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THE TRANSCENDENTAL MOTIVE OF A CUBIC FOURFOLD

MICHELE BOLOGNESI AND CLAUDIO PEDRINI

Abstract. In this note we introduce the transcendental part $t(X)$ of the motive of a cubic fourfold X and prove that it is isomorphic to the (twisted) transcendental part $h_2^{tr}(F(X))$ in a suitable Chow-Künneth decomposition for the motive of the Fano variety of lines $F(X)$. Then we prove that $t(X)$ is isomorphic to the *Prym motive* associated to the surface $S_l \subset F(X)$ of lines meeting a general line l. If X is a special cubic fourfold in the sense of Hodge theory, and $F(X) \simeq S^{[2]}$, with S a K3, then we show that $t(X) \simeq t_2(S)(1)$, where $t_2(S)$ is the transcendental motive. Therefore the motive $h(X)$ is finite dimensional if and only if S has a finite dimensional motive. If X is very general then $t(X)$ cannot be isomorphic to the (twisted) transcendental motive of a surface. We relate the existence of an isomorphism $t(X) \simeq t_2(S)(1)$ to conjectures by Hassett and Kuznetsov on the rationality of a special cubic fourfold. Finally we consider the case of cubic fourfolds X admitting a fibration over $\mathbf{P}^2$, whose fibers are either quadrics or del Pezzo surfaces of degree 6, and prove the isomorphism $t_2(S)(1) \simeq t(X)$, with S a K3 surface.

1. Introduction

We will work over the complex field. Cubic fourfolds are among the most mysterious objects in algebraic geometry. Despite the simplicity of the definition of such classically flavoured objects, the birational geometry of cubic fourfolds is extremely hard to understand and many modern techniques (Hodge theory, derived categories, etc. - see *e.g.* [Kuz, Has 2, AT] for details) have been successfully deployed in order to have a deeper understanding. In any case, the rationality of the generic cubic fourfold is still an open problem. Also the finite dimensionality of the motive $h(X)$ of a cubic fourfold, as conjectured by several authors (see [Ki] , [An]), is known to hold only in some scattered cases.

We will denote by $\mathcal{M}_{rat}(\mathbf{C})$ the (covariant) category of Chow motives (with $\mathbf{Q}$-coefficients), whose objects are of the form (X, p, n), where X is a smooth projective variety over $\mathbf{C}$ of dimension d, p is an idempotent in the ring $A^d(X \times X) = CH^d(X \times X) \otimes \mathbf{Q}$ and $n \in \mathbf{Z}$. If X and Y are smooth projective varieties over $\mathbf{C}$, then the morphisms $\operatorname{Hom}_{\mathcal{M}_{rat}}(h(X), h(Y))$ of their motives $h(X)$ and $h(Y)$ are given by correspondences in the Chow groups $A^*(X \times Y) = CH^*(X \times Y) \otimes \mathbf{Q}$. More precisely, in our covariant setting, we have

$$\operatorname{Hom}_{\mathcal{M}_{rat}}(X, p, m), (Y, q, n)) = q \circ A_{d+m-n}(X \times Y) \circ p \subset A_{d+m-n}(X \times Y)$$

where $d = \dim X$ and $\circ$ means composition of correspondences. The category $\mathcal{M}_{rat}(\mathbf{C})$ is additive,pseudo-abelian, rigid and has a tensor structure (see [KMP]). The unit motive is $\mathbf{1} = (\operatorname{Spec}(\mathbf{C}, 1, 0)$: it is a unit for the tensor structure. The *Lefschetz motive* $\mathbb{L}$ is defined via the motive of the projective line: $h(\mathbf{P}^1) = \mathbf{1} \oplus \mathbb{L}$ and there is an isomorphism $\mathbb{L} \simeq (\operatorname{Spec}(k), 1, 1)$ For every motive $M = (X, p, m)$

the Tate twist $M(r)$ is the motive $(X, p, m + r)$. Note that, with our covariant convention, $M(r) \simeq M \otimes \mathbb{L}^{\otimes r}$ for $r \geq 0$.
The Chow groups of a motive $(X, p, m) \in \mathcal{M}_{rat}(\mathbf{C})$ are defined as follows

$$A^i(X, p, m) = \mathrm{Hom}_{\mathcal{M}_{rat}}((X, p, m), \mathbb{L}^i) = p^* A^{i-m}(X)$$

$$A_i(X, p, m) = \mathrm{Hom}_{\mathcal{M}_{rat}}((\mathbb{L}^i, (X, p, m)) = p_* A_{i-m}(X).$$

A similar definition holds for the category $\mathcal{M}_{hom}(\mathbf{C})$ of homological motives, with respect to singular cohomology $H^*(X)$, where

$$H^i(X, p, m) = p^* H^{i-2m}(X) \ ; \ H_i(X, p, m) = p_* H_{i-2m}(X).$$

Let X be a smooth projective variety over $\mathbf{C}$. We say that its motive $h(X) \in \mathcal{M}_{rat}(\mathbf{C})$ has a *Chow-Künneth decomposition* (C-K for short) if there exist orthogonal projectors $\pi_i = \pi_i(X) \in \mathrm{Corr}_0(X, X) = A^d(X \times X)$, for $0 \leq i \leq 2d$, such that $cl^d(\pi_i)$ is the $(i, 2d - i)$-component of Δ_X in $H^{2d}(X \times X)$ and

$$[\Delta_X] = \sum_{0 \leq i \leq 2d} \pi_i.$$

This implies that in $\mathcal{M}_{\mathrm{rat}}$ the motive $h(X)$ decomposes as follows:

$$h(X) = \bigoplus_{0 \leq i \leq 2d} h_i(X),$$

where $h_i(X) = (X, \pi_i, 0)$. Moreover

$$H^*(h_i(X)) = H^i(X), \qquad H_*(h_i(X)) = H_i(X)$$

If we have $\pi_i = \pi^t_{2d-i}$ for all i, we say that the C-K decomposition is *self-dual.*

By the results in [KMP, 7.2.3] every smooth projective surface S has a *reduced C-K decomposition* $h(S) = \sum_{0 \leq 4} h_i(S)$ with

$$h_2(S) = h_2^{alg}(S) \oplus t_2(S) = (S, \pi_2^{alg}) \oplus (S, \pi_2^{tr}).$$

Here $h_2^{alg}(S) \simeq \mathbb{L}^{\rho(S)}$, where $\rho(S)$ is the rank of the Neron-Severi group and

$$H^2(S) = H^2_{alg}(S) \oplus H^2_{tr}(S) = \pi_2^{alg} H^2(S) \oplus \pi_2^{tr} H^2(S)$$

The motive $t_2(S)$ is called the *transcendental motive* of S. It is a birational invariant and

$$H^*(t_2(S)) = H^2(t_2(S) = T(S)_{\mathbf{Q}} \ ; \ A^2(t_2(S) \simeq K(S),$$

where $T(S)$ is the transcendental lattice and $K(S)$ is the Albanese kernel, i.e the kernel of the map $A_0(S)_{hom} \to \mathrm{Alb}(S)$.

We recall the definition of finite dimensionality introduced by S.Kimura in [Ki]. Let $M \in \mathcal{M}_{\mathrm{rat}}(\mathbf{C})$ and let Σ_n be the symmetric group of order n. Then $\wedge^n M$ is the image of $M(X^n)$ under the projector

$$(1/n!)(\sum_{\sigma \in \Sigma_n} sgn(\sigma)\Gamma_\sigma$$

while $S^n M$is its image under the projector

$$(1/n!)(\sum_{\sigma \in \Sigma_n} \Gamma_\sigma.$$

A motive M is said to be *evenly (oddly) finite-dimensional* if $\wedge^n M = 0$ ($S^n X = 0$) for some n. A motive M is finite-dimensional if it can be decomposed into a direct sum $M_+ \oplus M_-$ where M_+ is evenly finite-dimensional and M_- is oddly finite-dimensional. According to Kimura's conjecture in [Ki] all motives should be finite dimensional. The conjecture is known to hold for curves, rational surfaces, surfaces with $p_g(X) = 0$, which are not of general type, abelian varieties and some 3-folds. If $d = \dim X \le 3$, then the finite dimensionality of $h(X)$ is a birational invariant (see [GG, Lemma 7.1]), the reason being that in order to make regular a birational map $X \to Y$ between smooth projective 3-folds one needs to blow up only points and curves, whose motives are finite dimensional.

If X is a complex Fano threefold, then $h(X)$ is finite dimensional and of abelian type, i.e. it lies in the subcategory of $\mathcal{M}_{rat}(\mathbf{C})$ generated by the motives of abelian varieties, see [GG, Thm. 5.1]. The proof is based on the fact that all the Chow groups $A_i(X)_{alg}$ of algebraically trivial cycles are representable. More generally, if $M \in \mathcal{M}_{rat}(\mathbf{C})$ is a motive such that $A_i(M)_{alg}$ is representable, for all $i \ge 0$, then M is finite dimensional of abelian type, see [Vial 2].
In particular, if X is a cubic threefold in $\mathbf{P}^4_{\mathbf{C}}$, then $h(X)$ has the following Chow-Künneth decomposition

$$h(X) = \mathbf{1} \oplus \mathbb{L} \oplus N \oplus \mathbb{L}^2 \oplus \mathbb{L}^3.$$

Here $N = h_1(J) \otimes \mathbb{L} = h_1(J)(1)$, with J an abelian variety, isogenous to the intermediate Jacobian $J^2(X)$. Let l be a general line on X . Blowing up l we get a conic bundle $\tilde{X} \to \mathbf{P}^2$ whose discriminant curve C_l is degree 5. Let $\pi : \tilde{C}_l \to C_l$ be the double cover parametrizing irreducible components of singular conics. Then we have

$$J^2(X) \simeq \operatorname{Prym}(\tilde{C}_l/C_l).$$

where $\operatorname{Prym}(\tilde{C}_l/C_l)$ is the Prym variety of $\tilde{C}$ over C, i.e.the identity component of the fixed locus of the involution $\tau = -\sigma$ on the Jacobian variety $\operatorname{Jac}(\tilde{C})$. Here σ is the involution on $\tilde{C}$ induced by the double cover. The *Prym motive* associated to an étale double cover of curves $\tilde{C} \to C$ is the motive $(\tilde{C}, \pi)$, where π is the correspondence $\pi = (id - \sigma)/2 \in A^1(\tilde{C} \times \tilde{C})$, see [NS]. Then

$$\Delta_{\tilde{C}} = \pi + (\Delta_{\tilde{C}} + \sigma)/2 \in A^1(\tilde{C} \times \tilde{C})$$

If $h(\tilde{C}) = \mathbf{1} \oplus h_1(\tilde{C}) \oplus \mathbb{L}$ is a C-K decomposition there is an isomorphism

$$\phi_{\tilde{C}} : A^1(\tilde{C} \times \tilde{C})/\mathcal{I}(\tilde{C}) \simeq \operatorname{End}_{\mathcal{M}_{rat}}(h_1(\tilde{C})) \simeq \operatorname{End}_{Ab}(\operatorname{Jac}\tilde{C}) \otimes \mathbf{Q}.$$

Here $\mathcal{I}(\tilde{C})$ is the ideal of degenerate correspondence, that is generated by $[\tilde{C} \times P]$ and $[P \times \tilde{C}]$, with P a closed point and $\operatorname{End}_{Ab}(\operatorname{Jac}\tilde{C})$ is the group of endomorphisms as an Abelian variety, see [KMP, 7.4.4]. Therefore, under the map $\phi_{\tilde{C}}$, the Prym motive $\operatorname{Prym}(\tilde{C}/C)$ corresponds to the submotive $h_1(\tilde{C})^-$ of $h_1(\tilde{C})$ where the involution σ acts as -1.
As proved by Clemens and Griffiths, X is not rational, because $J^2(X)$ is not the Jacobian variety of a curve D and hence the Prym motive is not isomorphic to the mid-motive $h_1(D)(1)$.

Let X be a cubic fourfold in $\mathbf{P}^5_{\mathbf{C}}$. In Section 2, we show that the motive $h(X)$ has a reduced Chow-Künneth decomposition as follows

$$h(X) = \mathbf{1} \oplus \mathbb{L} \oplus (\mathbb{L}^2)^{\oplus \rho_2} \oplus t(X) \oplus \mathbb{L}^3 \oplus \mathbb{L}^4$$

where ρ_2 is the rank of $A^2(X)$ and all the summands of $h(X)$, but possibly $t(X)$, are finite dimensional, see (2.1). The motive $t(X)$ is the *transcendental motive* of X and

$$A_1(X)_{hom} = A_1(X)_{alg} = A_1(t(X)) \tag{1.1}$$

If l is a general line on X then the surface $S_l \subset F(X)$ of lines meeting l is smooth, with $q(S_l) = 0$, and geometric genus $p_g(S_l) = 5$ (see [Vois 1, Sect. 3]). Let $\pi : \tilde{X} \to \mathbf{P}^3$ be the conic bundle obtained by blowing up X along the line l. The surface S_l parametrizes irreducible components of the conics fibers of π. There is an involution σ on S_l with 16 isolated fixed points and the quotient $Y_l = S_l/\sigma$ is a quintic surface in $\mathbf{P}^3$. The involution σ induces a double cover $S_l \to Y_l$. Similarly to the case of the Prym motive associated to a conic bundle one can define the Prym motive

$$\mathrm{Pr}(S_l, \sigma) := t_2(S_l)^-,$$

where $t_2(S_l)^-$ is the direct summand of the transcendental motive $t_2(S_l)$ where σ acts as -1. In Prop. 2.7 we prove that

$$t_2(S_l)^-(1) \simeq t(X)$$

Therefore, similarly to the case of a cubic 3-fold, it is natural to ask if there is a surface Z such that the Prym motive $\mathrm{Pr}(S_l, \sigma)$ is isomorphic to the (twisted) transcendental motive of Z and hence $t(X) \simeq t_2(Z)(1)$. If X is very general and therefore conjecturally not rational, then this cannot happen, see Prop. 3.6 (i). On the other hand if X is special, then, assuming the Hodge conjecture and Kimura's conjecture, there exists a K3 surface S such that $t_2(S)(1) \simeq t(X)$, see Remark 3.7

In Sect. 2 we relate the transcendental motive of X with the motive of its Fano variety of lines $F(X)$ (F for short).

Theorem 1.2. *Let $h(F)$ be the motive of $F(X)$, endowed with a Chow-Künneth decomposition, and let $h_2(F) \cong h_2^{alg} \oplus h_2^{tr}$ be the standard decomposition of $h_2(F)$. Then we have an isomorphism*

$$h_2^{tr}(F)(1) \cong h_4^{tr}(X) = t(X).$$

and therefore

$$\mathrm{Mot}(X) = \mathrm{Mot}(F),$$

where, for a smooth projective variety Y we denote by $\mathrm{Mot}(Y)$ the full pseudo-abelian tensor subcategory of $\mathcal{M}_{rat}(\mathbf{C})$ generated by $h(Y)$ and the Lefschetz motive $\mathbb{L}$.

In some cases, we can say even more (see Sect. 3).

Theorem 1.3. *Suppose $F(X) \cong S^{[2]}$, with S a K3 surface. Then there is an isomorphism of motives*

$$t_2(S)(1) \cong t(X).$$

and hence

$$\mathrm{Mot}(X) = \mathrm{Mot}(F) = \mathrm{Mot}(S)$$

In particular X has a finite dimensional motive if and only if the motive of S is finite dimensional, in which case the transcendental motives of X and S are both indecomposable. Note that, according to Kimura's conjecture and a conjecture by Y. André (see [An]), the motives of a cubic fourfold and of a K3 surface should be of abelian type.

Added in proof: Some time after a first version of this paper appeared, T-H Bülles in [Bull] has given a different proof of Thm.1.2 and Thm. 1.3. His results also show that the isomorphism in Thm. 1.3 holds even when $F(X)$ is just birational to $S^{[2]}$, see [Bull, Prop. 1.4].

Let $\mathcal{C}_d$ be the Noether-Lefschetz divisor of special cubic fourfold of discriminant d, as defined by B.Hasset in [Has 1]. If d satisfies the condition

(**) d is not divisible by 4,9 or a prime $p \equiv 2(3)$

and X is a general member of $\mathcal{C}_d$ then, according to a conjecture of Kuznetsov [Kuz] and results of Addington-Thomas [AT], X should be rational. In Sect. 4 we prove that, assuming Kimura's conjecture, if $X \in \mathcal{C}_d$, with d satisfying (**), then there is an isomorphism $t_2(S)(1) \simeq t(X)$, where S is a K3 surface.

In Section 5, by adapting some results by Vial [Vial 1] about motives of fibrations with rational fibers, we showcase classes of cubic fourfolds with a K3 surface S such that $t(X) \simeq t_2(S)(1)$. More precisely we consider special cubic fourfolds that are genera members either of $\mathcal{C}_8$ or of $\mathcal{C}_{18}$. In the first case the fibers are quadrics, in the second case del Pezzo surfaces of degree 6.

In Sect.6 we consider the case of a cubic fourfold X, with an involution, and prove (see Prop. 6.5) that there is an isomorphism $t(X) \simeq t_2(S)(1)$, where $S \subset F(X)$ is a K3 surface.

2. The motive of a cubic fourfold

In this section we give a Chow-Künneth decomposition of the motive $h(X)$ of a cubic fourfold and show that its transcendental part $h_4^{tr}(X) = t(X)$ is isomorphic to the (twisted) transcendental motive $h_2^{tr}(F(X))(1)$ coming from a suitable Chow-Künneth decomposition of the motive of the Fano variety of lines $F(X)$ (see Thm. 2.5). Note that, by a result of R.Laterveer [Lat 1], if $h(X)$ is finite dimensional then also $h(F(X))$ is finite dimensional. Then we show that

$$A_1(X)_{hom} = A_1(X)_{alg} \simeq A_1(t(X)).$$

Every cubic fourfold X is rationally connected and hence $CH_0(X) \simeq \mathbf{Z}$. Rational, algebraic and homological equivalences all coincide for cycles of codimension 2 on X. Hence the cycle map $CH^2(X) \to H^4(X, \mathbf{Z})$ is injective and $A^2(X) = CH^2(X) \otimes \mathbf{Q}$ is a vector subspace of dimension $\rho_2(X)$ of $H^4(X, \mathbf{Q})$. By the results in [TZ] we have $A_1(X)_{hom} = A_1(X)_{alg}$. Moreover homological equivalence and numerical equivalence coincide for algebraic cycles on X, because the standard conjecture $D(X)$ holds true. Therefore $A_1(X)_{hom} = A_1(X)_{num}$.

A cubic fourfold X has no odd cohomology and $H^2(X, \mathbf{Q}) \simeq NS(X)_{\mathbf{Q}} \simeq A^1(X)$, because $H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0$. Let $\gamma \in A^1(X)$ be the class of a hyperplane section. Then $H^2(X, \mathbf{Q}) = A^1(X) \simeq \mathbf{Q}\gamma$ and $H^6(X, \mathbf{Q}) = \mathbf{Q}[\gamma^3/3]$. Here $< \gamma^2, \gamma^2 >= \gamma^4 = 3$, where $< \, , \, >$ is the intersection form on $H^4(X, \mathbf{Q})$.

Let $\pi_0=[X\times P_0], \pi_8=[P_0\times X]$, where P_0 is a closed point and $\pi_2=(1/3)(\gamma^3\times\gamma)$, $\pi_6=\pi_2^t=(1/3)(\gamma\times\gamma^3)$. Then

$$h(X)\simeq \mathbf{1}\oplus h_2(X)\oplus h_4(X)\oplus h_6(X)\oplus \mathbb{L}^4$$

where $\mathbf{1}\simeq (X,\pi_0)$, $\mathbb{L}^4\simeq (X,\pi_8)$, $h_2(X)=(X,\pi_2)$, $h_6(X)=(X,\pi_6)$ and $h_4(X)=(X,\pi_4)$, with $\pi_4=\Delta_X-\pi_0-\pi_2-\pi_6-\pi_8$. The above decomposition of the motive $h(X)$ is in fact integral, because

$$\gamma^3=3|l|$$

for a line $l\in F(X)$, see [SV 2, Lemma A3].

Let ρ_2 be the dimension of $A^2(X)$, i.e. the rank of the algebraic part in $H^4(X)$. Choosing 2-cycles $\{D_1,D_2\dots,D_{\rho_2}\}$ and their Poincaré dual cycles $\{D'_1,D'_2\dots,D'_{\rho_2}\}$ we get a splitting

$$h_4(X)=h_4^{alg}(X)\oplus h_4^{tr}(X)$$

where $\pi_4^{alg}=\sum_{1\le i\le\rho_2}[D_i,D'_i]$ and $h_4^{alg}(X)\simeq(\mathbb{L}^2)^{\rho_2}$. Therefore we get a *refined Chow-Künneth decomposition* of the motive $h(X)$

$$h(X)=\mathbf{1}\oplus\mathbb{L}\oplus(\mathbb{L}^2)^{\oplus\rho_2}\oplus t(X)\oplus\mathbb{L}^3\oplus\mathbb{L}^4. \tag{2.1}$$

Here $t(X)=h_4^{tr}(X)=(X,\pi_4^{tr})$ with $\pi_4^{tr}=\Delta_X-\pi_0-\pi_2-\pi_4^{alg}-\pi_6-\pi_8$. Then $H^*(t(X))=H^4(t(X))=T(X)_{\mathbf{Q}}$ where $T(X)$ is the transcendental lattice. All the motives in (2.1), different from $t(X)$, are isomorphic to a multiple of $\mathbb{L}^i$, for some i. Therefore in the decomposition (2.1) all motives, but possibly $t(X)$, are finite dimensional. It follows that the motive $h(X)$ is finite dimensional if and only if $t(X)$ is evenly finite dimensional.

Lemma 2.2. *Let X be a cubic fourfold and let $t(X)$ be the transcendental motive in the Chow-Künneth decomposition (2.1). Then $A^i(t(X))=0$ for $i\neq 3$ and $A^3(t(X))=A_1(X)_{hom}$*

Proof. The cubic fourfold X is rationally connected and hence $A^4(X)=A_0(X)=\mathbf{Q}$ that implies $A^4(t(X))=0$. Also from the Chow-Künneth decomposition in (2.1) we get $A^0(t(X))=0$ and $A^1(t(X))=\pi_4^{tr}A^1(X)=0$, because $A^1(h_2(X))=\pi_2(A^1(X))=A^1(X)$. Here and in the following we will denote by $\pi_iA^j(X)$the action of a correspondence π_i on the Chow groups.

We first show that $A^2(t(X))=0$. Let $\alpha\in A^2(X)$, with $\alpha\neq 0$. Then α is not homologically trivial, because $A^2(X)_{hom}=0$.

$$\pi_4^{tr}(\alpha)=\alpha-\pi_0(\alpha)-\pi_2(\alpha)-\pi_4^{alg}(\alpha)-\pi_6(\alpha)-\pi_8(\alpha),$$

where $\pi_0(\alpha)=\pi_8(\alpha)=0$. We also have

$$\pi_2(\alpha)=(1/3)[\gamma^3\times\gamma]_*(\alpha)=(1/3)(p_2)_*((\alpha\times X)\cdot[\gamma^3\times\gamma])$$

where $p_2:X\times X\to X$ and $\gamma^3\in A^3(X)$. Therefore $\pi_2(\alpha)=0$ in $A^2(X)$. Similarly

$$\pi_6(\alpha)=(1/3)[\gamma\times\gamma^3]_*(\alpha)=(1/3)(p_2)_*((\alpha\times X)\cdot[\gamma\times\gamma^3])$$

where $\alpha\cdot\gamma\in A_1(X)/A_1(X)_{hom}\simeq\mathbf{Q}[\gamma^3/3]$ and hence $\alpha\cdot\gamma=(a/3)[\gamma^3]$ with $a\in\mathbf{Q}$. Therefore $\pi_6(\alpha)=0$ in $A^2(X)$. Let $\{D_1,\dots,D_{\rho_2}\}$ be a $\mathbf{Q}$-basis for $A^2(X)$ and let $\alpha=\sum_{1\le i\le\rho_2}m_iD_i$, with $m_i\in\mathbf{Q}$. Then $\pi_4^{alg}(\alpha)=\sum_{1\le i\le\rho_2}\pi_{4,i}(\alpha)=\alpha$, because $(\pi_{4,i})_*(D_i)=D_i$. We get $\pi_4^{tr}(\alpha)=\alpha-\pi_4^{alg}(\alpha)=0$ and hence

$$A^2(t(X))=(\pi_4^{tr})_*A^2(X)=0.$$

Therefore we are left to show that $A_1(t(X)) = A_1(X)_{hom}$. Let $\beta \in A_1(X) = A^3(X)$. From the Chow-Künneth decomposition in (2.1) we get

$$\pi_4^{tr}(\beta) = \beta - \pi_0(\beta) - \pi_2(\beta) - \pi_4^{alg}(\beta) - \pi_6(\beta) - \pi_8(\beta),$$

where $\pi_0(\beta) = \pi_4^{alg}(\beta) = \pi_8(\beta) = 0$. We also have $\pi_2(\beta) = 0$ because $\pi_2 = (1/3)(\gamma^3 \times \gamma)$. Therefore

$$\pi_4^{tr}(\beta) = \beta - \pi_6(\beta) = \beta - (1/3)(\gamma \times \gamma^3)_*(\beta) = \beta - (1/3)(\beta \cdot \gamma)\gamma^3 \in A^3(X)$$

and hence

$$(\pi_4^{tr}(\beta) \cdot \gamma) = (\beta \cdot \gamma) - (1/3)(\beta \cdot \gamma)(\gamma^3 \cdot \gamma) = 0$$

because $\gamma^4 = 3$. Since γ is a generator of $A^1(X)$ it follows that the cycle $\pi_4^{tr}(\beta)$ is numerically trivial. Therefore we get

$$A_1(t(X)) = \pi_4^{tr} A_1(X) = A_1(X)_{num} = A_1(X)_{hom}.$$

□

The following Lemma follows from the results in [Vial 1, Thm. 3.18] and [GG, Lemma 1].

Lemma 2.3. *Let $f : M \to N$ be a morphism of motives in $\mathcal{M}_{rat}(\mathbf{C})$ such that $f_* : A^i(M) \to A^i(N)$ is an isomorphism for all $i \geq 0$. Then f is an isomorphism.*

Proof. Let $M = (X, p, m)$ and $N = (Y, q, n)$ and let $k \subset \mathbf{C}$ be a field of definition of f, which is finitely generated . Then $\Omega = \mathbf{C}$ is a universal domain over k. By [Vial 1, Thm. 3.18] the map f has a right inverse, because the map $f_* : A^i(M) \to A^i(N)$ is surjective. Let $g : N \to M$ be such that $f \circ g = id_N$. Then g has an image T which is a direct factor of M and hence f induces an isomorphism of motives in $\mathcal{M}_{rat}(\mathbf{C})$

$$f : M \simeq N \oplus T$$

From the isomorphism $A^i(M) \simeq A^i(N)$, for all $i \geq 0$, we get $A^i(T) = 0$ and hence $T = 0$, by [GG, Lemma 1]. □

Let X be a cubic fourfold and let $F(X) = F$ be its Fano variety of lines , which is a smooth fourfold. Let

$$\begin{array}{ccc} P & \xrightarrow{q} & X \\ {\scriptstyle p}\downarrow & & \\ F & & \end{array} \tag{2.4}$$

be the incidence diagram, where $P \subset X \times F$ is the universal line over X. Let $p_*q^* : H^4(X, \mathbf{Z}) \to H^2(F, \mathbf{Z})$ be the Abel-Jacobi map. Let $\alpha_1, \ldots, \alpha_{23}$ be a basis of $H^4(X, \mathbf{Z})$ and let $\tilde{\alpha}_i = p_*q^*(\alpha_i)$. Then, by a result of Beauville-Donagi in [BD], $\tilde{\alpha}_i = p_*q^*(\alpha_i)$ form a basis of $H^2(F, \mathbf{Z})$. The lattice $H^2(F, \mathbf{Z})$ is endowed with the Beauville-Bogomolov bilinear form q_F, see [SV 2, Sect. 19]. The Abel-Jacobi map induces an isomorphism between the primitive cohomology of $H^4(X, \mathbf{Z})_{prim}$ and the primitive cohomology $H^2(F, \mathbf{Z})_{prim}$. Here $H^2(F, \mathbf{Z})_{prim} =< g >^{\perp}$, with $g \in H^2(F, \mathbf{Z})$ the restriction to $F(X) \subset \mathrm{Gr}(2, 6)$ of the class on $\mathrm{Gr}(2, 6)$ defining the Plücker embedding. In particular $g = p_*q^*(\gamma^2)$. The Abel-Jacobi map induces an isomorphism between the Hodge structure of $H^4(X, \mathbf{C})_{prim}$ and the (shifted) Hodge structure of $H^2(F, \mathbf{C})_{prim}$.

The next result shows that the Abel-Jacobi map induces an isomorphism between $t(X)$ and the transcendental motive $h_2^{tr}(F)$ in a suitable Chow-Künneth decomposition for $h(F)$.

Theorem 2.5. *Let X be cubic fourfold and let $F(X)$ be its Fano variety of lines. Then there exists a Chow-Künneth decomposition*

$$h(F) = h_0(F) \oplus h_2(F) \oplus h_4(F) \oplus h_6(F) \oplus h_8(F)$$

with $h_2(F) \simeq h_2^{alg}(F) \oplus h_2^{tr}(F)$. The Abel-Jacobi map gives an isomorphism

$$h_2^{tr}(F)(1) \simeq h_4^{tr}(X) = t(X).$$

Proof. The hyperkähler manifold $F(X)$ is of $K3^2$-type, *i.e.* it is deformation equivalent to the Hilbert scheme of length-2 subschemes on a K3 surface. By the results in [SV 2, Sect. 19], there exists a cycle $L \in \mathrm{CH}^2(F \times F)$ whose cohomology class in $H^4(F \times F, \mathbf{Q})$ is the Beauville-Bogomolov class $\mathcal{B}$, i.e. the class corresponding to q_F^{-1}. Let us set $l := (i_\Delta)^* L \in CH^2(F)$, where $i_\Delta : F \to F \times F$ is the diagonal embedding. By [SV 2, Thm. 2], the Chow groups of the variety F have a *Fourier decomposition*. In particular the group $A^4(F) = A_0(F)$ has a canonical decomposition

$$A^4(F) = A^4(F)_0 \oplus A^4(F)_2 \oplus A^4(F)_4$$

with $A^4(F)_0 =< l^2 >$, $A^4(F)_2 = l \cdot L_* A^4(F)$ and $A^4(F)_4 = L_* A^4(F) \cdot L_* A^4(F)$. Here $< l^2 >= \mathbf{Q} c_F$, with c_F a special degree 1 cycle coming from a surface $W \subset F$ such that any two points on W are rationally equivalent on F, see [SV 2, Lemma A.3].

The Fourier decomposition of the Chow groups $A^*(F)$ is compatible with a Chow-Künneth decomposition of the motive $h(F)$ given by projectors

$$\{\pi_0(F), \pi_2(F), \pi_4(F), \pi_6(F), \pi_8(F)\},$$

as in [SV 1, Thm. 8.4]. Here $\pi_2(F) = \pi_2^{alg} \oplus \pi_2^{tr}, \pi_6(F) = \pi_6^{alg} \oplus \pi_6^{tr}$ and

$$\pi_4 = \Delta_F - (\pi_0 - \pi_2 - \pi_6 - \pi_8)$$

We have $h(F) = M \oplus N$ where

$$N = (F, \pi_0) \oplus (F, \pi_2^{alg}) \oplus (F, \pi_4^{alg}) \oplus (F, \pi_6^{alg}) \oplus (F, \pi_8)$$

and N is isomorphic to a direct sum of $\mathbb{L}^i$, for $i \geq 0$. We also have $A^*(F)_{hom} = A^*(M)$. Let us set $h_2(F) = h_2^{alg}(F) \oplus h_2^{tr}(F)$, where $h_2^{tr}(F) = (F, \pi_2^{tr}(F))$, with $\pi_2^{tr}(F) \in \mathrm{End}_{\mathcal{M}_{rat}} M$ and $H^*(h_2^{tr}(F)) = H^2_{tr}(F)$. Then

$$A^2(F) = \mathrm{Im}(\pi_4)_* \oplus \mathrm{Im}(\pi_2)_* = \mathrm{Im}(\pi_4)_* \oplus \mathrm{Im}(\pi_2^{tr})_*$$

because $\pi_2^{alg}(F)$ acts as 0 on $A^2(F)$.
Let us denote $\mathcal{A} = I_* A^4(F) \subset A^2(F)$, with I the incidence correspondence, i.e. $I = (p \times p)_*(q \times q)^* \Delta_X$. The group $\mathcal{A}_{hom}$ is generated by the classes $[S_{l_1}] - [S_{l_2}]$ where, for a line l on X, S_l denotes the surface in $F(X)$ of all lines meeting l, see [SV 2, Thm. 21.9]. By [SV 2, 21.10] the group $\mathcal{A}_{hom}$ coincides with the subgroup $A^2(F)_2$ in the Fourier decomposition $A^2(F) = A^2(F)_0 \oplus A^2(F)_2$.
The Abel-Jacobi map $q_* p^* : A^i(F) \to A^{i-1}(X)$ induces a surjective map $\Psi_0 : A^4(F) \to A^3(X) = A_1(X)$, where $A_1(X)$ is generated by the classes of lines, see [TZ]. The map induced by Ψ_0 on the subgroup $A^4(F)_{hom}$ has a kernel isomorphic

to $F^4A^4(F) = \mathcal{A}_{hom} \otimes \mathcal{A}_{hom}$, see [SV 2, Thm 20.2], where $F^4A^4(F) = \mathrm{Ker}\{I_* : A^4(F) \to A^2(F)\}$. The maps I_* and Ψ_0 yield two exact sequences

$$0 \longrightarrow F^4A^4(F) \longrightarrow A^4(F)_{hom} \xrightarrow{I_*} \mathcal{A}_{hom} \longrightarrow 0$$

$$0 \longrightarrow F^4A^4(F) \longrightarrow A^4(F)_{hom} \xrightarrow{\Psi_0} A_1(X)_{hom} \longrightarrow 0$$

where $(A^4(F))_{hom} = (A^4(F)_2)_{hom} \oplus (A^4(F)_4)_{hom}$, with $(A^4(F)_2)_{hom} \simeq \mathcal{A}_{hom}$ and $(A^4(F)_4)_{hom} \simeq \mathcal{A}_{hom} \cdot \mathcal{A}_{hom}$. Therefore we get the following isomorphisms

$$\mathcal{A}_{hom} \simeq A^4(F)_2 \simeq A_1(X)_{hom};$$
$$\mathcal{A}_{hom} \simeq A^2(F)_2 \simeq A_1(X)_{hom}.$$

By [SV 1, Proposition 7.7] we also have

$$A^2(F)_{hom} \simeq \mathcal{A}_{hom} \Longleftrightarrow \mathrm{Im}(\pi_2)_* = A^2(F)_{hom}.$$

Therefore $A^2(F)_{hom} = \mathrm{Im}(\pi_2^{tr})$ and we get an isomorphism

$$A^2(h_2^{tr}(F)) \simeq A_1(X)_{hom}.$$

The universal line P, viewed as a correspondence in $A_5(F \times X)$, yields a map in $\mathcal{M}_{rat}(\mathbf{C})$

$$P_* : h(F)(1) \to h(X).$$

By the results in [SV 2] the relation between the Chow groups of F and X is given via P. Therefore, by composing with the projection $h(X) \to t(X)$ and the inclusion $h_2^{tr}(F)(1) \subset h(F)(1)$, the correspondence P yields a map of motives

$$\bar{P}_* : h_2^{tr}(F)(1) \to t(X)$$

The above map induces a map of Chow groups

$$A^i(h_2^{tr}(F)(1)) \to A^i(t(X))$$

that is an isomorphism for all$i \geq 0$ because

$$A^3(h_2^{tr}(F)(1)) = A^2(h_2^{tr}(F)) \simeq A_1(X)_{hom} = A^3(t(X))$$

and $A^i(h_2^{tr}(F)) = A^i(t(X) = 0$ for $i \neq 3$. By Lemma 2.3 we get $h_2^{tr}(F)(1) \simeq t(X)$. $\square$

Remark 2.6. If the motive $h(F(X))$ is finite dimensional then, by Theorem 2.5, also $t(X)$ is finite dimensional and hence $h(X)$ is finite dimensional. Conversely if $h(X)$ is finite dimensional then, by [Lat 1], also $h(F(X))$ is finite dimensional .

Let X be a cubic fourfold and let $l \in F(X)$ be a general line. There exists a unique plane $P_l \subset \mathbf{P}^5$ containing l and which is everywhere tangent to X along l. Then

$$P_l \cdot X = 2[l] + [l_0]$$

Let $\phi : F \dashrightarrow F$ be the rational map defined by C.Voisin in [Vois 2]. The map ϕ sends a general line $l \subset X$ to its residual line with respect to the unique plane $\mathbf{P}^2 \subset \mathbf{P}^5$ tangent to X along l. Let $S_l \subset F$ be the surface of lines meeting a general line l. Then S_l is a smooth surface with $q(S_l) = 0$ and $p_g(S_l) = 5$ (see [Vois 1, Lemma 1 and 3]) and there is a natural involution $\sigma : S_l \to S_l$. If $[l'] \in S_l$ is a point different from $[l]$ then $\sigma([l'])$ is the residue line of $l \cup l'$, while $\sigma([l]) = [l_0]$. The involution σ has 16 isolated fixed points and the quotient $Y_l = S_l/\sigma$ is a quintic surface in $\mathbf{P}^3$ with 16 ordinary double points, see [Shen, Remark 4.4]. Let $\tilde{X}$ be the blow-up of X along l. Then the projection from the line l defines a conic bundle $\pi : \tilde{X} \to \mathbf{P}^3$. The surface S_l parametrizes lines in the singular fibers, the discriminant divisor $D \subset \mathbf{P}^3$ is the quintic surface Y_l and the induced map $S_l \to D$ is the double cover $f_l : S_l \to Y_l$ associated to the involution σ. The map $f_l : S_l \to Y_l$ induces a commutative diagram

$$\begin{array}{ccc} \tilde{S}_l & \xrightarrow{g_l} & S_l \\ {\scriptstyle \tilde{f}_l}\downarrow & & {\scriptstyle f_l}\downarrow \\ \tilde{Y}_l & \longrightarrow & Y_l \end{array}$$

where $\tilde{S}_l$ is the blow-up of the set of isolated fixed points of σ and $\tilde{Y}_l$ is a desingularization of Y_l. For a general l the quintic surface Y_l is normal with rational singularities, hence it has $p_g(Y_l) = 4$ and $q(Y_l) = 0$, see [Yang]. Since $t_2(-)$ is a birational invariant for smooth projective surfaces the above diagram yields a map

$$\theta : t_2(\tilde{S}_l) = t_2(S_l) \to t_2(\tilde{Y}_l)$$

which is a projection onto a direct summand. Since $q(S_l) = 0$ the motive $t_2(S_l)$ splits as follows, see [Ped, Prop. 1]

$$t_2(S_l) \simeq t_2(S_l)^+ \oplus t_2(S_l)^-$$

where $t_2(S_l)^+ = t_2(\tilde{Y}_l)$ and $t_2(S_l)^-$ are the direct summand of $t_2(S_l)$ on which the involution σ acts as $+1$ and -1 respectively . We also have $t_2(\tilde{Y}_l) \neq 0$, because $p_g(Y_l) \neq 0$ and

$$A^2(t_2(\tilde{Y}_l)) = A^2(t_2(S_l))^+ = A_0(S_l)_0^+ \ ; \qquad A^2(t_2(S_l))^- = A_0(S_l)_0^-;$$

where $A_0(S_l)_0 = A_0(S_l)_0^+ \oplus A_0(S_l)_0^-$. Here $A_0(S_l)_0$ is the group of 0-cycles of degree 0 (with $\mathbf{Q}$-coefficients) and $A_0(S_l)_0^+$ is the subgroup fixed by σ.

Let $\mathcal{C}_l$ be the total space of lines meeting l and let

$$\begin{array}{ccc} \mathcal{C}_l & \xrightarrow{q_l} & X \\ {\scriptstyle p_l}\downarrow & & \\ S_l \subset F & & \end{array}$$

be the incidence diagram. Then the Abel-Jacobi map Φ and the cylinder homomorphism Ψ induce

$$\Phi_l : A_i(X) \to A_{i-1}(S_l) \ ; \ \Psi_l : A_i(S_l) \to A_{i+1}(X).$$

The following result shows the relation between the transcendental motive $t(X)$ and the transcendental motive of the surface S_l.

Proposition 2.7. *Let X be a cubic fourfold and let S_l be the surface of lines meeting a general line $l \subset X$. Then*

(i) $t(X) \simeq t_2(S_l)(1)^-$

(ii) Φ_l induces an isomorphism: $H^4_{tr}(X, \mathbf{Q}) \simeq H^2(S_l, \mathbf{Q})^-$, with $H^2(S_l, \mathbf{Q})^-$ the subgroup where σ acts as -1.

Proof. **(i)** By [Shen, Thm. 4.7] the composition $\Phi_l \circ \Psi_l$ equals $\sigma - id$ and $\Psi_l \circ \Phi_l = -2$. The Abel-Jacobi map Φ_l induces an isomorphism between $A_1(X)$ and $\Pr(A_0(S_l)_0, \sigma)$, where $\Pr(A_0(S_l)_0, \sigma) = A_0(S_l)_0^- = A^3(t_2(S_l)^-(1))$ [Shen, Def. 3.6]. Therefore the map

$$\Phi_l : A_1(X) \simeq \mathbf{Q}[\gamma^3/3] \oplus A_1(X)_{hom} \to A_0(S_l)_0$$

yields an isomorphism between $A_1(X)_{hom}$ and $A_0(S_l)_0^-$.
In the incidence diagram $\mathcal{C}_l$ is a $\mathbf{P}^1$-bundle over S_l and hence $h(\mathcal{C}_l) \simeq h(S_l) \oplus h(S_l)(1)$. Since $q(S_l) = 0$ we the motive $hS_)$ has a C-K decomposition $h(S_l) = \mathbf{1} \oplus \mathbb{L}^{\otimes \rho} \oplus t_2(S_l) \oplus \mathbb{L}^2$ and hence we get a map

$$g : t_2(S_l)(1) \to t(X),$$

where $t_2(S_l) \simeq t_2(S_l)^+ \oplus t_2(S_l)^-$. The map g when restricted to $t_2(S_l)^-(1)$ gives

$$(g^-)_* : t_2(S_l)(1)^- \to t(X),$$

such that the induced map on Chow groups

$$(g)_* : A^i(t_2(S_l)^-(1)) \to A^i(t(X))$$

is an isomorphism for all $i \geq 0$, because $A^3(t_2(S_l)^-(1)) = A_0(S_l)_0^-$, $A^3(t(X)) = A_1(X)_{hom}$. while $A^i(t_2(S_l)^-(1)) = 0$ and $A^i(t(X)) = 0$ for $i \neq 3$ .Therefore g^- is an isomorphism in $\mathcal{M}_{rat}(\mathbf{C})$, see Lemma 2.3.

(ii) In [Shen, Cor. 4.8] it is proved that the Abel-Jacobi map $\Phi_l : H^4_{tr}(X, \mathbf{Z}) \to H^2(S_l, \mathbf{Z})^-$ is an isomorphism of lattices. □

Remark 2.8. (1) The isomorphism in (i) answers a question raised by M.Shen in a private communication. For a smooth projective surface S, with $q(S) = 0$ and $p_g(S) > 0$, equipped with an involution σ, we can define the *Prym motive* $\Pr(S, \sigma)$ to be the motive

$$\Pr(S, \sigma) = t_2(S)^-$$

where, as in [Ped, Prop 1], $t_2(S)^-$ is the direct summand of $t_2(S)$ where the involution σ acts as -1. The action of σ on $t_2(S)$ is defined via the homomorphism

$$\Psi_S : A^2(S \times S) \to \mathrm{End}_{\mathcal{M}_{rat}}(t_2(S))$$

which sends the correspondence $\Gamma_\sigma \in A^2(S \times S)$ to $\pi_2^{tr} \circ \Gamma_\sigma \circ \pi_2^{tr}$. Here $t_2(S) = (S, \pi_2^{tr})$ and hence the projector π_2^{tr} corresponds to the identity in $\mathrm{End}_{\mathcal{M}_{rat}}(t_2(S))$.

(2) If $l \in X$ is a general line then the blow-up $\tilde{X}$ of X along l is a conic bundle $\pi : \tilde{X} \to \mathbf{P}^3$ and $Y_l = S_l/\sigma$ is the discriminant divisor. The involution σ on S_l is given by the double cover $S_l \to Y_l$. Therefore (ii) may be viewed as a generalization of a result appearing in [NS] for a conic bundle $f : X \to \mathbf{P}^2$. In [NS] it is proved that the motive of $h(X)$ is determined by the Prym motive $\Pr(\tilde{C}/C)$, where the curve C is the discriminant of f and $\tilde{C} \to C$ is the usual double cover.

Remark 2.9. In the case $p_g(S) = 1$ (e.g. S a K3 surface) then $t_2(S)$ is indecomposable if $h(S)$ is finite dimensional (see [Vois 3, Cor 3.10]). Therefore the Prym motive of S is either 0 or it coincides with $t_2(S)$. If S is a K3 surface and σ is a symplectic involution then $t_2(S) \simeq t_2(S/\sigma)$ and hence σ acts as the identity on $t_2(S)$, i.e $\Pr(S, \sigma) = 0$. If σ is non-symplectic then the quotient surface S/σ is either an Enriques surface or a rational surface. In any case $t_2(S/\sigma) = 0$ and $\Psi_S(\Gamma_\sigma) = -id_{t_2(S)}$. Therefore $\Pr(S, \sigma) = t_2(S)$.

3. Special cubic fourfolds

In this section we prove (see Thm.3.2) that, if $F(X)$ is isomorphic to $S^{[2]}$, with S a K3 surface, then $t(X)$ is isomorphic to $t_2(S)(1)$. Therefore $h(X)$ is finite dimensional if and only if $h(S)$ is finite dimensional.
Recall that a cubic fourfold X is *special* if it contains a surface Z such that its cohomological class ζ in $H^4(X, \mathbf{Z})$ is not homologous to any multiple of γ^2. Therefore $\rho_2(X) > 1$. The discriminant d is defined as the discriminant of the intersection form $<,>_D$ on the sublattice D of $H^4(X, \mathbf{Z})$ generated by ζ and γ^2. B.Hassett in [Has 1] proved that special cubic fourfolds of discriminant d form an irreducible divisor $\mathcal{C}_d$ in the moduli space $\mathcal{C}$ of cubic fourfolds if and only if $d > 0$ and $d \equiv 0, 2(6)$.

Definition 3.1. Let X be a special cubic fourfold and let D be the sublattice of $H^4(X, \mathbf{Z})$ generated by ζ and γ^2. A polarized K3 surface S is *associated* to X if there is an isomorphism of lattices $K \simeq H^2(S, \mathbf{Z})_{prim}(-1)$, where $K = D^\perp$ and $H^2(S, \mathbf{Z})_{prim}$ denotes primitive cohomology with respect to a polarization $l \in H^2(S, \mathbf{Z})$.

If X is a generic special cubic fourfold with discriminant of the form $d = 2(n^2+n+1)$, where n is an integer ≥ 2, then the Fano variety of X is isomorphic to $S^{[2]}$, with S a K3 surface associated to X. Special cubic fourfolds of discriminant $d > 6$ have associated K3 surface S if and only if d is not divisible by 4 or 9 or any odd prime $p \equiv 2(3)$. In this case the transcendental lattice $T(X)$ is Hodge isometric to $T(S)(-1)$, see [Add].
In the case $d = 14$ the special surface is a smooth quartic rational normal scroll. By the results in [BD] and in [BRS] all the fourfolds X in $\mathcal{C}_{14}$ are rational (see also [ABBV] for details on the derived categories approach). Moreover if $X \in (\mathcal{C}_{14} - \mathcal{C}_8)$, then $F(X) \simeq S^{[2]}$, where S is the K3 surface of degree 14 and genus 8, parametrizing smooth quartic rational normal scrolls contained in X.

More generally, suppose that X is special and $F(X) \simeq S^{[2]}$, with S a K3 surface. Then the homomorphism $H^2(S, \mathbf{Q}) \to H^2(F, \mathbf{Q})$ induces an orthogonal direct sum decomposition with respect to the Beauville-Bogomolov form

$$H^2(F, \mathbf{Q}) \simeq H^2(S, \mathbf{Q}) \oplus \mathbf{Q}\delta,$$

with $q_F(\delta, \delta) = -2$ and q_F restricted to $H^2(S, \mathbf{Q})$ is the intersection form, see [SV 2, Rmk. 10.1]. Therefore

$$H^4_{tr}(X, \mathbf{Q}) \simeq H^2_{tr}(S, \mathbf{Q})$$

where $\dim H^4_{tr}(X, \mathbf{Q}) = 23 - \rho_2(X)$. Here $\rho_2(X) \geq 2$ and hence we get

$$\dim H^2_{tr}(F, \mathbf{Q}) = \dim H^2_{tr}(S, \mathbf{Q}) = 22 - \rho(S) = 23 - \rho_2(X) \leq 21,$$

where $\rho(S)$ is the rank of $NS(S)$.

Theorem 3.2. *Let X be a cubic fourfold and let $F = F(X)$ be the Fano variety of lines. Suppose that $F \simeq S^{[2]}$, with S a K3 surface. Let P be the correspondence in the incidence diagram (2.4). Then P_* induces a map of motives $\bar{q} : t_2(S)(1) \to t(X)$ in $\mathcal{M}_{rat}(\mathbf{C})$ which is an isomorphism.*

Proof. In (2.4) the universal line P as a correspondence in $A_5(F \times X)$ gives a map

$$P_* : h(F)(1) \to h(X)$$

By the results in [deC-M, Thm. 6.2.1] $h(S)$ is a direct summand of $h(S^{[2]}) = h(F)$. Therefore we get a map

$$h(S)(1) \longrightarrow h(F)(1) \xrightarrow{P_*} h(X)$$

Let

$$h(S) \simeq \mathbf{1} \oplus \mathbb{L}^{\oplus \rho(S)} \oplus t_2(S) \oplus \mathbb{L}^2$$

be a refined Chow-Künneth decomposition, as in [KMP, Sect. 7.2.2]. By composing with the inclusion $t_2(S)(1) \to h(S)(1)$ and the surjection $h(X) \to t(X)$ we get a map of motives in $\mathcal{M}_{\mathrm{rat}}(\mathbf{C})$,

$$P_* : t_2(S)(1) \to t(X)$$

For two distinct points $x, y \in S$ let us denote by $[x, y] \in F = S^{[2]}$ the point of F that corresponds to the subscheme $x \cup y \subset S$. If $x = y$ then $[x, x]$ denotes the element in $A^4(F)$ represented by any point corresponding to a non reduced subscheme of length 2 on S supported on x. With these notations the special degree 1 cycle $c_F \in A^4(F)$ (see [SV 2, Lemma A.3]), given by any point on a rational surface $W \subset F$, is represented by the point $[c_S, c_S] \in F$, where c_S is the Beauville-Voisin cycle in $A_0(S)$ such that $c_2(S) = 24c_S$. We also have (see [SV 2, Prop. 15.6])

$$(A^4(F)_2)_{hom} = < [c_S, x] - [c_S, y] > .$$

We claim that the map $\phi : A_0(S) \to A_0(S^{[2]}) = A^4(F)$ sending $[x]$ to $[c_S, x]$ is injective and hence

$$A_0(S)_0 \simeq (A^4(F)_2)_{hom}.$$

The variety $S^{[2]}$ is the blow-up of the symmetric product $S^{(2)}$ along the diagonal $\Delta \cong S$. Let $\tilde{S}$ be the inverse image of Δ in $S^{[2]}$. Then $\tilde{S}$ is the image of the closed embedding $s \to [c_S, s]$. By a result proved in [Ba, Thm. 2.1] the induced map of 0-cycles $A_0(\tilde{S}) \to A_0(S^{[2]})$ is injective. Therefore the map ϕ is injective.
From the isomorphism $A_0(S)_0 \simeq (A^4(F)_2)_{hom}$ we get

$$A^3(t_2(S)(1) = A^2(t_2(S)) = A_0(S)_0 \simeq A_1(X)_{hom} \simeq A^3(t(X))$$

Since $A^i(t_2(S)(1)) = A^i(t(X)) = 0$ for $i \neq 3$ the map $\bar{P} : t_2(S)(1) \to t(X)$ gives an isomorphism on all Chow groups .Therefore $t_2(S)(1) \simeq t(X)$ $\square$

Remark 3.3. Let X be a cubic fourfold such that there exist K3 surfaces S_1 and S_2 and isomorphisms $r_1 : F(X) \to S_1^{[2]}$ and $r_2 : F(X) \to S_2^{[2]}$ with $r_1^*\delta_1 \neq r_2^*\delta_2$, as in [Has 1, Def. 6.2.1], where $H^2(F, \mathbf{Q}) \simeq H^2(S_1, \mathbf{Q}) \oplus \mathbf{Q}\delta_1 \simeq H^2(S_2, \mathbf{Q}) \oplus \mathbf{Q}\delta_2$. Then, by Thm. 3.2, we get $t_2(S_1) \simeq t_2(S_2)$, and hence the motives $h(S_1)$ and $h(S_2)$ are isomorphic.

Corollary 3.4. *Let X be a cubic fourfold and let $F = F(X)$ be the Fano variety of lines. Suppose that $F \simeq S^{[2]}$, with S a K3 surface. Then $h(X)$ is finite dimensional if and only if $h(S)$ is finite dimensional in which case the motive $t(X)$ is indecomposable.*

Proof. If $h(X)$ is finite dimensional then also $t(X)$ is finite dimensional and hence, by Theorem 3.2, $t_2(S)$ is finite dimensional. Therefore $h(S)$ is finite dimensional. Conversely, if $h(S)$ is finite dimensional then also $t_2(S)$ and $t(X)$ are finite dimensional, by Theorem 3.2. From the Chow-Künneth decomposition in (2.1) we get that $h(X)$ is finite dimensional. If $h(S)$ is finite dimensional then the motive $t_2(S)$ is indecomposable, see [Vois 3, Cor 3.10], and hence also $t(X)$ is indecomposable. $\square$

Remark 3.5. If the motive $h(X)$ of a cubic fourfold is finite dimensional then the transcendental part $t(X)$ of $h(X)$ is, up to isomorphisms in $\mathcal{M}_{rat}(\mathbf{C})$, independent of the Chow-Künneth decomposition $h(X) = \sum_i h_i(X)$ in (2.1). If $h(X) = \sum_i \tilde{h}_i(X)$ is another Chow-Künneth decomposition, with $\tilde{h}_i(X) = (X, \tilde{\pi}_i)$, then, by [KMP, Thm. 7.6.9], there is an isomorphism $\tilde{h}_i(X) \simeq h_i(X)$ and $\tilde{\pi}_i = (1+Z) \circ \pi_i \circ (1+Z)^{-1}$, where $Z \in A^4(X \times X)_{hom}$ is a nilpotent correspondence. In particular

$$\tilde{\pi}_4 = (1+Z) \circ \pi_4 \circ (1+Z)^{-1} = (1+Z) \circ (\pi_4^{alg} + \pi_4^{tr}) \circ (1+Z)^{-1}$$

and hence $\tilde{h}_4(X)$ contains as a direct summand a submotive $\tilde{t}(X) = (X, (1+Z) \circ \pi_4^{tr} \circ (1+Z)^{-1})$ isomorphic to $t(X)$.
However, differently from the case of the transcendental motive $t_2(S)$ of a surface S, the motive $t(X)$ is not a birational invariant. In fact $t(X) \neq 0$ for a rational cubic fourfold X such that $F(X) \simeq S^{[2]}$, with S a K3 surface, while $\mathbf{P}^4_{\mathbf{C}}$ has no transcendental motive.

According to Cor. 3.4, if X is a special cubic fourfold with $F(X) \simeq S^{[2]}$, and $h(X)$ is finite dimensional, then $t(X)$ is indecomposable. The following proposition shows that, if X is not special and $h(X)$ is finite dimensional, then $t(X)$ is indecomposable.

Proposition 3.6. *Let X be a very general cubic fourfold, i.e. $\rho_2(X) = 1$. Then*
(i) The transcendental motive $t(X)$ is not isomorphic to $t_2(S)(1)$, for a smooth projective surface S;
(ii) If $h(X)$ is finite dimensional $t(X)$ is indecomposable.

Proof. (i) Suppose that there exists a smooth projective surface S such that $t(X) \simeq t_2(S)(1)$. Then

$$H^4(t(X)) = H^4_{tr}(X, \mathbf{Q}) \simeq H^4(t_2(S)(1)) = H^2_{tr}(S, \mathbf{Q}).$$

Since $h^{3,1}(X) = h^{1,3}(X) = 1$, we get $h^{2,0}(S) = h^{0,2}(S) = 1$ and therefore the surface S has geometric genus $p_g(S) = 1$. By the results in [Mo, Sect. 2] there exists a K3 surface $\tilde{S}$ such that $H^2_{tr}(S, \mathbf{Q}) \simeq H^2_{tr}(\tilde{S}, \mathbf{Q})$. Since the dimension of $H^2_{tr}(\tilde{S}, \mathbf{Q})$ is at most 21 we should also have $\dim H^4_{tr}(X, \mathbf{Q}) \leq 21$, while, for a very general cubic fourfold , $\dim H^{tr}(X, \mathbf{Q}) = 23 - 1 = 22$.
(ii) Let us define the primitive motive $h(X)_{prim} = (X, \pi_{prim}, 0)$ as in [Ki, Sect. 8.4], where

$$\pi_{prim} = \Delta_X - (1/3) \sum_{0 \leq i \leq 4} (\gamma^{4-i} \times \gamma^i).$$

and

$$H^*(h(X)_{prim}) = H^4(X, \mathbf{Q})_{prim}.$$

Since X is very general, $\rho_2(X) = 1$ and $A^2(X)$ is generated by the class γ^2. Therefore in the Chow-Künneth decomposition of $h(X)$ in (2.1) we have $h(X)_{prim} = h_4^{tr}(X) = t(X)$ and

$$h_4(X) = h_4^{alg}(X) + h_4^{tr}(X) \simeq \mathbb{L} \oplus h(X)_{prim}.$$

Let $\mathcal{M}_{hom}(\mathbf{C})$ be the category of homological motives and let $\widetilde{\mathcal{M}}_{hom}(\mathbf{C})$ be the subcategory generated by the motives of all smooth projective varieties V such that the Künneth components of the diagonal in $H^*(V \times V)$ are algebraic. The Hodge realization functor

$$H_{Hodge} : \mathcal{M}_{rat}(\mathbf{C}) \to HS_{\mathbf{Q}}$$

to the Tannakian category of $\mathbf{Q}$-Hodge structures induces a faithful functor $\widetilde{\mathcal{M}}_{hom}(\mathbf{C}) \to HS_{\mathbf{Q}}$. Let us denote $\bar{h}(X) := h^{hom}(X) \in \widetilde{\mathcal{M}}_{hom}(\mathbf{C})$. Since X is very general $\mathrm{End}_{HS}(H^4(X,\mathbf{Q})_{prim}) = \mathbf{Q}[id]$, see [Vois 2, Lemma 5.1]. Therefore $\mathrm{End}_{\mathcal{M}_{hom}}(\bar{h}(X)_{prim}) \simeq \mathbf{Q}[id]$ and hence

$$\mathrm{End}_{\mathcal{M}_{hom}}(\bar{h}^{tr}_4((X)) \simeq \mathrm{End}_{\mathcal{M}_{hom}}(\bar{h}(X)_{prim}) \simeq \mathbf{Q}[id]$$

If $h(X)$ is finite dimensional then the indecomposability of $\mathrm{End}_{\mathcal{M}_{hom}}(\bar{h}^{tr}_4(X))$ in $\mathcal{M}_{hom}(\mathbf{C})$ implies the indecomposability in $\mathcal{M}_{rat}(\mathbf{C})$. Therefore

$$\mathrm{End}_{\mathcal{M}_{rat}}(t(X)) \simeq \mathrm{End}_{\mathcal{M}_{rat}}(h(X)_{prim}) \simeq \mathbf{Q}[id]$$

and the transcendental motive of X is indecomposable.

□

Remark 3.7. Let X be a cubic fourfold and l a general line in X. By Prop. 2.7 there is an isomorphism of motives

$$\mathrm{Pr}(S_l/Y_l) \simeq t_2(S_l)^-(1) \simeq t(X)$$

Suppose that the Prym motive is isomorphic to the (twisted) transcendental motive of a smooth surface Z. Then, by Prop. 3.6 (i), X is special, i.e. $\dim A^2(X) \geq 2$. By the same argument as in the proof of Prop. 3.6 (i) the surface Z has geometric genus $p_g(Z) = 1$ and there exists a K3 surface S such that

$$H^2_{tr}(Z,\mathbf{Q}) \simeq H^2_{tr}(S,\mathbf{Q}).$$

By assuming that the Hodge conjecture holds for $Z \times S$, the above isomorphism is induced by a correspondence $\Gamma \in A^2(Z \times S)$, see [Mo]. Therefore Γ gives a map of transcendental motives $t_2(Z) \to t_2(S)$, that induces an isomorphism on the transcendental cohomology and hence, assuming Kimura's conjecture, is an isomorphism in $\mathcal{M}_{rat}(C)$. Then $t(X) \simeq t_2(S)(1)$.

4. Rationality conjectures

Let X be a cubic fourfold. It was conjectured in [Kuz] that X is rational if and only if there exists a semi-orthogonal decomposition of the derived category $\mathbf{D}^b(X)$ of bounded complexes of coherent sheaves

$$\mathbf{D}^b(X) = < \mathcal{A}_X, \mathcal{O}_X, \mathcal{O}_X(1), \mathcal{O}_X(2) >,$$

such that $\mathcal{A}_X$ is equivalent to the category $\mathbf{D}^b(S)$ where S is a K3 surface. If X has an associated K3 surface S , in the sense of Kuznetsov, then the motive $h(S)$ is uniquely determined, up to isomorphisms, by X. Let $\mathbf{D}^b(S_1)$ and $\mathbf{D}^b(S_2)$ be equivalent. It was conjectured by Orlov that this implies that the motives $h(S_1)$ and $h(S_2)$ are isomorphic. The conjecture has been proved in [DelP-P] in the case $h(S_1)$ (and hence also $h(S_2)$) is finite dimensional and recently extended by D.Huybrechts in [Huy 2] to all K3 surfaces over an algebraically closed field.
Let us denote by $\mathcal{C}$ the moduli space of smooth cubic fourfolds. As it is customary, we will denote by $\mathcal{C}_d \subset \mathcal{C}$ the irreducible divisors that parametrize special cubic

fourfolds with an intersection lattice whose determinant is d. Let X be a general cubic fourfold inside $\mathcal{C}_d$, where d satisfies the following condition:

(**) d is not divisible by 4,9 or a prime $p \equiv 2(3)$.

Hassett [Has 1] has shown that $X \in \mathcal{C}_d$ has an associated K3 surface, in the sense of Def. 3.1, if and only if satisfies (**). Then Addington and Thomas in [AT] proved that a general such X has an associated K3 surface in the sense of Kuznetsov. Therefore, for a general cubic fourfold, Kuznetsov conjecture is equivalent to the following conjecture, that has been certainly around for a while.

Conjecture 4.1. *A cubic fourfold $X \subset \mathbf{P}^5$ is rational if and only if it is contained in $\mathcal{C}_d$, with d satisifying (**).*

Proposition 4.2. *Let X be a cubic fourfold in $\mathcal{C}_d$, where d satisfies (**). Assuming Kimura's conjecture, there exists a K3 surface S and an isomorphism of motives $t_2(S)(1) \simeq t(X)$.*

Proof. By Theorem 1.2 in [AT] there exists a polarized K3 surface S of degree d and a correspondence $\Gamma \in A^3(S \times X)$ which induces an Hodge isometry between the (shifted) primitive cohomology of S and the lattice $< \gamma^2, Z >^{\perp}$ inside $H^4(X, \mathbf{Z})$. Here the class of Z is not homologous to γ^2. Let $PHS_{\mathbf{Q}}$ be the semisimple abelian category of polarized Hodge structures. Then Γ induces an isomorphism between the polarized Hodge structures $T(S)_{\mathbf{Q}}(1)$ and $T(X)_{\mathbf{Q}}$ in $PHS_{\mathbf{Q}}$, where $T(S)$ and $T(X)$ are the transcendental lattices of S and X respectively. Let $\mathcal{M}^B_{hom}(\mathbf{C})$ be the subcategory of $\mathcal{M}_{hom}(\mathbf{C})$ generated by the homological motives $h_{hom}(X)$ of smooth complex projective varieties X satisfying the standard conjecture $B(X)$. Since $B(X)$ implies the standard conjecture $D(X)$, for smooth varieties over $\mathbf{C}$, the category $\mathcal{M}^B_{hom}$ is contained in the category $\mathcal{M}_{num}(\mathbf{C})$ of numerical motives and hence it is semisimple. The Hodge realization functor

$$H_{Hodge} : \mathcal{M}_{rat}(\mathbf{C}) \to PHS_{\mathbf{Q}},$$

factors trough $\mathcal{M}^B_{hom}(\mathbf{C})$ and the induced functor $H_{Hodge} : \mathcal{M}^B_{hom}(\mathcal{C}) \to PHS_{\mathbf{Q}}$ is faithful and exact. Both the K3 surface S and the cubic fourfold X satisfy $B(X)$ and hence M_{hom} and N_{hom} belong to $\mathcal{M}^B_{hom}(\mathbf{C})$, where M_{hom} and N_{hom} are the images of $t_2(S)(1)$ and $t(X)$ in $\mathcal{M}_{hom}(\mathbf{C})$, respectively. Then M_{hom} and N_{hom} have isomorphic images in $PHS_{\mathbf{Q}}$ and hence the correspondence Γ induces an isomorphism between M_{hom} and N_{hom} in $\mathcal{M}^B_{hom}(\mathbf{C})$. By Kimura's conjecture on the finite dimensionality of motives the functor $F : \mathcal{M}_{rat}(\mathbf{C}) \to \mathcal{M}^B_{hom}(\mathbf{C})$ is conservative, i.e. it preserves isomorphisms, see [AK, Thm. 8.2.4]. Therefore the correspondence Γ gives an isomorphism between $t_2(S)(1)$ and $t(X)$ in $\mathcal{M}_{rat}(\mathbf{C})$. $\square$

Conjecture 4.1 and Prop. 4.2 clearly suggest the following

Conjecture 4.3. *If a cubic fourfold X is rational then there exist a K3 surface S and an isomorphism of transcendental motives $t(X) \simeq t_2(S)(1)$.*

5. Cubic fourfolds fibered over a plane

Let X be a rational cubic fourfold Then by considering the surfaces blown up in a birational map $\rho : \mathbf{P}^4 \dashrightarrow X$ one sees that there are smooth projective surfaces $S_1, \dots S_n$ such that the transcendental motive $t(X)$ is a direct summand of

$\sum_{1\le i\le n} t_2(S_i)(1)$, see [Has 2, Prop. 17]. Therefore the motive $h(X)$ is finite dimensional if all the surfaces S_i have finite dimensional motives. Y.Zarhin in [Za] gives a restriction for the types of surfaces that can appear when resolving the indeterminacy of ρ.

Let us check two examples where we have a K3 surface S, and $t(X) \simeq t_2(S)(1)$. This is the case for instance if X contains two planes. Then X is rational and S is a K3 surface, a complete intersection of hypersurfaces of bidegrees (1,2), that is the indeterminacy locus of a birational map $\rho : \mathbf{P}^2 \times \mathbf{P}^2 \dashrightarrow X$. The map ρ is defined by taking the unique line trough a point P joining the two planes and intersecting with X, see [Has 2, 1.2]. If X is a generic cubic fourfold in $\mathcal{C}_{14}$ then X is rational and there is a birational map $\rho : Q \to X$, where Q is a smooth quadric hypersurface in $\mathbf{P}^5$. The indeterminacy locus of ρ is a surface S' birationally equivalent to a K3 surface S such that $F(X) \simeq S^{[2]}$, see [BRS, Thm. 2.2]). Therefore, by Thm. 3.2, we get $t(X) \simeq t_2(S)(1) = t_2(S')(1)$.

In this section we consider the case of a cubic fourfold X endowed with a rational map $f : X \dashrightarrow \mathbf{P}^2$ such that the fibration $f : \tilde{X} \to \mathbf{P}^2$ obtained by resolving the base locus of f has rational fibers. Then, according to [Vial 1, (2)] the motive $h(\tilde{X})$ splits as follows

$$h(\tilde{X}) \simeq h(\mathbf{P}^2) \oplus h(\mathbf{P}^2)(1) \oplus h(\mathbf{P}^2)(2) \oplus M(1) \tag{5.1}$$

where M is isomorphic to a direct summand of the motive of some smooth surface Z. Suppose that f is obtained via a linear system having a smooth surface $T \subset X$ such that $t_2(T) = 0$ as base locus, and call $\tilde{X} \to X$ the blow-up along T. Then $h(\tilde{X}) \simeq h(X) \oplus h(T)(1)$ and

$$A_1(\tilde{X})_{hom} \simeq A_1(X)_{hom} \oplus A_0(T)_0, \tag{5.2}$$

where $A^3(\tilde{X}) = A^3(t(\tilde{X})) = A_1(\tilde{X})_{hom}$ and $A^3(X) = A^3(t(X)) = A_1(X)_{hom}$. Since $t_2(T) = 0$, by taking a reduced Chow-künneth decomposition $h(T) = \mathbf{1} \oplus h_1(T) \oplus h_2^{alg}(T) \oplus h_3(T) \oplus \mathbb{L}^2$, we get

$$A^3(h_3(T)(1)) = A^2(h(T)) = A^2(h_3(T)) = A_0(T)_0 \simeq (\operatorname{Alb} T)_{\mathbf{Q}}.$$

Therefore the isomorphism in (5.2) implies $A^3(t(\tilde{X})) \simeq A^3(t(X)) \oplus A^3(h_3(T)(1))$. Since the other Chow groups vanish on both sides we get an isomorphism of motives $t(\tilde{X}) \simeq t(X) \oplus h_3(T)(1)$. The motive $h_3(T)$, being a direct summand of the motive of $\operatorname{Alb} T$, is finite dimensional, see [MNP, 6.2.12]. Therefore the motive $h(X)$ is finite dimensional if and only if $h(\tilde{X})$ is finite dimensional that is the case if the surface Z appearing in (5.1) has a finite dimensional motive.
Examples of this situation are general cubic fourfolds X belonging either to $\mathcal{C}_8$ or to $\mathcal{C}_{18}$, that are conjecturally not rational. In the first case T is a plane and the fibers of π are quadrics, in the second case T is ruled elliptic and the fibers are del Pezzo surfaces of degree 6.
In order to identify the surfaces Z we will use the following proposition, that comes from the results in [Vial 1, Prop. 6.7].

Proposition 5.3. *Let X be a cubic fourfold containing a surface T with $t_2(T) = 0$ and let $\pi : \tilde{X} \to X$ be the blow-up of X along T with E exceptional divisor. Let*

$f : \tilde{X} \to \mathbf{P}^2$ be a surjective morphism. Let D be the discriminant curve of the fibration f and let $B^o = \mathbf{P}^2 - D$. Assume that for all $t \in B^o$, the fibers $\tilde{X}_t$ are smooth rational surfaces. Then there is a finite number of smooth surfaces $\tilde{B}_i$, for $i = 1 \dots n$, with surjective and finite maps $r_i : \tilde{B}_i \to \mathbf{P}^2$, such that the motive $h(X)$ is finite dimensional if all the motives $h(B_i)$ are finite dimensional. In this case the transcendental motive $t(X)$ is a direct summand of $\bigoplus_{1 \le i \le n} t_2(B_i)(1)$

Proof. Let $t \in \mathbf{P}^2$ and let $j_t : \tilde{X}_t \to \tilde{X}$ be the inclusion. The induced map on Chow groups $(j_t)_* : A_1(\tilde{X}_t) \to A_1(\tilde{X})$ fits into the following diagram

$$\begin{tikzcd} A_1(\tilde{X}_t) \arrow[r, "(j_t)_*"] \arrow[d] & A_1(\tilde{X}) \arrow[d, "cl"'] \\ H^2(\tilde{X}_t) \arrow[r, "(j_t)_*"] & H^6(\tilde{X}, \mathbf{Q}) \end{tikzcd}$$

Here $H^6(X, \mathbf{Q}) \simeq \mathbf{Q}[\gamma^3/3]$, with $\gamma \in A^1(X)$ a hyperplane section. From the long exact sequence of cohomology groups

$$\cdots \to H^n(X, \mathbf{Q})) \to H^n(T, \mathbf{Q})) \oplus H^n(\tilde{X}) \to H^n(E, \mathbf{Q}) \to H^{n+1}(X, \mathbf{Q}) \to \cdots$$

where E is the exceptional divisor of the blow-up, we get an isomorphism $H^6(\tilde{X}, \mathbf{Q}) \simeq H^6(X, \mathbf{Q}) \oplus \mathbf{Q}$, with $\mathbf{Q} \simeq H^6(E, \mathbf{Q})$. Therefore the image of $(j_t)_*$ lies in $A_1(\tilde{X})_{hom}$. From the isomorphism in (5.2) and the surjective homomorphism

$$A_1(X)_{hom} \oplus A_1(E)_{hom} \to A_1(\tilde{X})_{hom} \to 0$$

we get that the image of $A_1(E)_{hom}$ in $A_1(\tilde{X})$ is isomorphic to $A_0(T)_0$. Therefore the map

$$\bigoplus_{t \in \mathbf{P}^2} A_1(\tilde{X}_t) \xrightarrow{(j_t)_*} A_1(\tilde{X})_{hom},$$

when composed with the projection $A_1(\tilde{X})_{hom} \to A_1(X)_{hom}$, gives a surjective map

$$\bigoplus_{t \in \mathbf{P}^2} A_1(\tilde{X}_t) \to A_1(X)_{hom}. \tag{5.4}$$

Let $\mathcal{H} = \mathrm{Hilb}_1(\tilde{X}/\mathbf{P}^2)$ be the relative Hilbert scheme whose fibers parametrize curves in the fibers of f. Let

$$\begin{tikzcd} \mathcal{C} \arrow[r, "q"] \arrow[d, "p"'] & \tilde{X} \\ \mathcal{H} \arrow[d, "\pi"'] & \\ \mathbf{P}^2 & \end{tikzcd}$$

be the incidence diagram, where $\mathcal{C}$ is the universal family over $\mathcal{H}$, i.e. $\mathcal{C} = \{(C, x) | x \in C\} \subset \mathcal{H} \times \tilde{X}$. Then the map

$$p^* q_* : A_0(\mathcal{H}) \to A_1(\tilde{X})_{hom} \to A_1(X)_{hom}$$

factors trough $A_0(\mathcal{H}) \to A_1(\tilde{X}_t)$ and $f_t : A_1(\tilde{X}_t) \to A_1(X)_{hom}$, for every fiber $\tilde{X}_t$. By [Vial 1, Lemma 6.6] there is finite set $\mathcal{E} = \{\mathcal{H}_1, \dots, \mathcal{H}_n\}$ of irreducible components of $\mathrm{Hilb}_1(\tilde{X}/\mathbf{P}^2)$, such that they obey the following technical condition:

$\forall t \in B^o$, *the set* $\{cl(q_*[p^{-1}(u)]/u \in \mathcal{H}_i, t = \pi(u)\}$ *span* $H^2(\tilde{X}_t, \mathbf{Q})$. $\quad (*)$.

Let $f_i : \tilde{\mathcal{H}}_i \to \mathcal{H}_i$ be a resolution of singularities. By [Vial 1, Prop. 6.7], for all i there are smooth linear sections $\tilde{B}_i \to \tilde{\mathcal{H}}_i$ of dimension 2, such that, for every $i \in (1, \dots, n)$ the following composed map $r_i : \tilde{B}_i \to \mathbf{P}^2$ is surjective:

$$r_i : \tilde{B}_i \to \tilde{\mathcal{H}}_i \xrightarrow{f_i} \mathcal{H}_i \to \mathbf{P}^2 \tag{5.5}$$

The map r_i is finite and, for any $t \in \mathbf{P}^2$, $r_i^{-1}(t)$ contains a point in every connected component of the fiber of $\mathcal{H}_i$ over t. Let $\tilde{B} = \coprod_{1 \le i \le n} \tilde{B}_i$ be the disjoint union of the surfaces $\tilde{B}_i$. Again by [Vial 1, Prop. 6.7], there is a correspondence $\Gamma \in A^3(\tilde{B} \times \tilde{X})$ such that $\Gamma = \oplus_i \Gamma_i$ where $\Gamma_i \in A^3(\tilde{B}_i \times \tilde{X})$ is the class of the image of $\mathcal{C}_i$ inside $\tilde{B}_i \times \tilde{X}$ in the incidence diagram

$$\begin{array}{ccc} \mathcal{C}_i & \xrightarrow{q_i} & \tilde{X} \\ {\scriptstyle p_i}\downarrow & & \\ \tilde{B}_i & & \end{array} \tag{5.6}$$

Here $p_i : \mathcal{C}_i \to \tilde{B}_i$ is the pullback of the universal family $\mathcal{C} \to \mathcal{H}_i$ along $\tilde{B}_i \to \tilde{\mathcal{H}}_i \to \mathcal{H}_i$.The correspondence $\Gamma \in A^3(\tilde{B} \times X)$ gives a map of motives

$$(\Gamma)_* : h(\tilde{B})(1) = \sum_{1 \le i \le n} h(\tilde{B}_i)(1) \to h(X). \tag{5.7}$$

Let $h_i : \tilde{R}_i \to \tilde{R}_i$ be the normalization of the curve $R_i = r_i^{-1}(D)$, with r_i as in (5.5) and D the discriminant curve. For every $P \in D \subset \mathbf{P}^2$ and $i \in \{1, \dots n\}$, $r_i^{-1}(P)$ is a finite set of points, each one in a connected component of the fiber $\tilde{\mathcal{H}}_{i,P}$. Let us set

$$h : \tilde{R} := \coprod_{1 \le i \le n} \tilde{R}_i \to \tilde{B}$$

and $\Gamma_{\tilde{R}} := h^*(\Gamma) \in A^3(\tilde{R} \times \tilde{X}) = A_2(\tilde{R} \times \tilde{X})$. Then $\Gamma_{\tilde{R}}$ gives a map of motives $h(\tilde{R})(1) \to h(X)$. Let $h(\tilde{R}) = \mathbf{1} \oplus h_1(\tilde{R}) \oplus \mathbb{L}$ be a C-K decompositon and let $h_1(\tilde{R})(1) \to t(X)$ be the map induced by $\Gamma_{\tilde{R}}$. Then the associated maps on Chow groups vanish, because $A^p(h_1(\tilde{R})(1)) = A^{p-1}(h_1(\tilde{R})) \ne 0$ only for $p = 2$ while $A^2(t(X) = 0$. In particular the composite map

$$A_0(\tilde{R})_0 = \bigoplus\nolimits_{1 \le i \ \le n} A_0(\tilde{R}_i)_) \xrightarrow{h_*} A_0(\tilde{B})_0 \xrightarrow{\Gamma_*} A_1(X)_{hom} \tag{5.8}$$

is 0. Let

$$(j_D)_* : \bigoplus_{P \in D} A_1(\tilde{X}_P) \to A_1(X)_{hom}$$

be the sum of the $(j_P)_*$ and let us define $g_D : \bigoplus_{P \in D} A_1(\tilde{X}_P) \to A_0(\tilde{R})_0$ by sending a class $[C] \in A_1(\tilde{X}_P)$, lying on a connected component of the fiber $\mathcal{H}_{i,P}$, to the class of the corresponding point $x_C \in r^{-1}(P)$ in $A_0(\tilde{R})$. Then $(j_D)_*$ factors trough g_D and the map $A_0(\tilde{R})_0 \to A_1(X)_{hom}$ in (5.8). Therefore the map $(j_D)_*$ vanishes.

From 5.4 we get

$$A_1(X)_{hom} = \mathrm{Im}(\bigoplus_{t\in\mathbf{P}^2} A_1(\tilde{X}_t) \to A_1(X)_{hom}) = \mathrm{Im}(\bigoplus_{t\in B^o} A_1(\tilde{X}_t) \to A_1(X)_{hom}) \oplus$$

$$\oplus\, \mathrm{Im}(\bigoplus_{P\in D} A_1(\tilde{X}_P) \to A_1(X)_{hom});$$

where the map $\bigoplus_{P\in D} A_1(\tilde{X}_P) \to A_1(X)_{hom}$ is 0. Therefore from [Vial 1, Prop. 6.7] we get

$$A_1(X)_{hom} = \mathrm{Im}(\bigoplus_{t\in B^o} A_1(\tilde{X}_t) \to A_1(X)_{hom}) \subseteq \mathrm{Im}(\Gamma_* : A_0(\tilde{B})_0 \to A_1(X)_{hom}).$$

and thus

$$A_1(X)_{hom} = \mathrm{Im}(\Gamma_* : A_0(\tilde{B})_0 \to A_1(X)_{hom})$$

Therefore the map of motives in (5.7) induces a map on Chow groups

$$A^3(h(\tilde{B})(1)) = A_0(\tilde{B})_0 \to A^3(t(X)) = A_1(X)_{hom} \tag{5.9}$$

that is surjective. Let

$$h(\tilde{B}_i) = \mathbf{1} \oplus h_1(\tilde{B}_i) \oplus (\mathbb{L})^{\rho_i} \oplus h_2^{alg} \oplus t_2(\tilde{B}_i) \oplus h_3(\tilde{B}_i) \oplus \mathbb{L}^2$$

be a reduced C-K decomposition and let $h(\tilde{B}) = \sum_{1\leq i\leq n} h(\tilde{B}_i)$ be the corresponding decomposition for $\tilde{B}$. Then

$$A^3((\tilde{B})(1)) = A^3(t_2(\tilde{B})(1)) \oplus A^3(h_3(\tilde{B})(1) = A^2(t_2(\tilde{B}) \oplus A^2(h_3(\tilde{B})$$

Since the Chow groups $A^i(t_2(\tilde{B})(1)) \oplus A^i(h_3(\tilde{B})(1))$ and $A^i(t(X))$ vanish for $i \neq 3$, the transcendental motive $t(X)$ is a direct summand of

$$t_2(\tilde{B})(1) \oplus h_3(\tilde{B})(1) = \sum_{1\leq i\leq n} ((t_2(\tilde{B}_i)(1) \oplus h_3(\tilde{B}_i)(1)).$$

The motives $h_3(\tilde{B}_i)$, being of abelian type are finite dimensional. Therefore $t(X)$ is finite dimensional if $t_2(\tilde{B}_i)$ is finite dimensional for every $i \in \{1, \dots n\}$. Assume that $t_2(\tilde{B}_i)$ is finite dimensional : than $t(X)$ cannot be isomorphic to a direct summand of $M = h_3(\tilde{B})(1)$, hence it is a direct summand of $t_2(\tilde{B})(1)$. Suppose, on the contrary, that $M \simeq N \oplus t(X)$ and let $f : M \to t(X)$ and $g : t(X) \to M$ such that $f \circ g = id_{t(X)} = \pi_4^{tr}$, where $t(X) = (X, \pi_4^{tr})$. Since M and $t(X)$ are finite dimensional of different parity the map f is smash-nilpotent, see [MNP, Prop.5.3.1]. Smash-nilpotent correspondences form a bilateral ideal $\mathcal{I}$ in the category $\mathcal{M}_{rat}$, see [AK, 7.4.3], and hence we get $id_{t(X)} \circ f \circ g = id_{t(X)} \in \mathcal{I}$. Therefore, the projector $\pi_4^{tr} = id_{t(X)}$, being smash-nilpotent, vanishes and we get a contradiction: $t(X) = 0$. □

5.1. **Cubic fourfolds containing a plane.** Let X be a generic fourfold in $\mathcal{C}_8$. Then X contains a plane P. Call $\tilde{X}$ the blow-up of X along P and $\pi : \tilde{X} \to \mathbf{P}^2$ the morphism that resolves the projection off P. The morphism π is a fibration in quadric surfaces, whose fibers degenerate along a plane sextic C, which is smooth in the general case. The double cover $r : S \to \mathbf{P}^2$ ramified along C is a K3 surface. Recall that the relative Hilbert scheme of lines $\mathcal{H}(0,1)$ of the morphism π is an étale projective bundle over S. The Stein factorization of the map $\mathcal{H}(0,1) \to \mathbf{P}^2$ yields $\mathcal{H}(0,1) \to S \to \mathbf{P}^2$, where the first map is a $\mathbf{P}^1$-bundle.

Proposition 5.10. *If X is a general element in $\mathcal{C}_8$ the transcendental motive $t(X)$ is isomorphic to the motive $t_2(S)(1)$. Therefore if the motive of S is finite dimensional then also $h(X)$ is finite dimensional .*

Proof. Since $A_0(\mathbf{P}^2)_0 = 0$ we have $A_1(X)_{hom} = A_1(\tilde{X})_{hom}$. In order to show that $\mathcal{H}_1 = \mathcal{H}(0,1)$ is the only component that we need to apply Prop. 5.3, we need to check that the technical condition $(*)$ holds true for this Hilbert scheme. This is not hard to show, since the $H^2(\tilde{X}_t, \mathbf{Q})$, for $t \in \mathbf{P}^2 - C$, is generated by the classes of any line of the two rulings of the quadric. In fact the two irreducible components $\mathcal{H}_1^{(1)}$ and $\mathcal{H}_1^{(2)}$ of $\mathcal{H}(0,1)$ over the point $t \in \mathbf{P}^2$, not lying on the discriminant, parametrize the lines in each ruling. From (5.7) we get a map $t_2(S)(1) \to t(X)$ such that $A^3(t_2(S)(1)) = A_0(S)_0 \to A^3(t(X) = A_1(X)_{hom}$ is surjective and we are left to show that it is an isomorphism. By [SYZ, Theorem 3.6] there is an isomorphism $A_0(S)_0 \to A_0(F)_2$, where $F = F(X)$, that, together with the isomorphism $A_0(F)_2 \simeq A_1(X)_{hom}$ in Theorem 2.5, gives the isomorphism $A_0(S)_0 \simeq A^3(t(X)) = A_1(X)_{hom}$. □

Remark 5.11. The above example suggests that the statement in Conj. 4.3 cannot be inverted, in the sense that a generic element of $\mathcal{C}_8$ is conjecturally not rational and yet there is an isomorphism $t(X) \simeq t_2(S)(1)$, with S a K3 surface. A similar result appears in [Bull, Thm. 0.3], for all $X \in \mathcal{C}_d$, where d satisfies the following numerical condition : $d = k^2 d_0$, with $k \in \mathbf{Z}$ and $d | 2n^2 + 2n + 2$, with $n \in \mathbf{Z}$.

5.2. **Cubic fourfolds fibered in del Pezzo sextics.** Let X be a generic fourfold in $\mathcal{C}_{18}$. The fourfold X contains an elliptic ruled surface T of degree 6 such that the linear system of quadrics in $\mathbf{P}^5$ containing T is two dimensional. Let once again $r : \tilde{X} \to X$ be the blow-up of X at T and $\pi : \tilde{X} \to \mathbf{P}^2$ the (resolution of the) map induced by the linear system of quadrics containing T. The generic fiber of π is a del Pezzo surface of degree 6. The generic del Pezzo fibration π obtained from a cubic fourfold in $\mathcal{C}_{18}$ is a *good del Pezzo fibration* in the sense of [AHTV-A, Def. 11]. The discriminant curve D of π has two irreducible components, a smooth sextic C and a sextic $\bar{C}$ with 9 cusps. As in the previous case the double cover $S \to \mathbf{P}^2$ branched on C is a smooth K3 surface of degree 2. The goal of this section is to show that there is an isomorphism $t_2(S)(1) \simeq t(X)$. The main difference with the $\mathcal{C}_8$ case is that here the Picard rank of the generic fiber is higher, so we will need to consider surfaces inside two different Hilbert schemes of curves in order to obey the technical condition $(*)$ and hence to apply the constructions of Prop. 5.3.

Associated to the good del Pezzo fibration $\pi : \tilde{X} \to \mathbf{P}^2$ there is a non-singular degree 3 cover $f : Z \to \mathbf{P}^2$ branched along a cuspidal sextic $\bar{C}$ (see [AHTV-A]) where Z is a non singular surface. Let $\mathcal{H}(0,2) \to \mathbf{P}^2$ be the relative Hilbert scheme of connected genus 0 curves of anti canonical degree 2 on the fibers. The Stein factorization yields an étale $\mathbf{P}^1$-bundle $\pi_1 : \mathcal{H}(0,2) \to Z$. It is easy to see that, on every fiber, the $\mathbf{P}^1$-bundle is given by the strict transform of the lines through each of the 3 blown-up points $P_1, P_2, P_3 \in \mathbf{P}^2$ of the corresponding del Pezzo of degree 6. This gives a diagram

$$\begin{array}{c} \mathcal{H}(0,2) \\ {\scriptstyle \mathbf{P}^1}\big\downarrow \\ Z \\ {\scriptstyle f_1}\big\downarrow \\ \mathbf{P}^2 \end{array} .$$

Proposition 5.12. *The triple cover $Z \to \mathbf{P}^2$ is an elliptic ruled surface and hence $t_2(Z) = 0$ and $A_0(Z)_0 \simeq (\mathrm{Alb}\, Z)_{\mathbf{Q}} \simeq (\mathrm{Jac}E)_{\mathbf{Q}}$, with E an elliptic curve.*

Proof. Let $\bar{C} \subset \mathbf{P}^2$ be the ramification locus of the triple cover $f_1 : Z \to \mathbf{P}^2$. As it has been observed in [AHTV-A], for a generic cubic $X \in \mathcal{C}_{18}$, $\bar{C}$ is a cuspidal degree 6 curve with 9 cusps. It is well known [Mir] that such a triple cover is completely determined by the Tschirnhausen rank two vector bundle on $\mathbf{P}^2$ and a section of (a twist of) the relative $\mathcal{O}(3)$ on the associated projectivized $\mathbf{P}^1$-bundle. Let us denote V the Tschirnhausen module. From Prop. 4.7 of [Mir] we see that $\bar{C}$ belongs to the linear system $|-2c_1(V)|$, hence $c_1(V) = \mathcal{O}_{\mathbf{P}^2}(-3)$. Then, by [Mir, Lemma 10.1], the number of cusps is exactly $3c_2$, this means that $c_2(V) = 3$. With these data in mind we can use [Mir, Prop. 10.3] to compute the invariants of Z and get

$$\chi = 0, \quad K^2 = 0, \quad e(Z) = 0.$$

Now, by [Shi, Cor 2.3] we see that that $V \cong \Omega_{\mathbb{P}^2}$, hence by [Mir, Cor 10.6] we have $p_g(Z) = 0$, and $q(Z) = 1$. This easily implies that the surface Z is again an elliptic ruled surface. Note that such a triple plane being an elliptic ruled surface was first observed by Du Val in [DV] by different methods. Since $p_g(Z) = 0$ and Z is not of general type we get $t_2(Z) = 0$. The rest follows from the isomorphism $A_0(Z)_0 \simeq (\mathrm{Jac}E)_{\mathbf{Q}}$.

□

As we have already anticipated, in this case, considering just one Hilbert scheme will not be enough in order to apply Prop. 5.3, since the fibers of π have higher Picard rank. Hence we need to consider also $\mathcal{H}(0,3)$, the relative Hilbert scheme of curves of genus zero and canonical degree 3 inside the fibers. There are two 2-dimensional families of such curves on a del Pezzo sextic. One is given by the strict transforms of the lines in $\mathbf{P}^2$ that do not pass through any of the three base points. The second is given by conics passing through the three base points. We will call the former cubic curves of first type and the latter cubic curves of second type. The Stein factorization $\mathcal{H}(0,3) \to S \to \mathbf{P}^2$ of the natural projection $\pi_2 : \mathcal{H}(0,3) \to \mathbf{P}^2$ reflects this difference and displays $\mathcal{H}(0,3)$ as an étale $\mathbf{P}^2$-bundle over a smooth degree two K3 surface S [AHTV-A]. It is straightforward to see that one $\mathbf{P}^2$ parametrizes the curves of first type and the other those of second type.
In order to apply Prop. 5.3 to the fibration $\tilde{X} \to \mathbf{P}^2$, we prove the following lemma.

Lemma 5.13. *Let $\pi : \tilde{X} \to \mathbf{P}^2$ a del Pezzo fibration and let $D \subset \mathbf{P}^2$ be the discriminant curve. Then the two components $\mathcal{H}(0,2) = \mathcal{H}_1$ and $\mathcal{H}(0,3) = \mathcal{H}_2$ of* *the Hilbert scheme $\mathcal{H}/\mathbf{P}^2$ obey the technical condition $(*)$, i.e. $\forall t \in B^o = \mathbf{P}^2 - D$, the set $\{cl(q_*[p^{-1}(u)]/u \in \mathcal{H}_i, t = \pi(u)\}$ span $H^2(\tilde{X}_t, \mathbf{Q})$, where $\pi_i : \mathcal{H}_i \to \mathbf{P}^2$.*

Proof. Fix a point $p \in \mathbf{P}^2$, such that the fiber $\tilde{X}_p$ over p is a smooth del Pezzo sextic. Its Picard rank is 4 and the generators are the proper transform of a line and the three exceptional divisors. Let us denote H, E_1, E_2 and E_3 these divisor classes. Then, the fiber $(\mathcal{H}_2)_p \subset \mathcal{H}(0,3)$ over p contains at least a curve from the linear system $|H|$ and a curve from the linear system $|2H - E_1 - E_2 - E_3|$. On the other hand, the fiber $(\mathcal{H}_1)_p \subset \mathcal{H}(0,2)$ over p contain at least 3 curves from the linear systems $|H - E_1|$, $|H - E_2|$ and $|H - E_3|$. It is straightforward to see that linear combinations of these 5 divisor classes generate the whole $H^2(\tilde{X}_p, \mathbf{Q})$. □

Theorem 5.14. *Let X be a generic fourfold in $\mathcal{C}_{18}$. Then $t(X)$ is isomorphic to $t_2(S)(1)$.*

Proof. From Lemma 5.13 it follows that we can apply Prop. 5.3 to the Hilbert schemes $\mathcal{H}_1$ and $\mathcal{H}_2$. Hence, as in (5.5), we get smooth surfaces $\tilde{B}_1$ and $\tilde{B}_2$ with finite maps $r_1 : \tilde{B}_1 \to \mathbf{P}^2$ and $r_2 : \tilde{B}_2 \to \mathbf{P}^2$ that induce isomorphisms $A_0(\tilde{B}_1)_0 = A_0(Z)_0$ and $A_0(\tilde{B}_2)_0 = A_0(S)_0$. The discriminant curve $D \subset \mathbf{P}^2$ has two irreducible components, the cuspidal sextic $\bar{C}$ and the smooth sextic C. Let $\tilde{R}_1 \to R_1$ and $\tilde{R}_2 \to R_2$ be respectively the normalization of $R_1 = r_1^{-1}(\bar{C})$ and the normalization of $R_2 = r_2^{-1}(C)$. The surfaces Z and S have reduced Chow-Künneth decompositions with $t_2(Z) = 0$ and $h_3(S) = 0$. The curve $\tilde{R}_1$ is smooth of genus 1 and hence it is an elliptic curve birational (and hence isomorphic) to a curve E, such that Z is birational to the product $\mathbf{P}^1 \times E$. Therefore

$$A_0(\tilde{R}_1)_0 \simeq A_0(Z)_0 = (\mathrm{Jac}E)_{\mathbf{Q}}.$$

The K3 surface S is a double cover of $\mathbf{P}^2$ ramified along C. The curve $\tilde{R}_2$, is a constant cycle curve(see [Huy 1, 7.1]), hence the map $A_0(\tilde{R}_2)_0 \to A_0(S)_0$ is the 0-map. From (5.8) we get that the map

$$A_0(\tilde{R}_1)_0 \oplus A_0(\tilde{R}_2)_0 \to A_0(\tilde{B})_0 \simeq (A_0(Z)_0 \oplus A_0(S)_0) \to A_1(X)_{hom}$$

vanishes.Therefore the map $A_0(\tilde{R}_1) = A_0(Z)_0 \to A_1(X)_{hom}$ is 0. The map $(\Gamma)_*)$ in (5.7) gives a map

$$(\Gamma)_* : t_2(S)(1) \to t(X),$$

such that the associated map on Chow groups $\mathbf{A}_0(S)_0 \to A_1(X)_{hom}$ in 5.9 is surjective. In order to show that is also injective and hence the map $(\Gamma)_*$ gives an isomorphism of motives, we apply the same argument as in the proof of [SYZ, Thm. 3.6]. If $\Psi(\alpha) = 0$ in $A_1(X)_{hom}$, with $\alpha \in A_0(S)_0$ then $\sigma_*(\alpha) = 0$, where σ is the involution on S coming from the double cover $S \to \mathbf{P}^2$. Therefore $\alpha = 0$. □

Remark 5.15. G.Tabuada in [Tab, Thm. 1.7] proves a result, similar to Thm. 5.14, for a fibration $f : Y \to \mathbf{P}^2$, where Y is a smooth 4-fold and the fibers of f are sextic del Pezzo surfaces. There are finite flat morphisms $Z_2 \to \mathbf{P}^2$, $Z_3 \to \mathbf{P}^3$ of degree 3 and 2 respectively, such that, if the motives of Z_2 and Z_3 are *Schur-finite*, then the motive of Y is Schur-finite. Note that, since a finite dimensional motive is also Schur-finite, the motive of the surface Z in(5.12) is Schur-finite

6. Cubic fourfolds with an involution

Let σ be the involution on $\mathbf{P}^5$ defined by

$$[x_0, x_1, x_2, x_3, x_4, x_5] \to [x_0, x_1, x_2, x_3, -x_4, -x_5]$$

A cubic fourfold X fixed by σ has an equation of the form

$$C(x_0, x_1, x_2, x_3) + x_4^2 L_1 + x_5^2 L_2 + x_4 x_5 L_3 = 0 \tag{6.1}$$

where C has degree 3 and L_1, L_2, L_3 are linear forms in x_0, x_1, x_2, x_3. C. Camere shows that this is the unique automorphism of $\mathbf{P}^5$ inducing a symplectic involution on $F(X)$ [Ca, Sect. 7]. The locus of fixed points of σ on $\mathbf{P}^5$ is the disjoint union of a $\mathbf{P}^3$ defined by $x_4 = x_5 = 0$ and the line r joining the base points P_4 and P_5. The line r is contained in X and the fixed locus on $\mathbf{P}^3$ is the cubic surface $C = 0$. The symplectic involution σ_F on $F(X)$ has 28 isolated points, i.e. the line r and the 27 lines on the cubic surface, plus a K3 surface S, consisting of the lines joining a fixed point Q_1 on $\mathbf{P}^3$ and a point Q_2 on r (see again [Ca]). Let us now project with center the line r and let $\tilde{X}$ denote the blow-up of X along r. The projection resolves into a morphism $\delta : \tilde{X} \to \mathbf{P}^3$, which is well-known to be a conic bundle with quintic degeneration locus D.

Lemma 6.2. *The quintic hypersurface $D \subset \mathbf{P}^3$ has a cubic and a quadric irreducible components. For appropriate choices of the L_i and of C the sextic intersection curve is smooth and parametrizes rank one conics. For general choices of the L_i the quadric has rank 3.*

Proof. Let $p := [a : b : c : d] \in \mathbf{P}^3$, in order to study the conic over p we need to study the intersection of X with the plane $\mathbf{P}^2_p := \langle p, P_4, P_5 \rangle \subset \mathbf{P}^5$, where $\langle \cdot \rangle$ denotes as usual the linear span, and P_4, P_5 are the base points on r. Hence we substitute inside equation 6.1 the values $\lambda[0 : 0 : 0 : 0 : 1 : 0] + \mu[0 : 0 : 0 : 0 : 0 : 1] + \gamma[a : b : c : d : 0 : 0]$, with $\lambda, \mu, \gamma \in \mathbb{C}$. Recall that in this plane the equation of r is $\gamma = 0$. Then, dividing by γ the cubic equation in γ, λ and μ we obtain

$$\gamma^2 C(a, b, c, d) + \lambda^2 L_1(a, b, c, d) + \mu^2 L_2(a, b, c, d) + \lambda\mu L_3(a, b, c, d). \tag{6.3}$$

This is the conic obtained from the symmetric matrix

$$\begin{pmatrix} C(a,b,c,d) & 0 & 0 \\ 0 & L_1 & \frac{1}{2}L_3 \\ 0 & \frac{1}{2}L_3 & L_2 \end{pmatrix}.$$

Hence one easily sees that the equation of D is $C \cdot (L_1 L_2 - \frac{1}{4} L_3)$. The mere equation $L_1 L_2 - \frac{1}{4} L_3$ shows that the quadric has at most rank 3 and that for general L_i this is the case. Let us denote by Q the quadric surface. Suppose now $L_3 = x_0 - x_1 - x_2$, $L_1 = (t - z)$, $L_2 = (t + z)$ and C is the Fermat cubic. Then the quadric has equation $-(x - y - z)^2 + t^2 - z^2$ and rank 3. A quick Macaulay2 [Mac2] routine shows that the intersection with the Fermat cubic is a smooth sextic curve Y, and from the matrix representing the conic one sees that the sextic curves parametrizes conics of rank 1. $\square$

The surface S is a double cover of the cubic surface $C = 0$ ramified along the degree 6 curve Y. It is straightforward to see that S parametrizes irreducible (linear) components of degenerate conics, that are fixed by the involution. If one takes the double cover $W \stackrel{2:1}{\to} Q$, ramified along Y, this parametrizes the irreducible components of degenerate conics that are not fixed by the involution (except for double lines, parametrized by Y). It is a classical construction that double covers W of quadric cones, ramified along a smooth genus 4 sextic are del Pezzo surfaces of degree 1, and the double cover is induced by the linear system $|-2K_W|$, where K_W is the canonical bundle. We observe that by Kodaira vanishing it is easy to see that $q(W)=0$. The Abel-Jacobi map induces an isomorphism

$$H^{3,1}(X) \simeq H^{2,0}(F(X)) \simeq H^{2,0}(S)$$

and hence $H^2_{tr}(F,\mathbf{Q}) \simeq H^2_{tr}(S,\mathbf{Q})$. By [Lat 2, Thm. 3.1] there is a correspondence $\Gamma \in A^3(S \times X)$ inducing a surjective homomorphism $A_0(S)_0 \to A_1(X)_{hom}$. Let $h(S) = \mathbf{1} \oplus h_2^{alg} \oplus t_2(S) \oplus \mathbb{L}^2$ be a Chow-Künneth decomposition and let

$$\Gamma_* : t_2(S)(1) \to t(X) \tag{6.4}$$

be the map of motives induced by Γ. We have

$$A^3(t(X)) = A_1(X)_{hom} \ ; \ A^3(t_2(S)(1)) = A^2(t_2(S)) = A^2(S)_0$$

and $A^i(t(X)) = A^i(t_2(S)(1)) = 0$ for $i \neq 3$. Therefore Γ induces a surjective map on all Chow groups and hence $t(X)$ is a direct summand of $t_2(S)(1)$.

The following result shows that Γ_* is in fact an isomorphism.

Proposition 6.5. *The map of motives in 6.4 is an isomorphism.*

Proof. It is enough to show that the surjective map $A_0(S)_0 \to A_1(X)_{hom}$ is an isomorphism. Let $\alpha = [l_1] - [l_2] \in A_0(S)_0$ be such that $\Gamma_*(\alpha) = 0$. Then $[l_1] = [l_2]$ corresponds to a double line on a singular fiber of $\delta : \tilde{X} \to \mathbf{P}^3$. Let τ be the involution on S associated to the double cover $S \to C$, that is ramified along the sextic Y. Since Y parametrizes double lines on the singular fibers we have that $[l_1] = [l_2]$ belong to Y (more precisely to the branch locus inside S, which is isomorphic to Y). The cubic surface C is rational and hence $A_0(C)_0 = 0$. Therefore, by the same argument as in [Huy 1, 7.1], the fixed locus of τ is a constant cycle curve, i.e. the map $A_0(Y)_0 \to A_0(S)_0$ vanishes, and we get $\alpha = 0$ in $A_0(S)_0$. □

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