Birational geometry of some universal families of n-pointed Fano fourfolds

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ABSTRACT. The object of this note is the moduli spaces of cubic fourfolds (resp., Gushel-Mukai fourfolds) which contain some special rational surfaces. Under some hypotheses on the families of such surfaces, we develop a general method to show the unirationality of the moduli spaces of the n-pointed such fourfolds. We apply this to some codimension 1 loci of cubic fourfolds (resp., Gushel-Mukai fourfolds) appeared in the literature recently.

1. INTRODUCTION

One of the most active areas of research in algebraic geometry is related to the study of the birational geometry of Fano varieties, notably those of dimension four. In the last 20 years, algebraic geometers have been working on the problem of rationality of smooth cubic hypersurfaces in $\mathbb{P}^5$ (cubic fourfolds for short). Recall that cubic fourfolds are parametrized by an open subset $\mathcal{U}$ in the 55-dimensional projective space $\mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^5}(3)))$. The moduli space of cubic fourfolds is the GIT quotient $\mathcal{C} = \mathcal{U}//\mathrm{PGL}_6$, a quasi-projective variety of dimension $55 - 35 = 20$. It is classically well-known that all cubic fourfolds are unirational and that some of them are rational. While the general suspicion is that most cubic fourfolds are non rational, no cubic fourfold has yet been proven to be non rational.

Hassett, in his works [15, 16] (see also [17]), adopted a Hodge theoretic approach. He defined the *Noether-Lefschetz* locus as the subset of the moduli space $\mathcal{C}$ consisting of *special* cubic fourfolds, that is, fourfolds X containing an algebraic surface S that is not homologous to a complete intersection. The discriminant of X is defined as the determinant d of the intersection form on the saturated sublattice of $H^{2,2}(X, \mathbb{Z})$ generated by h^2 and $[S]$, where h denotes the hyperplane section class of X. Using the period map and the geometry of the period domain, Hassett proved that special cubic fourfolds form a countably infinite union of irreducible divisors $\mathcal{C}_d \subset \mathcal{C}$, corresponding to fourfolds having discriminant d, and where d runs over all integers $d \geq 8$ with $d \equiv 0, 2 \pmod 6$. For small values of the discriminant d, these divisors are characterized by the families of surfaces, not unique, that they contain (see [16, 28, 30, 32]). Moreover, for an infinity of values of d, cubic fourfolds in $\mathcal{C}_d$ are associated to a degree d polarized K3 surface via Hodge theory. This seems to relate strongly to the rationality of cubic fourfolds. In fact, it is conjectured that fourfolds with an associated K3 surface should be precisely the rational ones (see [22, 1, 23, 17, 5, 30, 32]).

Gushel-Mukai fourfolds (GM fourfolds, for short) are prime Fano fourfolds of degree 10 and index 2. By a result of Mukai [26], they can be realized as smooth dimensionally transverse intersections of a cone $C(\mathbb{G}(1,4)) \subset \mathbb{P}^{10}$ over the Grassmannian $\mathbb{G}(1,4) \subset \mathbb{P}^9$

with a linear subspace $\mathbb{P}^8 \subset \mathbb{P}^{10}$ and a quadric hypersurface $Q \subset \mathbb{P}^{10}$. The fourfolds for which the $\mathbb{P}^8 \subset \mathbb{P}^{10}$ does not pass by the vertex of the cone are called *ordinary*. These can be viewed as smooth quadric hypersurfaces in a smooth del Pezzo fivefold $\mathbb{Y}^5 = \mathbb{G}(1,4) \cap \mathbb{P}^8 \subset \mathbb{P}^8$, and are thus parametrized by an open subset $\mathcal{V}$ in the 39-dimensional projective space $\mathbb{P}(H^0(\mathcal{O}_{\mathbb{Y}^5}(2)))$. Recall also that all the fivefolds $\mathbb{Y}^5$ are projectively equivalent. The moduli space of GM fourfolds has dimension 24 and is denoted by $\mathcal{M}^4_{GM}$ (see [6]). The ordinary GM fourfolds correspond to the points of an open subset $\mathring{\mathcal{M}}^4_{GM}$ in $\mathcal{M}^4_{GM}$, which is the complement of an irreducible closed subset of codimension 2 in $\mathcal{M}^4_{GM}$. We can view $\mathring{\mathcal{M}}^4_{GM}$ as the quotient $\mathcal{V}//\mathrm{PGL}_9$.

From the point of view of birational geometry, GM fourfolds behave very much like cubic fourfolds and share many properties with them. They are again all unirational, rational examples are easy to construct, but no examples have yet been proven to be nonrational. Once again, by the study of the period map via Hodge theory, in [6] the authors introduced the *Noether-Lefschetz* locus inside the moduli space $\mathcal{M}^4_{GM}$, defined as the set of those fourfolds containing a surface whose cohomology class does not come from the Grassmannian $\mathbb{G}(1,4)$. This locus consists of a countable infinite union of divisors $(\mathcal{M}^4_{GM})_d \subset \mathcal{M}^4_{GM}$, labelled by the integers $d > 8$ with $d \equiv 0, 2, 4 \pmod 8$. The divisor $(\mathcal{M}^4_{GM})_d$ is irreducible if $d \equiv 0, 4 \pmod 8$, and it has two irreducible components $(\mathcal{M}^4_{GM})'_d$ and $(\mathcal{M}^4_{GM})''_d$ if $d \equiv 2 \pmod 8$; see [6, 8]. Recently, the third-named author [36] (see also [18]), inspired by the work of Nuer [28], gave an explicit description of the first irreducible components of this Noether-Lefschetz locus in terms of classes of rational smooth surfaces that the fourfolds contain.

In a fashion very similar to curves and K3 surfaces [11, 12, 24, 3], universal families were defined for cubic fourfolds in [2]. Since a general cubic fourfold lying in any divisor $\mathcal{C}_d$ does not have projective automorphism, universal cubic fourfolds $\mathcal{C}_{d,1} \to \mathcal{C}_d$ were introduced over divisors for $8 \leq d \leq 42$. These universal cubic fourfolds $\mathcal{C}_{d,1}$ correspond to the moduli space of 1-pointed cubic fourfolds. The authors prove [2, Theorem 4.10] the unirationality of $\mathcal{C}_{d,n}$ for $8 \leq d \leq 42$ and all n, using the presentation of the divisors $\mathcal{C}_d$ as cubics containing certain rational surfaces (see [28, 32]). This, combined with an induction argument based on a theorem of Kollár [20] on the unirationality of smooth cubic fourfolds over an arbitrary field, gives the result.

In this paper we propose a general, abstract method to show the unirationality of certain n-pointed universal (cubic and GM) fourfolds over their moduli spaces. By definition, the moduli space of n-pointed fourfolds corresponds to the universal family over the $(n-1)$-pointed fourfolds moduli space. Our method holds both for certain cubic fourfolds and for certain GM fourfolds. With respect to cubics, our argument is completely different from those of [2], and part of the results from [2] become now a special case of our main Theorem 1.1. In particular, Theorem 1.1 can be applied to any family $\mathcal{X}_{\mathcal{S}}$ of (cubic and GM) fourfolds that contains surfaces from a given family $\mathcal{S}$, under some hypotheses (see Remark 3.2) on $\mathcal{X}_{\mathcal{S}}$ and $\mathcal{S}$. In fact there are other higher codimensional loci in the moduli space of GM fourfolds to which our theorem could apply, but we refrained to describe them for sake of brevity.

Theorem 1.1. *The universal n-pointed fourfolds over the following irreducible codimension-one loci:*

$$\mathcal{C}_{14},\ \mathcal{C}_{26},\ \mathcal{C}_{38},\ \mathcal{C}_{42},\ (\mathcal{M}^4_{GM})'_{10},\ (\mathcal{M}^4_{GM})''_{10},\ (\mathcal{M}^4_{GM})_{20},$$

are unirational.

In the last section of the paper we restrict our attention to a codimension one locus $(\mathcal{M}^4_{GM})^{nod}_{20}$ inside $(\mathcal{M}^4_{GM})_{20}$, defined via certain genus 11 K3 surfaces contained in a Noether-Lefschetz divisor. By describing the birational geometry of the NL divisor in the moduli of K3 surfaces, and exploiting the relation between these surfaces and the GM fourfolds, we prove that $(\mathcal{M}^4_{GM})^{nod}_{20}$ and the universal family $(\mathcal{M}^4_{GM})^{nod}_{20,1}$ above it are rational.

Plan of the paper: in Section 2, we give explicit descriptions of certain divisors parametrizing (cubic or GM) fourfolds in their moduli space. We recall the constructions of several families $\mathcal{S}$ of surfaces characterizing these divisors and highlight some of their properties crucial for the next section. Section 3 is devoted to the proof of the main result of this paper. Finally, in Section 4 we describe the locus $(\mathcal{M}^4_{GM})^{nod}_{20} \subset (\mathcal{M}^4_{GM})_{20}$ and show its rationality as well as that of the universal family $(\mathcal{M}^4_{GM})^{nod}_{20,1}$.

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2. Explicit geometric descriptions of some Nother-Lefschetz divisors in the moduli space $\mathcal{C}$ of cubic fourfolds and in the moduli space $\mathcal{M}^4_{GM}$ of GM fourfolds

In this section, we shall recall some explicit descriptions of unirational irreducible families $\mathcal{S}$ in the Hilbert scheme of $\mathbb{P}^5$ (respectively in the Hilbert scheme of a fixed smooth del Pezzo fivevold $\mathbb{Y}^5 = \mathbb{G}(1,4) \cap \mathbb{P}^8 \subset \mathbb{P}^8$) such that the closure of the locus of cubic fourfolds (resp., GM fourfolds) containing a surface of the family $\mathcal{S}$ describes a Noether-Lefschetz divisor in the corresponding moduli space. Most of these properties must be well-known to the experts of the field. Due to the plethora of references that exist on these topics, and the fact that sometimes the properties we are interested in are not really pointed out, we decided to resume them in this section. The properties of these families will be an essential tool for our main results contained in Section 3.

We recall that the Hilbert scheme $\mathrm{Hilb}_{\mathbb{P}^n}$ of a projective space $\mathbb{P}^n$ parametrizes closed subschemes of projective space (see for example [21, Chapter I]). The Hilbert scheme decomposes as a disjoint union of pieces $\mathrm{Hilb}_{\mathbb{P}^5}(P)$. Each component parametrizes subschemes with given Hilbert polynomial P. It is not hard to generalize this definition to any projective variety.

We shall focus on the fact that starting from a pair (S, X), where S is a general member of the family $\mathcal{S}$ and X is a general fourfold containing S, we can build an explicit birational map $X \dashrightarrow^{\simeq} \mathbb{P}^4$, defined over the same field of definition as S and X.

2.1. Cubic fourfolds containing a quintic del Pezzo surface. A quintic del Pezzo surface is the image of $\mathbb{P}^2$ via the linear system of cubic curves with 4 base points in general position.

Definition 2.1. Let $S \subset \mathbb{P}^n$ a projective variety and C a curve in the same projective space. The curve C is said to be n-secant to S if the intersection $S \cap C$ is a 0-dimensional scheme of length n.

Theorem 2.2 ([10, 5]). *The cubic fourfolds containing a quintic del Pezzo surface describe the divisor $\mathcal{C}_{14} \subset \mathcal{C}$ of fourfolds of discriminant 14.*

Theorem 2.3 ([25, 10, 5]). *A quintic del Pezzo surface $S \subset \mathbb{P}^5$ admits a congruence of secant lines: through the general point of $\mathbb{P}^5$ there is a unique secant line to S.*

The general line of this congruence can be realized as the general fiber of the dominant map

$$\mu : \mathbb{P}^5 \dashrightarrow \mathbb{P}^4$$

defined by the linear system $|\mathcal{I}_S(2)|$ of quadric hypersurfaces through S.

If X is a general cubic fourfold containing S, then the restriction of μ induces a birational map $\mu|_X : X \dashrightarrow \mathbb{P}^4$.

2.2. **Cubic fourfolds containing a 3-nodal septic scroll.** Let $S \subset \mathbb{P}^5$ be the projection of a rational normal septic scroll $\Sigma_7 \subset \mathbb{P}^8$ from a plane spanned by three general points on the secant variety of Σ_7. Thus S is a rational septic scroll having 3 non-normal nodes.

Theorem 2.4 ([11]). *The family of rational 3-nodal septic scrolls, constructed as above, is irreducible, unirational and of dimension $44 = 74 + 5 \cdot 3 - \dim \mathrm{PGL}_9 + \dim \mathrm{PGL}_6$.*

The cubic fourfolds containing such a surface describe the divisor $\mathcal{C}_{26} \subset \mathcal{C}$ of fourfolds of discriminant 26.

Theorem 2.5 ([32, 31]). *Let $S \subset \mathbb{P}^5$ be a general rational 3-nodal septic scroll. Then S admits a congruence of 5-secant conics: through the general point of $\mathbb{P}^5$ there is a unique conic curve which is 5-secant to S.*

The general conic curve of this congruence can be realized as the general fiber of the dominant map

$$\mu : \mathbb{P}^5 \dashrightarrow \mathbb{P}^4$$

defined by the linear system $|\mathcal{I}_S^2(5)|$ of quintic hypersurfaces with double points along S.

If X is a general cubic fourfold containing S, then the restriction of μ induces a birational map $\mu|_X : X \dashrightarrow \mathbb{P}^4$.

2.3. **Cubic fourfolds containing a "generalized" Coble surface.** Let $S \subset \mathbb{P}^5$ be the image of $\mathbb{P}^2$ via the linear system of curves of degree 10 with 10 general triple points. We have that S is a smooth rational surface of degree 10 and sectional genus 6 cut out by 10 cubics.

Theorem 2.6 ([28]). *The surfaces $S \subset \mathbb{P}^5$ obtained as above form an irreducible unirational family $\mathcal{S}^{10,6} \subset \mathrm{Hilb}_{\mathbb{P}^5}$ of dimension $47 = 10 \cdot 2 - \dim \mathrm{PGL}_3 + \dim \mathrm{PGL}_6$.*

The cubic fourfolds containing a surface of the family $\mathcal{S}^{10,6}$ describe the divisor $\mathcal{C}_{38} \subset \mathcal{C}$ of fourfolds of discriminant 38.

Theorem 2.7 ([30, 32, 31]). *A general surface $[S] \in \mathcal{S}^{10,6}$ admits a congruence of 5-secant conics: through the general point of $\mathbb{P}^5$ there passes a unique conic curve which is 5-secant to S.*

The general conic curve of this congruence can be realized as the general fiber of the dominant map

$$\mu : \mathbb{P}^5 \dashrightarrow \mathbb{P}^4$$

defined by the linear system $|\mathcal{I}_S^2(5)|$ of quintic hypersurfaces with double points along S.

If X is a general cubic fourfold containing S, then the restriction of μ induces a birational map $\mu|_X : X \dashrightarrow^{\simeq} \mathbb{P}^4$

2.4. **GM fourfolds of discriminant** 10.

2.4.1. *τ-quadric surfaces.* A *τ-quadric* surface is a two-dimensional linear section of a Schubert variety $\Sigma_{1,1} \simeq \mathbb{G}(1,3) \subset \mathbb{G}(1,4)$. Thus the class of such a surface in $\mathbb{G}(1,4)$ is $\sigma_1^2 \cdot \sigma_{1,1} = \sigma_{3,1} + \sigma_{2,2}$. A standard parameter count (see [6, Proposition 7.4], and also [36]) shows that the closure inside $\mathcal{M}_{GM}^4$ of the family of fourfolds containing a τ-quadric surface forms the divisor $(\mathcal{M}_{GM}^4)'_{10} \subset \mathcal{M}_{GM}^4$, one of the two irreducible components of the Noether-Lefschetz locus in $\mathcal{M}_{GM}^4$ parametrizing fourfolds of discriminant 10. In particular, since the family of τ-quadric surfaces in $\mathbb{G}(1,4)$ is unirational, we deduce that the divisor $(\mathcal{M}_{GM}^4)'_{10}$ is also unirational.

Theorem 2.8 ([6]; see also [18])**.** *The projection of a general fourfold $[X] \in (\mathcal{M}_{GM}^4)'_{10}$ containing a τ-quadric surface S, from the linear span $\langle S \rangle \simeq \mathbb{P}^3$ of S, gives a birational map $X \dashrightarrow \mathbb{P}^4$.*

2.4.2. *Quintic del Pezzo surfaces.* A quintic del Pezzo surface can be realized as a two-dimensional linear section of $\mathbb{G}(1,4)$. Thus the class of such a surface in $\mathbb{G}(1,4)$ is $\sigma_1^4 = 3\sigma_{3,1} + 2\sigma_{2,2}$. A standard parameter count (see [6, Proposition 7.7], and also [36]) shows that the closure inside $\mathcal{M}_{GM}^4$ of the family of fourfolds containing a quintic del Pezzo surface forms the divisor $(\mathcal{M}_{GM}^4)''_{10} \subset \mathcal{M}_{GM}^4$; one of the two irreducible components of the Noether-Lefschetz locus in $\mathcal{M}$ of fourfolds of discriminant 10. In particular, since the family of quintic del Pezzo surfaces in $\mathbb{G}(1,4)$ is unirational, we deduce that the divisor $(\mathcal{M}_{GM}^4)''_{10}$ is also unirational.

Theorem 2.9 ([29]; see also [6, 9, 35])**.** *The projection of a general fourfold $[X] \in (\mathcal{M}_{GM}^4)''_{10}$ containing a quintic del Pezzo surface S, from the linear span $\langle S \rangle \simeq \mathbb{P}^5$ of S, induces a dominant map $X \dashrightarrow \mathbb{P}^2$ whose generic fiber is a quintic del Pezzo surface. In particular, X is rational. Indeed, by a classic result of Enriques, a quintic del Pezzo surface defined over an infinite field K is rational over K.*

2.5. **GM fourfolds of discriminant** 20. Throughout this subsection, we continue to let $\mathbb{Y}^5 \subset \mathbb{P}^8$ denote a fixed del Pezzo fivefold.

Recall first two well-known ways to parametrize $\mathbb{Y}^5$ over its field of definition.

(1) If $P \subset \mathbb{Y}^5$ is a plane in $\mathbb{Y}^5$ with class $\sigma_{2,2}$ in $\mathbb{G}(1,4)$, then the projection of $\mathbb{Y}^5$ from P gives a birational map $\mathbb{Y}^5 \dashrightarrow \mathbb{P}^5$, whose inverse is defined by the linear system of quadrics through a rational normal cubic scroll $\Sigma_3 \subset \mathbb{P}^4 \subset \mathbb{P}^5$.

(2) If $C \subset \mathbb{Y}^5$ is a conic such that its linear span P is not contained in $\mathbb{Y}^5$, then the projection of $\mathbb{Y}^5$ from P gives a birational map $\mathbb{Y}^5 \dashrightarrow \mathbb{P}^5$, whose inverse is defined by the linear system of cubics through a rational quartic scroll $\Theta_4 \subset \mathbb{P}^5$, obtained as a general projection of a rational normal threefold scroll in $\mathbb{P}^6$.

Now we breafly recall the construction due to [32, 18] of a 25-dimensional unirational family $\mathcal{S}^{9,2} \subset \mathrm{Hilb}_{\mathbb{Y}^5}$ of smooth rational surfaces of degree 9 and genus 2 having class $6\sigma_{3,1} + 3\sigma_{2,2}$ in the Chow ring of $\mathbb{G}(1,4)$.

Let $T \subset \mathbb{P}^6$ be the image of the plane via the linear system of quartic curves having 8 general base points $p_1, \ldots, p_8$. Thus T is a smooth rational surface of degree 8 and sectional genus 3 cut out by 7 quadrics. These 7 quadrics define a *special* Cremona transformation

$$\varphi : \mathbb{P}^6 \dashrightarrow \mathbb{P}^6$$

of type $(2,4)$, which has been classically studied in [34] (see also [19]).

Let us recall a bit of geometry from the papers [34, 19]. The pencil of plane cubics through the 8 base points $p_1, \ldots, p_8$ yields a pencil of elliptic normal quartic curves on T passing through a special point $q \in T$. The union of the linear spans of these curves gives a cone of vertex q over a Segre threefold $\mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$.

Let $H \simeq \mathbb{P}^5 \subset \mathbb{P}^6$ be a general hyperplane in $\mathbb{P}^6$. The restriction of φ to H gives a birational map

$$\varphi|_H : \mathbb{P}^5 \dashrightarrow Z \subset \mathbb{P}^6$$

onto a quartic hypersurface $Z \subset \mathbb{P}^6$, whose base locus, that is the intersection of T with the hyperplane H, is an octic curve $C \subset \mathbb{P}^5$ of arithmetic genus 3 contained in a Segre threefold $\Sigma \simeq \mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$. The image $\overline{\varphi(\Sigma)}$ is a smooth quadric surface $Q \subset Z \subset \mathbb{P}^6$ (which, considered with a double scheme structure, is a component of the base locus of the inverse of $\varphi|_H$). The pullback via the restriction $\varphi|_H$ of a line in one of the two pencils of lines on Q is a $\mathbb{P}^2$ of the ruling of Σ, while the pullback of a general line in the other pencil of lines on Q is a smooth quintic del Pezzo surface containing C. In particular, the curve C is the base locus of a pencil $\{D_\lambda\}_\lambda$ of quintic del Pezzo surfaces contained in Σ and whose general member is smooth.

Everything we have said about C continues to hold true even if we take the hyperplane $H \subset \mathbb{P}^6$ to be general among the hyperplanes containing a general tangent plane to T. But in this case (and only in this case), the curve C has a node and it can be embedded in a rational quartic scroll $\Theta_4 \subset \mathbb{P}^5$ as the one considered above. Indeed, such a nodal curve C can be realized as a nodal projection of a smooth curve of degree 8 and genus 2 contained in a smooth rational normal quartic scroll threefold in $\mathbb{P}^6$; see [32, Section 4] for more details on this last step. Then the birational map $\mathbb{P}^5 \dashrightarrow \mathbb{Y}^5 \subset \mathbb{P}^8$ defined by the linear system of cubics through Θ_4 induces an isomorphism between a general quintic del Pezzo surface of the pencil $\{D_\lambda\}_\lambda$ with a smooth rational surface $S \subset \mathbb{Y}^5 \subset \mathbb{P}^8$ of degree 9 and sectional genus 2, cut out by 19 quadrics, and having class $6\sigma_{3,1} + 3\sigma_{2,2}$ in $\mathbb{G}(1,4)$.

Theorem 2.10 ([32], see also [18])**.** *The surfaces $S \subset \mathbb{Y}^5$ produced by the construction above form an irreducible unirational family $\mathcal{S}^{9,2} \subset \mathrm{Hilb}_{\mathbb{Y}^5}$ of dimension 25.*

The closure of the family of quadric hypersurfaces in $\mathbb{Y}^5$ containing a surface of the family $\mathcal{S}^{9,2}$, after passing to the quotient modulo PGL_9, describes the divisor $(\mathcal{M}^4_{GM})_{20} \subset \mathcal{M}^4_{GM}$, the irreducible component of the Noether-Lefschetz locus in $\mathcal{M}^4_{GM}$ of fourfolds of discriminant 20.

Remark 2.11. An implementation of the construction of the family $\mathcal{S}^{9,2}$ is provided by the *Macaulay2* package *SpecialFanoFourfolds* [14, 37]. In particular, one is able to find explicit equations of a general member of the family.

Remark 2.12 ([36]). Let $\Sigma_3 \subset \mathbb{P}^4 \subset \mathbb{P}^5$ be a rational cubic scroll surface, and let $\psi : \mathbb{P}^5 \dashrightarrow \mathbb{Y}^5 \subset \mathbb{P}^8$ be the birational map defined by the quadrics through Σ_3. Take $D \subset \mathbb{P}^5$ to be a quintic del Pezzo surface intersecting Σ_3 along a hyperplane section of Σ_3. Then the restriction of ψ induces an isomorphism between D and a surface S belonging to the family $\mathcal{S}^{9,2}$. However, the surfaces S obtained by this "simplified" construction do not describe the whole family $\mathcal{S}^{9,2}$.

Theorem 2.13 ([18]). *Let $S \subset \mathbb{Y}^5$ be a surface corresponding to a general member of the family $\mathcal{S}^{9,2}$. Then S admits inside $\mathbb{Y}^5$ a congruence of 3-secant conic curves, that is, through the general point of $\mathbb{Y}^5$ there passes a unique conic which is 3-secant to S and is contained in $\mathbb{Y}^5$.*

The general conic curve of this congruence can be realized as the general fiber of the dominant map

$$\mu : \mathbb{Y}^5 \dashrightarrow \mathbb{P}^4$$

defined by the linear system $|\mathcal{I}^2_{S,\mathbb{Y}^5}(5)|$ of quintic hypersurfaces in $\mathbb{Y}^5$ with double points along S.

If X is a general quadric hypersurface in $\mathbb{Y}^5$ containing S, then the restriction of μ induces a birational map $\mu|_X : X \overset{\simeq}{\dashrightarrow} \mathbb{P}^4$.

Remark 2.14. The above theorem suggests an alternative construction for the general surface $S \subset \mathbb{Y}^5$ of the family $\mathcal{S}^{9,2}$, which historically was the first to be discovered [18]. Indeed, the inverse map of $\mu|_X : X \overset{\simeq}{\dashrightarrow} \mathbb{P}^4$ is defined by the linear system $|\mathcal{I}^2_{U,\mathbb{P}^4}(9)|$ of hypersurfaces of degree 9 singular along a surface U which is obtained as an internal projection of a triple projection of a minimal K3 surface of degree 20 and genus 11 in $\mathbb{P}^{11}$. We can reverse this construction by starting with a general K3 surface of degree 20 and genus 11 in $\mathbb{P}^{11}$.

2.6. **Cubic fourfolds of discriminant** 42. The 25-dimensional family $\mathcal{S}^{9,2} \subset \mathrm{Hilb}_{\mathbb{Y}^5}$ considered in the previous subsection and the 3-dimensional family of planes in $\mathbb{Y}^5$ with class $\sigma_{2,2}$ can be combined together to get a family of surfaces in $\mathbb{P}^5$ of dimension $48 = 25 + 3 - \dim \mathrm{Aut}(\mathbb{Y}^5) + \dim \mathrm{Aut}(\mathbb{P}^5)$. Indeed, let $S \subset \mathbb{Y}^5$ be a surface corresponding to a general member of the family $\mathcal{S}^{9,2}$, and let $P \subset \mathbb{Y}^5$ be a general plane with class $\sigma_{2,2}$ in $\mathbb{G}(1,4)$. Then the projection of S from P gives a rational surface $\widetilde{S} \subset \mathbb{P}^5$ of degree 9 and sectional genus 2, cut out by 9 cubics and having 5 non-normal nodes. Let us denote by $\widetilde{\mathcal{S}^{9,2}} \subset \mathrm{Hilb}_{\mathbb{P}^5}$ the (closure of the) family of surfaces $\widetilde{S}$ obtained as above.

Theorem 2.15 ([32]). *The family $\widetilde{\mathcal{S}^{9,2}} \subset \mathrm{Hilb}_{\mathbb{P}^5}$ is irreducible and unirational of dimension 48.*

The cubic fourfolds containing a surface of the family $\widetilde{\mathcal{S}^{9,2}}$ describe the divisor $\mathcal{C}_{42} \subset \mathcal{C}$ of fourfolds of discriminant 42.

Theorem 2.16 ([32]). *Let $\widetilde{S} \subset \mathbb{P}^5$ be a surface corresponding to a general member of the family $\widetilde{\mathcal{S}^{9,2}}$. Then $\widetilde{S}$ admits a congruence of 8-secant twisted cubic curves, that is, through the general point of $\mathbb{P}^5$ there passes a unique twisted cubic which is 8-secant to $\widetilde{S}$.*

The general cubic curve of this congruence can be realized as the general fiber of the dominant map

$$\mu : \mathbb{P}^5 \dashrightarrow W \subset \mathbb{P}^7$$

onto a smooth del Pezzo fourfold $W = \mathbb{G}(1,4) \cap \mathbb{P}^7$, defined by the linear system $|\mathcal{I}^3_{S,\mathbb{P}^5}(8)|$ of octic hypersurfaces with triple points along $\widetilde{S}$.

If X is a general cubic fourfold containing S, then the restriction of μ induces a birational map $\mu|_X : X \overset{\sim}{\dashrightarrow} W$.

Remark 2.17. It follows from well-known classic results that a del Pezzo fourfold $W = \mathbb{G}(1,4) \cap \mathbb{P}^7$ defined over an infinite field K is rational over K.

3. Main Theorem

The goal of this Section is to prove the following result.

Theorem 3.1. *The moduli spaces of n-pointed fourfolds over the following moduli loci are unirational:*

- (1) $\mathcal{C}_{14}$: *cubic fourfolds containing a quintic del Pezzo surface (Subsection 2.1);*
- (2) $\mathcal{C}_{26}$: *cubic fourfolds containing a 3-nodal septimic scroll (Subsection 2.2);*
- (3) $\mathcal{C}_{38}$: *cubic fourfolds containing a generalized Coble surface (Subsection 2.3);*
- (4) $\mathcal{C}_{42}$: *cubic fourfolds containing a 5-nodal rational surface of degree 9 and sectional genus 2 (Subsection 2.6);*
- (5) $(\mathcal{M}^4_{GM})'_{10}$: *GM fourfolds containing a τ-quadric surface (Subsection 2.4.1);*
- (6) $(\mathcal{M}^4_{GM})''_{10}$: *GM fourfolds containing a quintic del Pezzo surface (Subsection 2.4.2);*
- (7) $(\mathcal{M}^4_{GM})_{20}$: *GM fourfolds containing a smooth rational surface of degree 9 and sectional genus 2 (Subsection 2.5);*

Remark 3.2. Recall that we denoted by $\mathcal{U} \subset H^0(\mathcal{O}_{\mathbb{P}^5}(3))$ (respectively, $\mathcal{V} \subset H^0(\mathcal{O}_{\mathbb{Y}^5}(2))$) the open set parametrizing smooth cubic hypersurfaces in $\mathbb{P}^5$ (respectively, smooth quadric hypersurfaces in $\mathbb{Y}^5$). Let $\mathcal{S}$ be a family of surfaces in the the Hilbert scheme $\mathrm{Hilb}_{\mathbb{P}^5}$ of $\mathbb{P}^5$ (respectively, in the Hilbert scheme $\mathrm{Hilb}_{\mathbb{Y}^5}$ of $\mathbb{Y}^5$). We will abuse slightly of notation by denoting by $X_{\mathcal{S}}$ both the cubic fourfold incidence correspondence $\{(S,X) \in \mathcal{S} \times \mathcal{U}|S \subset X\}$ and the GM incidence correspndence $X_{\mathcal{S}} := \{(S,X) \in \mathcal{S} \times \mathcal{V}|S \subset X\}$. Accordingly, we will denote by $\mathcal{X}_{\mathcal{S}}$ the image of $X_{\mathcal{S}}$ inside $\mathcal{U}$ (respectively, inside $\mathcal{V}$) via the second projection, and define $\widetilde{\mathcal{X}_{\mathcal{S}}}$ as the closure $\overline{\mathcal{X}_{\mathcal{S}}//\mathrm{SL}_6}$ inside the moduli space $\mathcal{C}$, or, respectively, $\overline{\mathcal{X}_{\mathcal{S}}//\mathrm{Aut}(\mathbb{Y}^5)}$ inside $\mathcal{M}^4_{GM}$. We observe that the families of fourfolds described in Section 2 (and object of Theorem 3.1) all share the following properties:

- (1) $\mathcal{S}$ is irreducible and unirational; so that the same holds true for the corresponding family $\mathcal{X}_{\mathcal{S}}$, and hence for $\widetilde{\mathcal{X}_{\mathcal{S}}}$.
- (2) If (S,X) is a couple where S is a general member of the family $\mathcal{S}$ and X is a general fourfold containing S, then we are able to build, starting from the pair (S,X), an explicit birational map $\psi_{(S,X)} : \mathbb{P}^4 \overset{\sim}{\dashrightarrow} X$, defined over the same field of definition as S and X.

Remark 3.3. The family $\mathcal{X}_{\mathcal{S}}$ carries the universal 1-pointed fourfold $\mathcal{X}_{\mathcal{S},1} \to \mathcal{X}_{\mathcal{S}}$. And inductively one can define a tower of maps

$$\cdots \to \mathcal{X}_{\mathcal{S},n} \to \mathcal{X}_{\mathcal{S},n-1} \to \cdots \to \mathcal{X}_{\mathcal{S},1} \to \mathcal{X}_{\mathcal{S}}.$$

By quotienting out by the automorphisms of $\mathbb{P}^5$ or $\mathbb{Y}^5$, we can give straight away the following definition.

Definition 3.4. We will denote by $\widetilde{\mathcal{X}_{\mathcal{S},n}}$ the moduli space of n-pointed (cubic or GM) fourfolds, that contain one (or more) surfaces from the family $\mathcal{S}$. It is the quotient of $\mathcal{X}_{\mathcal{S},n}$ by the automorphisms of the ambient fivefold ($\mathbb{P}^5$ or $\mathbb{Y}^5$ for the corresponding fourfold.

Remark 3.5. The existence over an open subset of $\widetilde{\mathcal{X}_{\mathcal{S}}}$ of the moduli spaces in Definition 3.4 is guaranteed by the fact that the very general cubic fourfold and GM fourfold has no nontrivial automorphisms ([13, Theorem 3.8], [7, Proposition 3.21]). In fact, a family of cubic fourfolds with at least a non-trivial projective automorphism has dimension at most 14 [13, Theorem 3.8]. Since we are working in the birational category, it will be enough to consider the subset given by smooth cubic or smooth GM fourfolds with no non-trivial automorphism. This subset carries by definition a universal family, and contains an open subset of the moduli space. The universal family over this open subset is clearly birational to the universal family over the full moduli space.

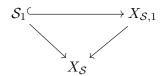Of course there are forgetful maps

$$\widetilde{\mathcal{X}_{\mathcal{S},n}} \overset{\pi_n}{\to} \widetilde{\mathcal{X}_{\mathcal{S},n-1}} \overset{\pi_{n-1}}{\to} \widetilde{\mathcal{X}_{\mathcal{S},n-2}} \to \cdots,$$

with evident meaning.

Proof of Theorem 3.1. Let us consider the unirational family $\mathcal{S}$ of surfaces inside $\mathbb{P}^5$ (resp., $\mathbb{Y}^5$). Over the function field of $\mathcal{S}$, we can write the equations of the surface $S \in \mathcal{S}$, and notably describe their ideal. Call m the dimension of $H^0(\mathbb{P}^5, \mathcal{I}_S(3))$ (resp., $H^0(\mathbb{Y}^5, \mathcal{I}_S(2))$). By semicontinuity, this is generically constant over $\mathcal{S}$. This implies that the incidence correspondence $X_{\mathcal{S}}$ is birational to a $\mathbb{P}^{m-1}-$bundle over $\mathcal{S}$ and hence it is unirational.

By definition $X_{\mathcal{S}}$ sits naturally inside the product $\mathcal{S} \times \mathcal{U}$, and has two natural projections to the two components. The image under the second projection is $\mathcal{X}_{\mathcal{S}} \subset \mathcal{U}$ and is unirational as well.

Now we observe that, by definition, over $X_{\mathcal{S}}$ there is a natural *double* universal family. That is: since $X_{\mathcal{S}}$ parametrizes the couples (S, X), then on one side we have the universal family $\mathcal{S}_1$ of 1-pointed surfaces - that is, the universal family of surfaces over $\mathcal{S}$, pulled-back to $X_{\mathcal{S}}$, on the other we have the universal family $X_{\mathcal{S},1}$ of 1-pointed cubic (resp., GM) fourfolds with the forgetful map $X_{\mathcal{S},1} \overset{\pi}{\to} X_{\mathcal{S}}$. By definition there is a fiberwise inclusion:

$$
\begin{array}{ccc}
\mathcal{S}_1 & \hookrightarrow & X_{\mathcal{S},1} \\
& \searrow \quad \swarrow & \\
& X_{\mathcal{S}} &
\end{array}
$$

Remark that different points of $X_{\mathcal{S}}$ may parametrize the same Fano fourfold but a different surface. We observe that $X_{\mathcal{S},1}$ lives naturally inside $\mathcal{S} \times \mathcal{U} \times \mathbb{P}^5$ and it is easy to see that the family $\mathcal{X}_{\mathcal{S},1}$ is the image of $X_{\mathcal{S},1}$ into $\mathcal{U} \times \mathbb{P}^5$ via the map that forgets the surface. That is, we have a commutative diagram

$$
\begin{array}{ccc}
X_{\mathcal{S},1} & \longrightarrow & \mathcal{X}_{\mathcal{S},1} \subset \mathcal{U} \times \mathbb{P}^5 \\
\downarrow & & \downarrow \\
X_{\mathcal{S}} & \longrightarrow & \mathcal{X}_{\mathcal{S}} \subset \mathcal{U}.
\end{array}
$$

Now, by assumption (2) of our working hypotheses at the beginning of this section, we have that - relatively over X_S - we can define a linear system on $X_{\mathcal{S},1}$ (with base locus supported on $\mathcal{S}_1$) that defines a birational map from $X_{\mathcal{S},1}$ to a $\mathbb{P}^4$-bundle over $X_{\mathcal{S}}$, that we denote $\mathbb{P}^4_{X_{\mathcal{S}}}$. Now $\mathbb{P}^4_{X_{\mathcal{S}}}$ is rational over $X_{\mathcal{S}}$ and $X_{\mathcal{S}}$ is unirational, hence the universal family $X_{\mathcal{S},1}$ is unirational, since it is birational to $\mathbb{P}^4_{X_{\mathcal{S}}}$.

Exactly as one does for $X_{\mathcal{S}}$, we can construct a universal cubic (resp., GM) fourfold over $X_{\mathcal{S},1}$, just by taking the pull-back $\pi^* X_{\mathcal{S},1}$ over $X_{\mathcal{S},1}$. We denote by $X_{\mathcal{S},2}$ this family, and we observe that it tautologically contains $\pi^* \mathcal{S}_1$, as the following diagram follows.
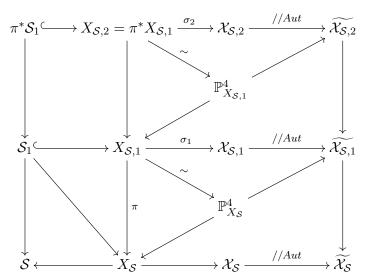
$$
\begin{array}{ccc}
\pi^* \mathcal{S}_1 & \hookrightarrow & X_{\mathcal{S},2} = \pi^* X_{\mathcal{S},1} \\
\downarrow & & \downarrow \\
\mathcal{S}_1 & \hookrightarrow & X_{\mathcal{S},1}
\end{array}
$$

Thus $X_{\mathcal{S},2}$ has the same property (2) as $X_{\mathcal{S},1}$ and one can define a relative linear system defining the birationality between $X_{\mathcal{S},2}$ and a $\mathbb{P}^4$-bundle over $X_{\mathcal{S},1}$ - that we denote by $\mathbb{P}^4_{X_{\mathcal{S},1}}$. By the same argument as above, since $X_{\mathcal{S},1}$ is unirational, $X_{\mathcal{S},2}$ is unirational as well. Then, inductively, the same argument shows the unirationality of the universal families $X_{\mathcal{S},n}$, for all n.

Now, the maps $\sigma_i : X_{\mathcal{S},i} \to \mathcal{X}_{\mathcal{S},i}$ forgetting the surfaces followed by the natural classifying maps, given by the quotient by the automorphisms groups, make the following diagram commute (that of course can be indefinitely vertically extended by induction).

$$\begin{array}{ccccccc} \pi^*\mathcal{S}_1 \hookrightarrow X_{\mathcal{S},2} = \pi^*X_{\mathcal{S},1} & \xrightarrow{\sigma_2} & \mathcal{X}_{\mathcal{S},2} & \xrightarrow{//Aut} & \widetilde{\mathcal{X}_{\mathcal{S},2}} \\ & \searrow{\sim} & & \nearrow & \\ & & \mathbb{P}^4_{X_{\mathcal{S},1}} & & \\ \mathcal{S}_1 \hookrightarrow X_{\mathcal{S},1} & \xrightarrow{\sigma_1} & \mathcal{X}_{\mathcal{S},1} & \xrightarrow{//Aut} & \widetilde{\mathcal{X}_{\mathcal{S},1}} \\ & \searrow{\sim} & & \nearrow & \\ & & \mathbb{P}^4_{X_{\mathcal{S}}} & & \\ \mathcal{S} \leftarrow X_{\mathcal{S}} & \longrightarrow & \mathcal{X}_{\mathcal{S}} & \xrightarrow{//Aut} & \widetilde{\mathcal{X}_{\mathcal{S}}} \end{array}$$

It is then clear that also the corresponding moduli spaces $\widetilde{\mathcal{X}_{\mathcal{S},n}}$, corresponding to the families $\mathcal{X}_{\mathcal{S},n}$ are unirational.

To conclude the proof we observe that, thanks to the properties (1) and (2), we can plug any one of the seven loci mentioned in the claim inside this construction, and get the result. □

Remark 3.6. There are other special families of cubic (resp., GM) fourfolds verifying the hypotheses required in this section. We nevertheless decided to concentrate on certain particular descriptions of codimension one loci.

In fact we did not only choose some codimension 1 loci, but we also chose a particular description of them. For example, for cubic fourfolds in $\mathcal{C}_{14}$ we could have chosen quartic scrolls as surfaces defining the divisor. We remark however that in that case our argument would not have worked since the quartic scroll defines a birational map to a 4-dimensional quadric, and a quadric bundle is not automatically rational over its base.

A celebrated codimension two locus of cubic fourfold for which our argument would work is the locus of cubic fourfolds containing two planes [38].

Remark 3.7. A few words are in order to describe the difference between our result here and [2, Theorem 4.10], where the unirationality of $\mathcal{C}_{d,n}$ is proven for any n and $8 \le d \le 42$. The result of [2] holds for a larger set of Hassett divisors (for instance our proof here does not work for $\mathcal{C}_8$ since the rationality of all cubics is essential to us), but on the other hand it does only work for cubic fourfolds, relying on a result of Kollár for cubic hypersurfaces. Theorem 3.1 is more abstract in its formulation, it only depends on some general features of the Fano varieties considered, and on the families of surfaces they contain. This implies that it holds also for some classes of GM fourfolds, and we believe that, because of its generality, by mimicking its strategy one could obtain similar results for other Fano fourfolds of K3 type, or even higher dimensional Fano varieties. Of course a necessary condition is the knowledge of families of rational surfaces contained in other Fano fourfolds (or possibly rational threefolds in Fano sixfolds, etc.). A wealth of good candidates can be certainly found in the recent preprint [4].

4. A RATIONAL NOETHER-LEFSCHETZ DIVISOR OF GENUS 11 K3 SURFACES, AND THEIR ASSOCIATED GM FOURFOLDS.

In this section we will consider a codimension 1 locus of $(\mathcal{M}_{GM}^4)_{20}$, in order to show that, once we restrict the family of GM-fourfolds to this locus, stronger rationality statements hold.
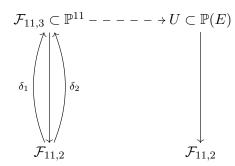
As observed in Remark 2.14 (see also [18, 32]), for the generic $X \in (\mathcal{M}_{GM}^4)_{20}$ there exists a birational map $\mathbb{P}^4 - - \to X$, defined by the linear system of hypersurfaces of degree 9 having double points along a surface U, which is a projection of a genus 11 K3 surface.

More precisely, one starts from a K3 surface $Z \subset \mathbb{P}^{11}$ of degree 20 and sectional genus 11. We take two points $p, q \in Z$, and perform first a triple projection from p (*i.e.* a linear projection with center a $\mathbb{P}^5 \subset \mathbb{P}^{11}$ intersecting Z with multiplicity 3 in p) to $\mathbb{P}^5$, then a simple projection from q to $\mathbb{P}^4$. The image is the required surface U.

With this in mind, one can prove the following result in a new fashion. We remark that this was already a part of our Theorem 3.1, but here we give a sketch of this different proof, based on the birational geometry of certain moduli spaces of K3 surfaces, since it will be necessary in the following.

Theorem 4.1. *The universal family $(\mathcal{M}_{GM}^4)_{20,1}$ of 1-pointed GM-fourfolds is unirational.*

Proof. The philosophy is to mimic the above rationality construction over $\mathcal{F}_{11,2}$. We will then consider the moduli space $\mathcal{F}_{11,2}$ of polarized K3 surface of genus 11 with two marked points. The moduli space $\mathcal{F}_{11,3}$ comes equipped with an embedding inside a $\mathbb{P}^{11}$-bundle over $\mathcal{F}_{11,2}$ and with two sections $\delta_1, \delta_2 : \mathcal{F}_{11,2} \to \mathcal{F}_{11,3}$. Performing a relative triple projection from the image of δ_1 and a simple one from the image of δ_2 we obtain a $\mathbb{P}^4$-bundle $\mathbb{P}(E)$ over $\mathcal{F}_{11,2}$ containing the family U of degree 10 surfaces.

$$\mathcal{F}_{11,3} \subset \mathbb{P}^{11} - - - - - \to U \subset \mathbb{P}(E)$$

Since $\mathcal{F}_{11,2}$ is unirational [3, Theorem 0.1], the projective bundle $\mathbb{P}(E)$ is unirational. The relative linear system of degree 9 hypersurfaces, with multiplicity two along U gives a rational dominant map between the $\mathbb{P}^4$-bundle $\mathbb{P}(E)$ and $(\mathcal{M}_{GM}^4)_{20,1}$ □

In [18], Hoff and the third named author also consider a codimension one subfamily of genus 11 K3 surfaces, that forms a Noether-Lefschetz (NL in what follows) divisor inside $\mathcal{F}_{11}$. This divisor seems particularly interesting under our point of view, since the wealth of geometry going on here allows us to strengthen our rationality results concerning the

corresponding universal families of GM fourfolds related to these K3 surfaces. But let us give a couple more details about these surfaces.

We start from a smooth Fano threefold Y of type $X_{22} \subset \mathbb{P}^{13}$. It is well known that the generic tangent hyperplane sections of Y are one-nodal (a double point) K3 surfaces (see [26], [33]). The projection from the node of such a K3 surface gives a K3 surface in $\mathbb{P}^{11}$, of degree 20 and sectional genus 11, containing a further conic (the exceptional divisor over the node). In fact, such a construction gives a Noether-Lefschetz divisor inside the 19-dimensional moduli space of K3 surfaces of genus 11, and the intersection lattice of these surfaces contains a sublattice of type

$$\begin{pmatrix} 20 & 2 \\ 2 & -2 \end{pmatrix}.$$

Before studying the universal family of GM fourfolds obtained from these special K3 surfaces, we need to show some results on the birational geometry of their Noether-Lefschetz locus. We will denote by $\mathcal{V}_n^{nod}$ the moduli space of $n-$pointed one nodal K3 surfaces of sectional genus 12, obtained by cutting a X_{22}-type 3fold with tangent hyperplanes as above. The generic element of $\mathcal{V}_n^{nod}$ is represented by a vector $(Y, p, H, q_1, \ldots, q_n)$, where Y is a Fano threefold of type X_{22}, p is a point of Y, H is a hyperplane tangent to Y in p, and $q_1, \ldots, q_n$ are n points on the surface $S_H := Y \cap H$. We will also denote by $\mathcal{X}_{22}$ the (rational, see [27]) moduli space of Fano threefolds of type X_{22}. All these Fano threefolds are rational, and we will need to fix one $\widetilde{X}_{22} \in \mathcal{X}_{22}$.

Theorem 4.2. *The moduli space $\mathcal{V}_n^{nod}$ is rational if $n \leq 9$.*

Proof. Let us consider the rational map

$$(4.1) \qquad\qquad \varphi : \mathcal{V}_n^{nod} \;\;\rightarrow\;\; \mathcal{X}_{22} \times \widetilde{X}_{22}^{n+1}$$
$$(4.2) \qquad\qquad (Y, p, H, q_1, \ldots, q_n) \;\;\mapsto\;\; (Y, p, q_1, \ldots, q_n).$$

Remark that $\mathcal{X}_{22} \times \widetilde{X}_{22}^{n+1}$ is rational (and of dimension $3n+9$) since it is the product of rational varieties. Then, the fiber of φ over $(Y, p, q_1, \ldots, q_n)$ is exactly the linear system of hyperplanes in $\mathbb{P}^{13}$ that are tangent to Y in p, and pass through $q_1, \ldots, q_n$. This shows that $\mathcal{V}_n^{nod}$ is birational to a $\mathbb{P}^{9-n}-$projective bundle over $\mathcal{X}_{22} \times \widetilde{X}_{22}^{n+1}$, and hence is rational if $n \leq 9$. $\qquad\square$

We recall that the projection from the node sends birationally $\mathcal{V}_n^{nod}$ onto a $(18+2n)$-dimensional NL locus inside $\mathcal{F}_{11,n}$. Let us denote by $(\mathcal{M}_{GM}^4)_{20}^{nod}$, the moduli space of GM fourfolds obtained from the NL K3 surfaces described above, and by $(\mathcal{M}_{GM}^4)_{20,1}^{nod}$ the universal family above, obtained by restricting the construction of Theorem 4.1. The moduli space $(\mathcal{M}_{GM}^4)_{20}^{nod}$ is of dimension 22, and is contained in $(\mathcal{M}_{GM}^4)_{20}$.

Corollary 4.3. *The universal family $(\mathcal{M}_{GM}^4)_{20,1}^{nod}$ is rational. The moduli space $(\mathcal{M}_{GM}^4)_{20}^{nod}$ is rational.*

Proof. The moduli space $\mathcal{V}_3^{nod}$ of nodal, 3-pointed K3 surfaces can be embedded in a $\mathbb{P}^{12}$-bundle, and endowed with two sections $\delta_1, \delta_2 : \mathcal{V}_2^{nod} \to \mathcal{V}_3^{nod}$, over $\mathcal{V}_2^{nod}$. Since we are working in the birational category, we can even consider (at least an open subset of)

$\mathcal{V}_2^{nod}$ as contained in $\mathcal{F}_{11,2}$. Now, we project fiberwise off the node, obtaining a family of NL K3 surfaces in a $\mathbb{P}^{11}$-bundle, with two sections, over $\mathcal{V}_2^{nod}$. Then, as we did in Theorem 4.1, we perform the two projections off the sections and we obtain a $\mathbb{P}^4$-bundle over $\mathcal{V}_2^{nod}$, containing a family $\mathcal{T}$ of degree 10 surfaces. The moduli space $\mathcal{V}_2^{nod}$ is rational, hence the $\mathbb{P}^4$-bundle is rational as well. Then, by applying the relative linear system of degree 9 hypersurfaces through $\mathcal{T}$ as in Theorem 4.1, we obtain a rational family of GM fourfolds over $\mathcal{V}_2^{nod}$, hence $(\mathcal{M}_{GM}^4)_{20,1}^{nod}$ is rational. By construction $(\mathcal{M}_{GM}^4)_{20}^{nod}$ is birational to $\mathcal{V}_2^{nod}$ and hence rational. $\qquad\square$

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