  $\varphi_{D,N}$  through the maps  $h_N$  and  $\pi_N$  of Proposition 5.3. Let us now describe the set

$$\text{Sec}^{N'} := \text{Sec}^N \setminus \{\Gamma \cup \text{Sec}^{g-2}(N)\}.$$

This set is empty for  $g = 3$ , and the map  $\varphi_{D,N}$  is exactly the composition of  $h_N$  and  $\pi_N$ , as described in Section 6. In higher genus, the existence of nonempty additional base locus  $\text{Sec}^N$  corresponds to the fact that the map  $\varphi_{D,N}$  may not be exactly the composition of the maps  $h_N$  and  $\pi_N$ . In other words, the map  $l_N$  from Diagram 6 may not be nontrivial in higher genus.

This supplementary base locus is given by the intersections of  $\mathbb{P}_N^{2g-2}$  with  $(g - 2)$ -dimensional  $(g - 1)$ -secant planes of  $C$  in  $\mathbb{P}_D^{3g-2}$ , which are not already supported on  $\text{Sec}^{g-2}(N)$  and  $\Gamma$ . According to Lemma 4.10, these intersections are given by effective divisors  $L_{g-1}$  on  $C$  of degree  $g - 1$ , not contained in  $\mathbb{P}_N^{2g-2}$ , and satisfying  $\dim |L_{g-1}| \geq 1$ . Again by Lemma 4.10, we obtain  $\dim(\langle L_{g-1} \rangle \cap \mathbb{P}_N^{2g-2}) = \dim |L_{g-1}| - 1$ .

We will now give account of the situation in low genera.

**Case  $g = 4$**

In this case, the divisor  $N$  is of degree 8 and the map  $\varphi_{D,N} : \mathbb{P}_N^6 \dashrightarrow |2\Theta| = \mathbb{P}^{15}$  is given by the restriction of the linear system  $|\mathcal{I}_C^3(4)|$  to  $\mathbb{P}_N^6$ . This map factors through the map  $\pi_N$  which coincides with the one-osculating projection  $\Pi_w^1$ , where  $w = h_N(\Gamma)$ .

We are looking for degree 3 divisors  $L_3$  with  $\dim |L_3| \geq 1$ . These satisfy all  $\dim |L_3| = 1$  and are of the form

$$L_3 = h + q \quad \text{for } q \in C,$$

where  $h$  is the hyperelliptic divisor. Let  $p$  be a point of  $C$ . Then  $L_3 = p + i(p) + q$ . Since  $\dim |L_3| = 1$ , the secant plane  $\mathbb{P}_{L_3}^2$  in  $\mathbb{P}_D^{10}$  spanned by  $p, i(p)$  and  $q$  intersects  $\mathbb{P}_N^6$  in a point. But this point necessarily lies on  $\Gamma$ , since the line passing through  $p$  and  $i(p)$  is already contained in this plane. Hence, we do not obtain any additional locus.

**Case  $g = 5$** 

In this case, the divisors  $L_4$  of degree 4 are all of the form

$$L_4 = h + q + r \quad \text{for } q, r \in C,$$

and satisfy  $\dim |L_4| = 1$ . Thus, the corresponding secant  $\mathbb{P}_{L_4}^3$  spanned by  $p, i(p), q$  and  $r$  intersects  $\mathbb{P}_N^8$  in a point. As before, this point lies on  $\Gamma$ , thus we do not obtain any additional locus. The upshot is the following

**PROPOSITION 7.1.** *Let  $C$  be a hyperelliptic curve of genus 4 or 5, then  $\varphi_D$  is defined by a (possibly equal) linear subsystem of the linear system defining  $\pi_N \circ h_N$ , and set-theoretically the base locus of  $\pi_N \circ h_N$  coincides with  $\text{Sec}^{g-2}(C) \cap \mathbb{P}_N^{2g-2}$ .*

**Case  $g = 6$** 

Here we have, as in the genus 5 case, the divisors of the form

$$L_3 = h + q \quad \text{for } q \in C,$$

which do not give rise to any additional base locus. But there is a new family of divisors

$$L_5 = 2h + r \quad \text{for } r \in C.$$

These divisors satisfy  $\dim |L_5| = 2$ . In particular, the intersection of the  $\mathbb{P}_{L_5}^4$  spanned by  $p, i(p), q, i(q)$ , and  $r$ , for  $p, q \in C$ , with  $\mathbb{P}_N^{10}$  is a line  $m$  in  $\mathbb{P}_N^{10}$ . The line  $l_1$  (resp.  $l_2$ ) spanned by  $p$  and  $i(p)$  (resp.  $q, i(q)$ ) intersects  $\Gamma$  in a point  $\tilde{p}$  (resp.  $\tilde{q}$ ). In particular, the line  $m$  is secant to  $\Gamma$  and passes through  $\tilde{p}$  and  $\tilde{q}$ . Since every point of  $\Gamma$  comes as an intersection of a secant line in  $C$  with  $\mathbb{P}_N^{10}$ , we obtain the following description of the base locus of  $\varphi_{D,N}$ :

**PROPOSITION 7.2.** *Let  $C$  be a curve of genus  $g = 6$ . Then, the base locus of the restricted map  $\varphi_{D,N}$  contains the ruled three-fold  $\text{Sec}^1(\Gamma)$ .*

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