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ABSTRACT. Let $\mathcal{C}_d$ denote Hassett's Noether-Lefschetz divisor in the moduli space of cubic fourfolds given by cubics with intersection form of discriminant d. By imposing certain arithmetic conditions on the indexes d_i, we construct families of rational cubic fourfolds with an associated K3 surface with any prescribed Néron-Severi rank, contained in the intersection $\bigcap \mathcal{C}_{d_i}$ of a finite number of Hassett divisors. Then we apply this result to show that there exist one dimensional families of cubic fourfolds with finite dimensional Chow motive of Abelian type inside every divisor of special cubic fourfolds. This also implies Abelianity and finite dimensionality of the motive of related HyperKähler varieties, such as the Fano variety of lines and the LLSvS 8fold. A similar remark allows us to show the Abelianity of the motive of an infinity of LSV 10folds, and of other HyperKähler 10folds associated to the twisted intermediate Jacobian fibration of cubic fourfolds with an associated K3 surface. After that, starting from certain 4-dimensional families of K3 surfaces, we construct two families of Fano varieties whose motive either contains the finite dimensional motive of a K3 surface, or it is itself finite dimensional. Varieties from the first family are some quadric surface fibrations, and contain the finite dimensional motive of a K3 surface. Varieties from the second family are singular, and their motives are Schur-finite and Abelian in Voevodsky's triangulated category of motives.

CONTENTS

1. Introduction

Let $\mathcal{M}_{rat}(\mathbf{C})$ be the (covariant) category of Chow motives with $\mathbf{Q}$-coefficients and let $\mathcal{M}^{Ab}_{rat}(\mathbf{C})$ be the strictly full, thick, rigid, tensor subcategory of $\mathcal{M}_{rat}(\mathbf{C})$ generated by the motives of Abelian varieties. All the examples of motives that have been proven to be finite-dimensional, in the sense of Kimura-O'Sullivan, belong to the category $\mathcal{M}^{Ab}_{rat}(\mathbf{C})$. More precisely, the following classes of smooth projective varieties are known to have motives belonging to $\mathcal{M}^{Ab}_{rat}(\mathbf{C})$:

(1) projective spaces, Grassmannian varieties, projective homogeneous varieties, toric varieties;
(2) smooth projective curves;
(3) Kummer K3 surfaces;
(4) K3 surfaces with Picard numbers at least 19;
(5) K3 surfaces with a non-symplectic group of automorphisms acting trivially on the algebraic cycles: K3 surfaces satisfying these conditions have Picard numbers equal to $2, 4, 6, 10, 12, 16, 18, 20$, see [48];
(6) Hilbert schemes of points on Abelian surfaces;
(7) Fermat hypersurfaces ;
(8) Cubic 3-folds and their Fano surfaces of lines, see [25] and [20].

In this paper, we consider the case of cubic fourfolds, in relation with certain families of K3 surfaces.

According to Kimura's Conjecture on finite dimensional motives and by the results in [5] the Chow motive $h(S)$ of a complex K3 surface S should be of Abelian type. Also, by the work of Kuga and Satake, if one assumes the Hodge conjecture, every K3 surface over $\mathbf{C}$ is of Abelian Hodge type, i.e. there exists an algebraic correspondence between any K3 surface S and an associated Abelian variety, the *Kuga-Satake variety* $K(S)$. This would imply that the motive of any such K3 surface is Abelian, i.e. it lies in the subcategory $\mathcal{M}^{Ab}_{rat}(\mathbf{C})$ of the (covariant) category $\mathcal{M}_{rat}(\mathbf{C})$ of Chow motives generated by the motives of curves. On the other hand, cubic fourfolds have been a very active field of research in the last few years, for several reasons. The rationality of the generic cubic is a very classical, and still unanswered, question in algebraic geometry. A lot of energy has been spent in order to find a good invariant that would detect the required birational properties, and tentatives have been made via Hodge theory, derived categories, Chow motives, algebraic cycles, etc. In most of these papers K3 surfaces appear as an important presence (in the cohomology, derived category, related HyperKähler varieties, ...) whenever the cubic fourfolds are rational, or suspected to be rational.

Hassett [28] introduced the notion of "*special*" cubic fourfold, that is a cubic that contains an algebraic surface not homologous to a complete intersection. These fourfolds form a countable infinite union of divisors $\mathcal{C}_d$ called *Hassett's Noether-Lefschetz divisors* (for short Hassett divisors) inside the moduli space $\mathcal{C}$ of smooth cubic fourfolds, which is a 20-dimensonal quasi-projective variety. Hassett showed that $\mathcal{C}_d$ is irreducible and nonempty if and only if $d \geq 8$ and $d \equiv 0, 2$ [6]. Only few $\mathcal{C}_d$ have been defined explicitly in terms of surfaces contained in a general element of these divisors (see [28], [27], [50], [49] , [3], [16]). More recently, the first and second named authors have deployed similar techniques to study the birational geometry of universal cubics over certain divisors $\mathcal{C}_d$ [10]. Hassett [28] proved that, for an infinite set of values of d, one can associate a polarized K3 surface (S, f) of degree d to a cubic fourfold in $\mathcal{C}_d$. This is true for d satisfying $4 \nmid d, 9 \nmid d$, and $p \nmid d$ for any odd prime number $p \equiv 2$ [3], and the association between a cubic X and S is essentially an isomorphism of Hodge structures between certain subgroups of their middle cohomologies (see Sect. 2). A natural conjecture, supported by Hassett's Hodge theoretical work ([28], [27], [29]) and Kuznetsov's derived categorical work [34], is that any rational cubic fourfold ought to have an associated K3 surface. A link between rationality and the transcendental motive of a cubic fourfold has also been described in [15]. For now, every fourfold in $\mathcal{C}_{14}$, $\mathcal{C}_{26}$, $\mathcal{C}_{38}$ and $\mathcal{C}_{42}$ has been proved to be rational (see [16], [50], [49]). These cubics all have an associated K3 surface.

Several recent papers [27, 3, 8, 9] study the intersection of couples of Hassett divisors in the moduli space, especially with applications to rationality. In particular, Yang and Yu [63] proved that any two Hassett divisors intersect, and more recently [64] that *all* Hassett divisors intersect. The first result of this paper lies in a similar frame of ideas, but it aims rather at constructing cubic fourfold with associated K3 surfaces with prescribed Néron-Severi lattice. Using lattice theoretical computations and a result of [63], we are able to prove the following:

Theorem 1.1. *Let $3 \leq n \leq 20$, and $\mathcal{C}_{d_1}, \ldots, \mathcal{C}_{d_n}$ n different Hassett divisors. Let us set $d_k \geq 8$, $d_k \equiv 0, 2 \ (mod\ 6)$ and $d_3, .., d_n = 6\prod_i p_i^2$ or $6\prod_i p_i^2 + 2$, p_i a prime number, then:*

1. *There exist families of cubic fourfolds $\mathcal{F} \subseteq \bigcap_{k=1}^{n} \mathcal{C}_{d_k} \neq \emptyset$, of dimension $20 - n$, such that for any cubic fourfold $X \in \mathcal{F}$ has $rk(A(X)) \geq n + 1$ (and $rk(A(X)) = n + 1$ for the generic one).*
2. *In addition, if at least one of the divisors $\mathcal{C}_{d_k}$ in the intersection parametrizes cubic fourfolds with an associated K3 surface, then cubic fourfolds in the families $\mathcal{F}$ have an associated K3 surface with a Néron-Severi group of rank greater than or equal to n.*

At least one of these 20 divisors can be chosen in a way such that its cubics have associated polarized K3 surface. This implies that, if one of the divisors has this feature, cubics in the intersection of the 20 divisors have an associated K3 with Néron-Severi group of rank 20, those that lie in the intersection of 19 divisors have associated K3 with rank 19 and so on.

In fact, few examples of cubic fourfolds belonging to the category $\mathcal{M}^{Ab}_{rat}(\mathbf{C})$ are known, mainly due to the beautiful work of Laterveer [39, 38, 37, 40]. Moreover, one would be interested to understand how these cubics are positioned in the geography of the moduli space, that is whether they belong to any of the divisors $\mathcal{C}_d$. In the second part of this paper, we start to fill these gaps, by constructing one dimensional families of cubic fourfolds with finite dimensional, Abelian motive. Moreover, these families are quite ubiquitous in the moduli space, since any Hassett divisor contains an infinity of them.

The construction of these families is basically the combination of three ingredients. First, the intersection theoretical machinery developed in the first part of the the paper, notably Thm. 1.1, allows us to construct one dimensional families of cubic fourfolds with associated K3 surfaces with NS rank 19. These families can be arbitrarly constructed inside any Hassett divisor. Then, we combine this with results of the second and third named authors [15] and Bülles [17], about the Chow-Künneth decomposition of the Chow motive of a cubic fourfold X. These allow us to reduce the finite dimensionality and Abelianity of $h(X)$ to the same properties of the associated K3 surface. Finally we apply results of the third named author [48] to obtain our second main theorem:

Theorem 1.2. *Every Hassett divisor $\mathcal{C}_d$ contains a countable infinity of one dimensional families of cubic fourfolds, whose Chow motive is finite dimensional and Abelian.*

Remark 1.3. *We observe that our construction allows us to construct some families of cubics, with properties as in Thm. 1.2, whose members are all rational. If Kuznetsov's conjecture [34] about the rationality of cubic fourfolds holds true, all the families we can construct would be made up of rational cubic fourfolds.*

Remark 1.4. *It is not hard to show, and we do so in Sect. 4.4, that cubic fourfolds with finite dimensional motive are dense for the complex topology inside Hassett divisors.*

As a consequence of our Theorem 1.2 we will also observe the finite dimensionality and Abelianity of the motives of certain related HyperKähler varieties, namely the Fano variety $F(X)$ of lines contained in a cubic fourfold X, and the Lehn-Lehn-Sorger-van Straten 8fold $L(X)$ constructed in [42].

In Section 5 we procced our study of motives of HyperKähler varieties. By combining results from Laza-Saccà-Voisin, Floccari-Fu-Zhang and some observations about the topology of $\mathcal{C}_{14}$ from [16], we show the existence of an infinity of examples of Laza-Saccà-Voisin HyperKähler 10-folds with finite dimensional and Abelian Chow motive. Adding to the picture some recent results of Li-Pertusi-Zhao [43] about stability conditions on the Kuznetsov component of the derived category of cubic fourfolds, we manage to show the same result for the twisted version of the HyperKähler 10fold, introduced by Voisin [62]. For the twisted version we find an infinite number of examples in each divisor $\mathcal{C}_d$ where cubic fourfolds have associated K3 surfaces.

In the two last sections of the paper, we somehow do reverse engineering. Starting from two explicit examples of families $\mathcal{F}$ and $\mathcal{G}$ of K3 surfaces, we manage to construct two families of Fano fourfolds, whose motives have remarkable properties.

The Fano fourfolds from the first family are smooth, notably they are quadric surface fibrations over $\mathbb{P}^2$. The relation with K3 surfaces is given by the Chow-Künneth decomposition of the motive of a quadric fibration due to Vial [58], and some results of Kuznetsov and Căldăraru. The corresponding K3 surfaces are in fact a 4-dimensional family $\mathcal{F}$ of smooth octic surfaces in $\mathbb{P}^5$, that are complete intersections of 3 quadrics (with particularly simple equations) in $\mathbb{P}^5$. Notably Q_1, Q_2, Q_3 are hypersurfaces in $\mathbb{P}^5$ [24, 10.2], whose equations are

$$\sum_{0\le i\le 5} a_i x_i^2 = 0; \sum_{0\le i\le 5} b_i x_i^2 = 0 \ ; \ \sum_{0\le i\le 5} c_i x_i^2 = 0,$$

with complex parameters a_i, b_i, c_i and $i = 0, \cdots, 5$. See Sect. 6.1 for details.

Theorem 1.5. *Let S be a general smooth K3 surface in $\mathcal{F}$ and let Q_1, Q_2 and Q_3 be the three quadrics in $\mathbb{P}^5$ cutting out S. Let $\mathcal{Q}' \to \mathbb{P}^2$ be the quadric surface fibration obtained as hyperbolic reduction of the quadric fourfold fibration $\mathcal{Q} \to \mathbb{P}^2$ defined by the quadrics Q_i. Then $h(S)(1)$ is a finite dimensional submotive of $h(\mathcal{Q}')$.*

Hyperbolic reduction is a classical way to obtain a quadric fibration of relative dimension $n-2$, starting from a quadric fibration of relative dimension n with a section. See Sect. 6 for more details.

The fourfolds from the second family are singular cubics, hence rational. For any cubic X of this family, there exists in fact a birational map $\psi_X : \mathbb{P}^4 \dashrightarrow X$ given by the full linear system of cubics through one 15-nodal, sextic K3 surface in $\mathbb{P}^4$ (see Sect. 7.3). Let us denote by $\mathcal{G}$ the family of these sextic surfaces; it is not hard to see that it is 4-dimensional. The motive of a desingularization of such a surface has been recently proven to be finite dimensional and of Abelian type [31]. This, combined with some not difficult birational transformations, allows us to show that the motive of the corresponding cubic fourfold is Schur-finite inside Voevodsky's triangulated category of motives $\mathbf{DM}_{\mathbf{Q}}(\mathcal{C})$.

Theorem 1.6. *Let S be any surface from the family $\mathcal{G}$, and X the cubic fourfold obtained via ψ_X, then X has Schur finite motive in $\mathbf{DM}_{\mathbf{Q}}(\mathcal{C})$, belonging to the subcategory $\mathbf{DM}^{Ab}_{\mathbf{Q}}$, generated by the motives of curves.*

Special families of cubic foufolds with the same property have been described by R. Laterveer in [38] and [37]

When needed, computations were done using Macaulay2 [26] and Sagemath [56].

Plan of the paper: In Section 2, we recall some generalities about the moduli space of cubic fourfolds. In Section 3, based on some lattice theoretical computations, we study the intersection of Hassett divisors proving Theorem 1.1. Most of the proofs of this section are contained in the final Appendix. In Section 4, we start by recalling some results about the Chow motives of cubic fourfolds and K3 surfaces. Then, by using also results from Sect. 3, we prove Theorem 1.2 and a couple of similar results about HyperKähler varieties. In Sect. 5 we recall that every K3 surface S in the first family $\mathcal{F}$ has a motive of Abelian type and there is a Kuga-Satake correspondence between $h(S)$ and the motive of a Prym variety P of dimension 4. Then we describe the second family $\mathcal{G}$ of K3 surfaces S in $\mathbb{P}^4$ with 15 nodes and show that the motive of a desingularization of S is of Abelian type. In Sect. 6 we show how to construct cubic fourfolds from K3 surfaces from the families $\mathcal{F}$ and $\mathcal{G}$, and show that their motives are of Abelian type as well.

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2. Generalities on cubic fourfolds

2.1. **Moduli space of cubic fourfolds.** A cubic fourfold X is a smooth complex cubic hypersurface in $\mathbb{P}^5$. The coarse moduli space of cubic fourfolds $\mathcal{C}$ is a 20-dimensional quasi-projective variety. It can be described as a GIT quotient $\mathcal{C} := \mathcal{U}//PGL(6, \mathbf{C})$, where $\mathcal{U}$ is the Zariski open subset of $|\mathcal{O}_{\mathbb{P}^5}(3)|$ parametrizing smooth cubic hypersurfaces in $\mathbb{P}^5$.

2.2. **Lattices of cubic fourfolds.** The cohomology of cubic fourfolds is torsion-free and the middle cohomology $H^4(X,\mathbb{Z})$ is the one containing nontrivial information about the geometry of X (see [13]). Let L be the cohomology group $H^4(X,\mathbb{Z})$, known as the *cohomology lattice*, and $L_{prim} = H^4(X,\mathbb{Z})_{prim} :=< h^2 >^{\perp}$, the *primitive cohomology lattice*, with h the class of a hyperplane section of X. Hassett [28] computed explicitly these lattices as follows:

$$\begin{aligned} L \simeq &\quad E_8^1 \bigoplus E_8^2 \bigoplus U^1 \bigoplus U^2 \bigoplus I_{3,0} \\ \simeq &\qquad\qquad (+1)^{\bigoplus 21} \bigoplus (-1)^{\bigoplus 2}, \end{aligned}$$

$$L_{prim} \simeq \quad E_8^1 \bigoplus E_8^2 \bigoplus U^1 \bigoplus U^2 \bigoplus A_2.$$

Let us explain shortly the notation used in the formulas here above:

(1) $I_{3,0}$ is a rank 3 lattice whose bilinear form is the identity matrix of rank 3 with
$$h^2 = (1,1,1) \in I_{3,0};$$

(2) E_8^1 and E_8^2 are two copies of E_8, the unimodular positive definite even rank 8 lattice associated to the corresponding Dynkin diagram represented by the following matrix in the basis $< t_k^i >$, for $k = 1,..,8$, $i = 1,2$:

$$E_8 = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix};$$

(3) U^1 and U^2 are two copies of the hyperbolic plane $U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, the basis of U^k consists of vectors e_1^k and e_2^k, $k = 1,2$ such that
$$(e_1^k, e_1^k) = 0, (e_1^k, e_2^k) = 1, (e_2^k, e_2^k) = 0;$$

(4) A_2 is a rank 2 lattice with bilinear form $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ in the canonical basis $< a_1,a_2 >$.

These notations will be useful later in the paper for lattice computations.

2.3. **Special cubic fourfolds.** Hassett ([28], [27]) studied cubic fourfolds via Hodge theory and introduced the notion of *special* cubic fourfolds, that is those containing an algebraic surface whose cohomology class is linearly independent of h^2, where h is the class of a hyperplane section. Let $A(X) = H^4(X,\mathbb{Z}) \cap H^{2,2}(X)$ be the positive definite lattice of integral middle Hodge classes that coincides with $CH_2(X)$ the Chow group of 2-cycles on X. Since the integral Hodge conjecture holds for cubic fourfolds (see [60]), X is *special* if and only if the rank of $A(X)$ is at least 2.

Definition 2.1. *A labelling of a special cubic fourfold is a rank 2 saturated sublattice $K_d \subseteq A(X)$ containing h^2. Its discriminant d is the determinant of the intersection form on K_d.*

Special cubic fourfolds with labelling of discriminant d form a countably infinite union of divisors $\mathcal{C}_d \subset \mathcal{C}$, called Hassett divisors. Hassett[28, Theorem 1.0.1] showed that $\mathcal{C}_d$ is irreducible and nonempty if and only if $d \geq 8$ and $d \equiv 0,2$ [6] $(*)$. Moreover, Hassett described how, in certain cases, one can associate to a cubic fourfold a K3 surface. More precisely, there exists a polarized K3 surface S of degree d such that $K_d^{\perp} \subset H^4(X,\mathbb{Z})$ is Hodge-isometric to $H^2_{prim}(S,\mathbb{Z})(-1)$ if and only if d is not divisible by 4, 9, or any odd prime number $p \equiv 2$ [3] $(*')$.

Moreover, for infinitely many values of d, and for the generic cubic fourfold $X \in \mathcal{C}_d$, the Fano variety $F(X)$ of lines on the cubic fourfold is isomorphic to the Hilbert scheme of length two subschemes $S^{[2]}$ of the associated K3 surface S. This holds if $d = 2(n^2+n+1)$ for an integer $n \geq 2$.

In the next section we will proceed with the lattice theoretical computations that will show the existence of cubic fourfolds with associated K3 of any Néron-Severi rank.

3. Intersection of divisors

In this section, we will study properties of cubics inside the intersection of Hassett divisors $\mathcal{C}_d \in \mathcal{C}$. Recall that the Hassett divisor $\mathcal{C}_d \in \mathcal{C}$ is an irreducible and nonempty divisor if and only if

$$d \geq 8 \text{ and } d \equiv 0, 2[6] \qquad (*).$$

In particular, we will set the discriminants d_k for which the intersection $\bigcap\limits_k \mathcal{C}_{d_k}$ contains special families of cubic fourfolds with prescribed rank of $A(X)$. First of all we recall two interesting facts, proved recently by Yang and Yu.

Proposition 3.1. [63, Theorem 3.1] *Any two Hassett divisors intersect i.e $\mathcal{C}_{d_1} \cap \mathcal{C}_{d_2} \neq \emptyset$ for any integers d_1 and d_2 satisfying $(*)$.*

Theorem 3.2. [64, Thm. 1.2] *The intersection of all Hassett divisors is non-empty and contains the Fermat cubic fourfold.*

Remark 3.3. *If X is a cubic fourfold in $\mathcal{C}_{d_1} \cap \mathcal{C}_{d_2}$, then there exists infinitely many divisors $\mathcal{C}_d$ containing X. In fact, $A(X)$ is a rank 3 lattice generated by h^2 and both surfaces S_1 and S_2 defining respectively $\mathcal{C}_{d_1}$ and $\mathcal{C}_{d_2}$. For every vector u in the sublattice $< S_1, S_2 >$, $u^\perp \subset A(X)$ is of rank 2. This defines a labelling (thus a divisor containing X), and moving u appropiately we obtain as many different labellings as we want.*

We denote by $(**)$ the following condition on the discriminant d:

$$(**) \qquad d = 6\prod_i p_i^2 \text{ or } d = 6\prod_i p_i^2 + 2, \text{ where } p_i \text{ is a prime number.}$$

(Prime numbers appearing in previous formulas are not necessarily distinct)

For sake of simplicity, the proof of the Theorem 1.1 will be divided into different steps.
First, in Lemma 3.4, we construct a dimension 17 family of cubics inside the intersection of any three divisors $\mathcal{C}_{d_1}$, $\mathcal{C}_{d_2}$ and $\mathcal{C}_{d_3}$ for d_1, d_2 and d_3 satisfying $(*)$ and d_3 satisfying $(**)$ as well. The proof is divided into 4 parts covering all possible cases, depending on the values of the discriminants. Then in Prop. 3.5 we construct the required families of cubics inside $\bigcap_{1\leq k\leq n} \mathcal{C}_{d_k}$, for any $1 \leq n \leq 20$, for a particular choice of the integers d_k. Finally in Theorem 3.6 we prove the full statement of Theorem 1.1. In order to preserve the readability of the paper, and since the proofs of some of these results involve heavy computations, we collected their proofs in the Appendix at the end of the paper.

Lemma 3.4. *For any integers d_1, d_2, d_3 satisfying (*) such that d_3 satisfies (**), $\mathcal{C}_{d_1} \cap \mathcal{C}_{d_2} \cap \mathcal{C}_{d_3}$ contains a 17 dimensional family $\mathcal{H}_3$ of cubic fourfolds, such taht all $X \in \mathcal{H}_3$ has $rk(A(X)) \geq 4$, and the generic cubic has $rk(A(X)) = 4$.*

In the following proposition, using a strategy similar to the one developed in the previous Lemma, but assuming that all discriminants verify $d_k \equiv 0[6]$, we prove that analogous families exist inside the intersection of up to 20 divisors $\mathcal{C}_{d_k}$ in $\mathcal{C}$.

Proposition 3.5. *Assume that* $\forall d_k \equiv 0[6]$ *and that* $d_3, .., d_{20} = 6\prod_i p_i^2$*, for some prime numbers* p_i*. Then, for* $4 \leq n \leq 20$ *the intersection* $\bigcap_{k=1}^{n} \mathcal{C}_{d_k}$ *contains a family* $\mathcal{H}_n$ *of dimension* $20 - n$ *of cubic fourfolds, whose generic member has* $rk(A(X)) = 1 + n$*, and all* $X \in \mathcal{H}_n$ *have* $rk(A(X)) \geq 1 + n$

Next Theorem is our first main Theorem 1.1. Its proof is again divided into multiple cases. In fact, Lemma 3.4 and Proposition 3.5 are particular cases of the next theorem, which generalizes them. In this theorem, we will showcase families of cubics inside the intersection of up to 20 divisors $\mathcal{C}_{d_i}$, whenever $d_i \equiv 0, 2[6]$, for all i. Also its proof is contained in the Appendix.

Theorem 3.6. *Let* d_k *be positive integers indexes satisfying* $(*)$ *and let* $d_3, .., d_{20} = 6\prod_i p_i^2$ *or* $6\prod_i p_i^2 + 2$*, with* p_i *prime numbers. Let us set* $1 \leq n \leq 20$*. Then, inside* $\bigcap_{k=1}^{n} \mathcal{C}_{d_k}$*, there exist a family* $\mathcal{H}_n$ *of dimension* $20 - n$ *of cubics, whose generic element has* $rk(A(X)) = 1 + n$*, and s.t. all* $X \in \mathcal{H}_n$ *have* $rk(A(X)) \geq 1 + n$*.*

3.1. **Applications.**

3.1.1. *Lower Néron-Severi ranks.* In the constructions of the preceding section, one can choose all these 20 divisors in a way such that cubic fourfolds inside the intersection have associated K3 surfaces.

Corollary 3.7. *Suppose that we are under the hypotheses of Thm 3.6 and that at least one of the divisors* $\mathcal{C}_{d_k}$ *parametrizes cubic fourfolds with an associated K3 surface. Let* $\mathcal{H}_n$ *denote the family of dimension* $20 - n$ *of cubic fourfolds inside* $\bigcap_{k=1}^{n} \mathcal{C}_{d_k}$ *constructed in the preceding section. Then, the generic cubic fourfold in* $\mathcal{H}_n$ *has associated K3 of Néron-Severi rank equal to* n*.*

Proof. Generically the cubic fourfolds inside $\mathcal{H}_n \subset \bigcap_{k=1}^{n} \mathcal{C}_{d_k}$ have $rk(A(X)) = n + 1$, hence the primitive cohomology of the associated K3 surface has rank $n - 1$, and the Néron-Severi group has rank n. □

In particular, the family $\mathcal{H}_{19}$ inside the nonempty intersection of 19 divisors in $\mathcal{C}$ is a curve that intersects infinitely many other divisors. If one of the 19 divisors parametrizes cubics with an associated K3 surface, then all the cubics in the family $\mathcal{H}_{19}$ have associated singular K3 surfaces. As it is well known, this means that the NS rank of the associated K3 is 19 or 20.

3.1.2. *Fano varieties of lines.* As we already said in 2.3, for a generic cubic fourfold $X \in \mathcal{C}_d$ such that $d = 2(n^2 + n + 1)$ where n is an integer ≥ 2, we have that $F(X) \cong S^{[2]}$. We observe that, if we take $d_1 = 14$ in Cor. 3.7, the examples in the intersection that we produce in Corollary 3.7 verify this isomorphism.

The singular K3 surface associated to cubic fourfolds X as in Corollary 3.7 admits a Shioda-Inose structure. In particular, there is a birational map from the K3 surface to a Kummer surface which is the quotient of a product-type Abelian variety by an involution (see [46], [52]). This implies that $F(X)$ is birational to the Hilbert scheme of length two subschemes of a Kummer surface.

4. Abelianity of the motive of families of cubic fourfolds

In this section, we will mention first some recent results about Chow motives of cubic fourfolds and K3 surfaces. Then we will combine them with the intersection theoretical results from the preceding section to construct families of cubic fourfolds with finite dimensional Chow motive of Abelian type.

4.1. Chow motives of cubic fourfolds and K3 surfaces.

The first result we need to recall from [15] is the existence of a Chow-Künneth decomposition for the motive of the cubic fourfold X. Namely, we have

$$h(X) = \mathbf{1} \oplus \mathbf{L} \oplus (\mathbf{L}^2)^{\rho_2(X)} \oplus t(X) \oplus \mathbf{L}^3 \oplus \mathbf{L}^4, \tag{1}$$

where $\rho_2(X) = \mathrm{rank}(CH_2(X))$, with $CH_2(X) \subset H^4(X, \mathbb{Z})$, and $t(X)$ is the transcendental motive of X, i.e. the realization functor gives $H^*(t(X)) = H^4_{tr}(X, \mathbf{Q})$.

Let now X be a special cubic fourfold contained in a divisor $\mathcal{C}_d$. In [17], Bülles shows that for certain values of d, there exists a K3 surface S such that

$$t(X) \simeq t_2(S)(1). \tag{2}$$

Here $t_2(S)$ is the transcendental motive of S , i.e.

$$h(S) = \mathbf{1} \oplus \mathbf{L}^{\rho(S)} \oplus t_2(S) \oplus \mathbf{L}^2,$$

where $\rho(S)$ is the rank of the Néron-Severi group $NS(S)$. More precisely, the isomorphism (2) holds whenever d satisfies the following numerical condition

$$(***) : \exists f, g \in \mathbb{Z} \textit{ with } g|(2n^2 + 2n + 2) \; n \in \mathbb{N} \textit{ and } d = f^2 g.$$

Therefore, in this case, $h(X) \in \mathcal{M}^{Ab}_{rat}(\mathbf{C})$ if and only if $h(S) \in \mathcal{M}^{Ab}_{rat}(\mathbf{C})$. Note that an isomorphism $t(X) \simeq t_2(S)(1)$ can never hold if X is not special, i.e. if $\rho_2(X) = 1$, see [15, Prop.3.4]. Remark moreover that $d = 14$ satisfies $(***)$ with $f = 1,\ n = 2$ and $g = 14$.

On the other hand, finite dimensionality of motives of K3 surfaces has been addressed in [48]. In particular the following is proved:

Theorem 4.1. *Let S be a smooth complex projective K3 surface with $\rho(S) = 19, 20$. Then the motive $h(S) \in \mathcal{M}_{rat}(\mathbf{C})$ is finite dimensional and lies in the subcategory $\mathcal{M}^{Ab}_{rat}(\mathbf{C})$.*

4.2. **The divisor $\mathcal{C}_{14}$.** In the rest of this section, we will need to consider a particular Hassett divisor in order to construct families of cubic fourfolds with associated K3 surface. It is the divisor $\mathcal{C}_{14}$. This divisor can be defined mainly in three different ways, let us recall them.

(i) as the closure in $\mathcal{C}$ of the locus of Pfaffian cubic fourfolds, i.e. cubics defined by the Pfaffian of a 6×6 alternating matrix of linear forms.
(ii) as the closure of the locus of cubics containing a del Pezzo quintic surface;
(iii) as the closure of the locus of cubics containing a quartic scroll.

The three descriptions are strictly related (see [13, 16, 29]). All cubics in $\mathcal{C}_{14}$ have an associated genus 8 K3 surface, and they are all rational [16].

4.3. **Construction of the families of cubic foufolds.** Applying the previous results, we give now the proof of the following theorem:

Theorem 4.2. *Every Hassett divisor $\mathcal{C}_d$ contains a one dimensional family of cubic fourfolds, whose Chow motive is finite dimensional and Abelian.*

Proof. Let us consider $\mathcal{C}_d \subset \mathcal{C}$ any divisor of special cubic fourfolds. By Theorem 3.6, we can chose appropriately 17 divisors $\mathcal{C}_{d_1}, \dots \mathcal{C}_{d_{17}}$ such that the intersection

$$\mathcal{C}_d \cap \mathcal{C}_{14} \cap (\bigcap_{k=1}^{17})\mathcal{C}_{d_k}$$

contains a one dimensional family $\mathcal{B}$ of cubic fourfolds with special intersection theoretical features. First, $\mathcal{B}$ is an algebraic subvariety of $\mathcal{C}_d$, that is, a family of cubic fourfolds of discriminant d. On the other hand, by construction, the family $\mathcal{B}$ is also contained in $\mathcal{C}_{14}$, hence all the cubics in $\mathcal{B}$ have an associated K3 surface. Moreover, by the results of Section 3, we observe that cubic fourfolds in $\mathcal{B}$ have associated K3 surfaces with Néron-Severi rank $\rho(S) \geq 19$. More precisely: the generic cubic fourfold in $\mathcal{B}$ has an associated K3 surface with $\rho(S) = 19$; the intersection points of $\mathcal{B}$ with certain further Hassett divisors represent cubic fourfolds whose associated K3 surface has $\rho(S) = 20$. By Thm. 4.1 the Chow motives of these K3 surfaces are finite dimensional and Abelian. Now we need to evoke the isomorphism of Eq. 2. The divisor $\mathcal{C}_{14}$ is among those whose cubic fourfolds have Chow motive that decomposes as follows

$$h(X) = \mathbf{1} \oplus \mathbf{L} \oplus (\mathbf{L}^2)^{\rho_2(X)} \oplus t_2(S)(1) \oplus \mathbf{L}^3 \oplus \mathbf{L}^4,$$

where $t_2(S)$ is the transcendental part of the motive $h(S)$ of the associated K3 surface. Now, if the motive of the associated K3 surface is finite dimensional or Abelian, then also the motive of the cubic fourfold has the same property. This means in turn that, by Thm. 4.1, all the cubics in $\mathcal{B}$ have finite dimensional and Abelian Chow motive, since the associated K3 surfaces have Néron-Severi group of rank bigger or equal to 19. □

Remark 4.3. *Let us point out that, since all the family $\mathcal{B}$ is contained in $\mathcal{C}_{14}$, then, by results of* [16]*, all cubic fourfolds in $\mathcal{B}$ are rational.*

In the proof of Thm. 4.2, we chose $\mathcal{C}_{14}$ for simplicity, since it is the first of the series that has d verifying condition $(***)$, and where cubics have associated K3 surfaces. Any other divisor $\mathcal{C}_t$ obeying $(***)$, such that cubics in $\mathcal{C}_t$ have associated K3 surfaces, would have worked. Hence we can say even more.

Corollary 4.4. *Every Hassett divisor $\mathcal{C}_d$ contains a countable infinity of one dimensional families of cubic fourfolds, whose Chow motive is finite dimensional and Abelian.*

Proof. Just consider, in $(***)$, $f=1$ and $g=2n^2+2n+2$, for any $n \in \mathbb{N}$. This gives an infinite series of values of d, such that cubics in $\mathcal{C}_d$ have associated K3 surfaces. □

4.4. **Density of cubic fourfolds with Abelian motive.** More generally, let $\mathcal{G}_d$ the moduli space of polarized K3 surfaces of degree d. This is a quasi-projective 19-dimensional algebraic variety.

Theorem 4.5. *Let d be not divisible by 4, 9, or any odd prime number $p \equiv 2$ [3]. Then there exists a countable, dense (in the complex topology) set of points in a non-empty Zariski open subset inside $\mathcal{C}_d$ such that the corresponding fourfolds have finite dimensional Chow motive.*

Proof. Let $X \in \mathcal{C}_d$, for d in the range of the claim here above. That is: X has one (or two, see [28]) associated polarized K3 surface S_X in $\mathcal{G}_{\frac{d+2}{2}}$. Then the map

$$\begin{aligned} &(3) && \mathcal{G}_{\frac{d+2}{2}} \quad \to \quad \mathcal{C}_d; \\ &(4) && S_X \quad \mapsto \quad X; \end{aligned}$$

is rational and dominant. Hence, if d is in the range here above, there exists an open set $\mathcal{U}_d$ of $\mathcal{C}_d$ such that for every $X \in \mathcal{U}_d$ there exists a K3 surface S_X of degree d associated to X. We observe also that singular K3 surfaces form a (countable) subset of the moduli space $\mathcal{G}_{\frac{d+2}{2}}$ which is dense in the complex topology. The proof of this fact goes along the same lines as the proof of the density of all K3 surfaces in the period domain (see [12, Corollary VIII.8.5]). By the dominance of the map in (3), this directly implies the claim. □

4.5. **Some remarks on HyperKähler varieties.** In this section, we draw consequences on Abelianity and finite dimensionality of the motive of some HyperKähler varieties related to cubic fourfolds from the preceding results.

Notably, we will consider $F(X)$ - the Fano variety of lines - and $L(X)$ - the 8-fold constructed in [42] from the space of twisted cubic curves on a cubic fourfold not containing a plane. The 4-dimensional $F(X)$ is deformation equivalent to the Hilbert scheme $S^{[2]}$, with S a K3 surface, while $L(X)$ is deformation equivalent to $S^{[4]}$. For every even complex dimension $2n$ there are two known deformations classes of irreducible holomorphic symplectic varieties: the Hilbert scheme $S^{[n]}$ of n-points on a K3 surface S and the generalized Kummer. A generalized Kummer variety Y is of the form $Y = K^n(A) = a^{-1}(0)$, where A is an Abelian surface and $a : A^{[n+1]} \to A$ is the Albanese map. In dimension 10 there is also an example, usually referred as OG10, discovered by O'Grady. The HyperKähler variety OG10 is not deformation equivalent to $S^{[5]}$.

Let $\mathcal{M}_A(\mathbf{C})$ be the category of André motives which is obtained from the category of homological motives $\mathcal{M}_{hom}(\mathbf{C})$ by formally adjoining the Lefschetz involutions $*_L$ associated to the Lefschetz isomorphisms $L^{d-i} : H^i(X) \to H^{2d-i}(X)$, where L^{d-i} is induced by the hyperplane section. By the Standard Conjecture $B(X)$, for every $i \leq d$ there exists an algebraic correspondence inducing the isomorphism $H^{2d-i}(X) \to H(X)$ inverse to L^{d-i}. Therefore, under $B(X)$ the category of André motives coincides with $\mathcal{M}_{hom}(\mathbf{C})$. The André motive of a K3 surface S and of a cubic fourfold X belong to the full subcategory $\mathcal{M}_A^{Ab}(\mathbf{C})$ generated by the motives of Abelian varieties, see [5, 10.2.4.1].

In [53] it is proved that the André motive of a HyperKähler variety which is deformation equivalent to $S^{[n]}$ lies in $\mathcal{M}_A^{Ab}(\mathbf{C})$. Soldatenkov [54] proves that if X_1 and X_2 are deformation equivalent projective HyperKähler manifolds then the André motive of X_1 is Abelian if and only if the André motive of X_2 is Abelian. In a recent preprint (see [21]) it is proved that also the André motive of OG10 lies in $\mathcal{M}_A^{Ab}(\mathbf{C})$. Therefore the André motives of all the known deformation classes of HyperKähler varieties lie in $\mathcal{M}_A^{Ab}(\mathbf{C})$. These results suggest the following conjecture:

Conjecture 4.6. *The motive of a HyperKähler manifold is of Abelian type in $\mathcal{M}_{rat}(\mathbf{C})$.*

By [19, Sect. 6], the Hilbert scheme $S^{[n]}$ of a K3 surface with finite dimensional (or Abelian) motive has finite dimensional (or Abelian) motive. Now, recall that our family $\mathcal{B}$ of cubic fourfolds from Thm. 4.2 entirely lies in $\mathcal{C}_{14}$. Moreover, for all cubics X in $\mathcal{C}_{14}$, we have an birational map $F(X) \sim S^{[2]}$ [28], where S is the associated K3 surface. Since birational HyperKähler varieties have isomorphic Chow motives, it is straightforward to check that we have the following:

Proposition 4.7. *All HyperKähler fourfolds $F(X)$, $X \in \mathcal{B}$, have finitely generated and Abelian Chow motive.*

Remark 4.8. *Once again, we can play the same game as before by taking $\mathcal{C}_{\frac{2n^2+2n+2}{a^2}}$, $n, a \in \mathbb{Z}$, instead of $\mathcal{C}_{14}$. By [1, Thm. 2], having $d = \frac{2n^2+2n+2}{a^2}$ is equivalent to having a birational equivalence between $F(X)$ and the Hilbert square of the associated K3 surface. Hence, everything runs the same way, and we have a countably infinite set of families of Fano varieties $F(X)$ with finite dimensional and Abelian motive, whose cubic fourfolds all lie in a fixed $\mathcal{C}_d$.*

On the other hand, let us now consider the HyperKähler 8fold $L(X)$. In order to define properly $L(X)$, we need to assume that X does not contain a plane, i.e. $X \notin \mathcal{C}_8$. Then, the analogue of Prop. 4.7 is the following:

Proposition 4.9. *All HyperKähler 8folds $L(X)$, $X \in \mathcal{B}$, have finitely generated and Abelian Chow motive.*

Proof. Let $X \in \mathcal{C}_d$ be a cubic fourfold not containing a plane, and S its associated K3 surface. In [2, Thm. 3] the authors show that the 8fold $L(X)$ is birational to $S^{[4]}$ if and only if

$$(***') \quad d = \frac{6n^2+6n+2}{a^2}, \quad n, a \in \mathbb{Z}.$$

The first integer of the list is once again 14, hence for all the cubic fourfolds of our family $\mathcal{B}$ we have $L(X) \overset{birat}{\cong} S^{[4]}$. Since birational HyperKähler varieties have isomorphic Chow motives, the results from [19] complete the proof, and we have a one-dimensional family of $L(X)$ with finite dimensional and Abelian motive for all $\mathcal{C}_d$, with $d \neq 8$. $\square$

Remark 4.10. *In fact, we observe that $d = 14$ verifies both condition $(***)$ (with $f = 1$) and $(***')$ (with $n = a = 1$). This means that for cubic fourfolds X in $\mathcal{B}$ both $F(X)$ and $L(X)$ 8folds have finite dimensional and Abelian motive.*

5. A remark on the finite dimensionality of the motive of the LSV 10 dimensional HyperKähler manifold

5.1. **The intermediate Jacobian fibration.** Let $X \subset \mathbb{P}^5$ be a smooth cubic fourfold, and $\mathcal{J}(X)$ the 10 dimensional HyperKähler manifold constructed in [41]. The variety $\mathcal{J}(X)$ is a compactification of the fibration over $\mathbb{P}^{5*}$, whose fibers are the intermediate Jacobians of the hyperplane sections of X, and it is deformation equivalent to O'Grady ten dimensional example OG10 [41, Cor. 6.3].

Let X be a Pfaffian cubic fourfold. Recall from a lot of references ([13, 35, 28]) that the K3 surface associated to a Pfaffian cubic is the "orthogonal" section of the Grassmannian $G(2,6)$. Let W be a 6 dimensional vector space. If X is a pfaffian cubic defined by choosing a 6 dimensional vector space V and an embedding $\mathbb{P}(V) \subset \mathbb{P}(\wedge^2 W^*)$, then we can consider the 9 dimensional annihilator $V^\perp$ in $\wedge^2(W)$. The projectivized $\mathbb{P}^8$ cuts out a degree 14 K3 S from the $G(2,6)$ that naturally lives in $\mathbb{P}(\wedge^2 W)$, and S is the K3 associated to X.

An argument similar to those appearing in the preceding section will allow us to show that there exist several examples of $\mathcal{J}(X)$ with finite dimensional, Abelian Chow motive. First of all we need to recall the following result [41, Thm. 6.2].

Theorem 5.1. *If X is a smooth Pfaffian cubic fourfold, then $\mathcal{J}(X)$ is birational to the O'Grady moduli space $\mathcal{M}_{2,0,4}(S)$ parameterizing rank-2 semi-stable sheaves on S with $c_1 = 0$ and $c_2 = 4$.*

Recall that O'Grady HyperKähler example OG10 is a birational desingularization of $\mathcal{M}_{2,0,4}(S)$. The main result of this section is the following.

Theorem 5.2. *There are infinitely many Pfaffian cubic fourfolds $X \in \mathcal{C}_{14}$ such that that the motive $h(\mathcal{J}(X))$ of the associated LSV 10-fold is finite dimensional and Abelian.*

Proof. Recall from [16, Rmk. 4.8] that the locus of Pfaffian cubics inside $\mathcal{C}_{14}$ is a constructible set. This means that it contains open subset, that we will denote by U. On the other hand, we know [28] that there is a birational map

$$g : \mathcal{C}_{14} \dashrightarrow \mathcal{G}_8$$

sending a cubic fourfold onto its associated K3 surface. This means that there exists an open subset $A \subset \mathcal{C}_{14}$, which is isomorphically sent onto an open set $g(A) \subset \mathcal{G}_8$. Let us consider the intersection $A \cap U \subset \mathcal{C}_{14}$. The image $g(A \cap U)$ is still open in $\mathcal{G}_8$ and all K3 surfaces in $g(A \cap U)$ are associated to a smooth Pfaffian cubic fourfold. Now, it is well known that K3 surfaces with $rk(NS) \geq 19$ are dense (in the complex topology) in the 19-dimensional moduli space $\mathcal{G}_d$ of degree d polarized K3 surfaces, for all d. This means that there are also K3 surfaces of this kind inside our open subset $g(A \cap U)$. It is also straightforward to see that there are infinitely many, by a density argument. In turn, this means that there exist Pfaffian cubic fourfolds, whose associated K3 has Néron-Severi group of rank at least 19. Hence, by [48], they have a finite dimensional and Abelian Chow motive.

The existence of the LSV HK 10fold, compactifying the intermediate Jacobian fibration, is verified in [41] only for the generic cubic fourfold. More recently, Saccà [51] has shown the existence of the HK compactification for all cubics, hence it is in this framework that we will place ourselves. Now, by [21, Cor. 4.5 and 4.7], we know that the Chow motive of the OG10 HyperKähler associated to a K3 surface S is finite dimensional and Abelian whenever the $h(S)$ is. Then, if the cubic fourfold X is Pfaffian, by Thm. 5.1 $\mathcal{J}(X)$ is birational to the OG10 associated to its K3 S. This implies that over the open subset $A \cap U$ there are infinitely many $\mathcal{J}(X)$ that, due to their birationality to the corresponding OG10, have finite dimensional and Abelian motive. □

5.2. **The twisted jacobian fibration.** This section is the development of a suggestion of Giulia Saccà. In the last few years, Voisin [62, 61] introduced a twisted version of the intermediate jacobian fibration (see [62, Sect. 3] for details), parametrizing 1-cycles of degree 1 in the fibers of the cubic 3fold fibration. By the results of Voisin [62] and Saccà [51], it is now known that, for any cubic fourfold X, there exist a HyperKähler compactification of the twisted intermediate Jacobian fibration. The main result of this section goes shows that even in the twisted case, the motive of these HK compactifications can be often finite dimensional.

Theorem 5.3. *For d not divisible by 4, 9, or any odd prime number $p \equiv 2[3]$, there exists an infinity of cubic fourfolds in $\mathcal{C}_d$ s.t. the twisted Jacobian fibration has a HK compactification with finite dimensional Chow motive.*

Proof. We observe that the condition on d is nothing but the fact that the cubic fourfold X has an associated K3 surface. Let $Ku(X)$ be the Kuznetsov component of the derived category of the cubic 4fold X. In the recent paper [43], the authors studied the moduli space $M(X)$ of semistable objects of Mukai vector $2\lambda_1 + 2\lambda_2$ in $Ku(X)$. Recall in fact from [4] that the algebraic Mukai lattice of $Ku(X)$ contains two distinguished classes λ_i, $i = 1, 2$, that span an A_2-lattice. With an approporiate choice of a stability condition on $Ku(X)$ (that we will not detail here, see [43, Sect. 2 and 3] for details), the authors manage to show that the moduli space $M(X)$ has a symplectic resolution $\widetilde{M}(X)$, which is a projective HyperKähler manifold, deformation equivalent to O'Grady 10 dimensional examples. More importantly, they show that there exists a HK birational model $N(X)$ of $\widetilde{M}(X)$ that provides a compactification of the twisted intermediate Jacobian fibration. Of course $N(X)$ and $\widetilde{M}(X)$ have isomorphic Chow motive, since they are birational.

The same argument as in [21, Sect. 5] (see also [22, Thm. 1.3 and Rmk 1.4]) shows that $\widetilde{M}(X)$ has finite dimensional Chow motive whenever $h(X)$ is finite dimensional. This means once again that, if X has an associated singular K3 surface, then the motive $h(N(X))$ of the HK compactified twisted Jacobian fibration is finite dimensional. Since singular K3 surfaces are dense in the moduli space of genus $\frac{d+2}{2}$ K3 surfaces, by the same argument as in the untwisted case, this completes the proof.

□

6. K3 surfaces with Abelian Chow motive

In this section, we construct higher dimensional families of cubics with Abelian Chow motive starting from families of K3 surfaces with particular shapes. It is well known that among the examples of K3 surfaces with a motive of Abelian type there are Kummer surfaces and K3 surfaces with Picard rank ≥ 19 (see [48]). In this section we consider two families $\mathcal{F}$ and $\mathcal{G}$ of K3 surfaces with Picard rank 16, constructed in [24].

In [24] the authors construct families of K3 surfaces S, with $\rho(S) = 16$, admitting a symplectic action of the group $G = (\mathbb{Z}_2)^4$. The following lemma shows that the transcendental motive $t_2(S)$, as defined in [32], only depends on the motive of the quotient surface S/G. Note that the motive $h(S/G)$ can be represented in $\mathcal{M}_{rat}(\mathbf{C})$ see [23, 16.1.13].

Lemma 6.1. *Let S be a K3 surface over $\mathbf{C}$ with a finite group G of symplectic automorphisms. Let Y be a minimal desingularization of the quotient surface S/G. Then*

$$t_2(S) \simeq t_2(Y)$$

in $\mathcal{M}_{rat}(\mathbf{C})$.

Proof. From the results in [30], every symplectic automorphism $g \in G$ acts trivially on $A_0(S)$, so that $A_0(S)^G = A_0(S)$. Since G is symplectic there are a finite number $\{P_1, \cdots, P_k\}$ of isolated fixed points for G on S. Let Y be a minimal desingularization of the quotient surface S/G. The maps $f : S \to S/G$ and $g : Y \to S/G$ yield a rational map $S \dashrightarrow Y$ which is defined outside $\{P_1, \cdots, P_k\}$. Since $t_2(-)$ is a birational invariant for smooth projective surfaces we get a map $\theta : t_2(S) \to t_2(Y)$ such that θ is the map onto a direct summand, see [48, Prop.1]. Hence we can write

$$t_2(Y) = t_2(S) \oplus N.$$

Since $A_i(t_2(S)) = 0$, for $i \neq 0$, and $A_0(S)^G = A_0(S)$, we get $A_i(N) = 0$, for all i. From [25, Lemma 1] we get $N = 0$, hence $t_2(S) = t_2(S)^G \simeq t_2(Y)$.

□

6.1. **Intersection of three quadrics.** The moduli space of K3 surfaces with Picard number 16, admitting a symplectic action of the group $G = (\mathbb{Z}_2)^4$, has a countable number of connected components of dimension 4. The generic element of one of these connected components, that we will denote by $\mathcal{F}$, can be realized by considering a complete intersection S of three quadrics Q_1, Q_2, Q_3 in $\mathbb{P}^5$, see [24, 10.2], whose equations are

$$\sum_{0 \le i \le 5} a_i x_i^2 = 0; \sum_{0 \le i \le 5} b_i x_i^2 = 0 \ ; \ \sum_{0 \le i \le 5} c_i x_i^2 = 0, \tag{5}$$

with complex parameters a_i, b_i, c_i and $i = 0, \cdots, 5$. The group G is realized as the transformations of $\mathbb{P}^5$ changing an even number of signs in the coordinates $\{x_0 : x_1 : x_2 : x_3 : x_4 : x_5\}$. Let us denote by $\mathcal{F}$ the family of such K3 surfaces. The dimension of the moduli space of these K3 surfaces is 4, see [24, 10.2].

By Lemma 6.1 we get $t_2(S) = t_2(Y)$, with Y a desingularization of S/G. An important feature of these surfaces is described in [36, Thm.3.1].

Theorem 6.2. *A K3 surface belonging to the family $\mathcal{F}$ has a motive of Abelian type.*

More precisely one can show, using the results in [47], that the motive $h(S)$ belongs to the subcategory of $\mathcal{M}^{ab}_{rat}$ generated by the motive of a Prym variety associated to the surface Y. By [24, 10.2] the quotient S/G is a double cover of $\mathbb{P}^2$ branched at six lines l_i meeting in 15 points. The vector space $NS(Y) \otimes \mathbf{Q}$ of the desingularization Y of S/G is generated by the 15 classes $E_{i,j}$ of the 15 exceptional curves over the intersection points $l_i \cap l_j$ and by the class h of the inverse image of a general line in $\mathbb{P}^2$. The motives of the surfaces S and Y have a Chow-Künneth decomposition as in [32, 7.2.2]

$$h(S) = \mathbf{1} \oplus \mathbf{L}^{\oplus \rho(S)} \oplus t_2(S) \oplus \mathbf{L}^2; \tag{6}$$

$$h(Y) = \mathbf{1} \oplus \mathbf{L}^{\oplus \rho(Y)} \oplus t_2(Y) \oplus \mathbf{L}^2; \tag{7}$$

where $\rho(S) = \rho(Y) = 16$ and $t_2(Y) = t_2(S)$, by 6.1. Therefore $h(Y) = h(S)$.
By the results in [47] there exists a surface W which is a desingularization of the quotient $(C \times C)/F$ with C a curve and F a finite group such that $Y = W/i$, where i is a symplectic involution. The curve C has genus 5 and has an automorphism h of order 4 such that the quotient C/h is an elliptic curve E. The finite group F acting on $C \times C$ is generated by the automorphism (h, h^{-1}) and the involution $(c_1, c_2) \to (c_2, c_1)$. Let P be the connected component of the identity in the kernel of the

natural homomorphism Jac $C \to \operatorname{Pic} E$. The Prym variety P is an Abelian variety of dimension 4 and the Kuga-Satake variety $K(Y)$ is a sum of copies of P.

Proposition 6.3. *The motive $h(S)$ belongs to the subcategory of $\mathcal{M}_{rat}^{Ab}$ generated by the motive $h(P)$.*

Proof. By the Main Theorem in [47] there exists an algebraic cycle $\Gamma \in A^2(P \times Y)$ such that the associated map $\Gamma^* : H^2(Y, \mathbf{Q}) \to H^2(P, \mathbf{Q})$ induces an inclusion between the lattices of transcendental cycles. Therefore the vector space $H^2_{tr}(Y, \mathbf{Q})$ is a direct summand of $H^2_{tr}(P, \mathbf{Q})$. Since P is an Abelian variety its Chow motive $h(P)$ has a C-K decomposition

$$h(P) = \sum_{0 \le i \le 8} h_i(P).$$

The correspondence $\Gamma \in A_4(P \times Y)$ gives a map $h(P) \to h(Y)$, in the covariant category $\mathcal{M}_{rat}(\mathbf{C})$. By composing with the inclusion $h_2(P) \subset h(P)$ and the projection $h(Y) \to t_2(Y)$ we get a map of motives

$$\gamma : h_2(P) \to t_2(Y) \tag{8}$$

such that the corresponding map at the level of cohomology is split surjective. The motives $h(P)$ and $t_2(Y) = t_2(S)$ lie in the subcategory $\mathcal{C} \subset \mathcal{M}_{rat}(\mathbf{C})$ generated by finite dimensional motives. Let $\mathcal{N}$ be the largest tensor ideal in $\mathcal{C}$ such that the quotient category $\mathcal{C}/\mathcal{N}$ is semi simple. The ideal $\mathcal{N}$ corresponds to numerical equivalence of cycles, the functor $\mathcal{C} \to \mathcal{C}/\mathcal{N}$ is conservative and reflects split epimorphisms, see [6, 1.4.4 and 8.2.4]. Therefore the map γ is split surjective in $\mathcal{M}_{rat}(\mathbf{C})$. This proves that $t_2(Y) = t_2(S)$ is a direct summand of $h_2(P)$, hence the motive $h(S) = h(Y)$ lies in the subcategory of $\mathcal{M}_{rat}^{Ab}$ generated by the motive of the Prym variety P. $\square$

6.2. **Degree six surfaces in $\mathbb{P}^4$.** A further example of a family $\mathcal{G}$ of K3 surfaces with a motive of Abelian type is given by degree 6 surfaces S in $\mathbb{P}^4$ with 15 ordinary nodes. The desingularization of these surfaces has Picard rank 16 as well. By a result in [24, 8.2] these surfaces are the quotient of a K3 surface X by the action of a symplectic group $G \simeq (\mathbb{Z}_2)^4$ of automorphisms. We have a diagram

$$\begin{tikzcd} \tilde{X} \arrow[r] \arrow[d, "\pi"'] & X \arrow[d, "f"] \\ Y \arrow[r, "g"] & S \end{tikzcd} \tag{9}$$

where $\tilde{X}$ is the blow-up of the fixed points under the action of G and Y contains 15 rational curves coming from the resolution of the singularities of $S = X/G$. The map π is $16:1$ outside the branch locus. By Lemma 6.1, the maps in 9 induce an isomorphism of motives $h(X) \simeq h(Y)$.

The results of Paranjape have been recently extended in [31, Cor. 6.16].

Theorem 6.4. *Let S be general K3 surface of degree 6 with 15 ordinary nodes, i.e singularities of type A_1. Then the motive of a desingularization of S is finite dimensional and of Abelian type.*

In particular every surface in the family $\mathcal{G}$ has a motive of Abelian type.

7. Fano fourfolds with associated K3 surfaces in $\mathcal{F}$ or $\mathcal{G}$

In this section we use finite dimensionality of K3 surfaces belonging to the families $\mathcal{F}$ and $\mathcal{G}$ to construct Fano fourfolds either with a finite dimensional motive, or that contain a significant finite dimensional submotive (actually the motive of a K3 surface). This is similar to the approach of *categorical representability* followed in [14]. In Thm. 7.2 we consider a family of special Fano fourfolds fibrated in quadric surfaces. On the other hand, in Prop. 7.3 we study singular cubic fourfolds associated to K3 surfaces belonging to $\mathcal{G}$. In the second case the motive of the singular cubic 4-fold is Schur-finite in Voevodsky's triangulated category of motives $\mathbf{DM}_{\mathbf{Q}}(\mathbf{C})$.

7.1. **Fano fourfolds associated to the net of quadrics of an octic K3 surface.** Let Q_1, Q_2 and Q_3 the three quadrics defined in Eq. 5. They generate a net of quadrics, whose total space is a quadric fibration $\sigma : \mathcal{Q} \to \mathbb{P}^2$ of relative dimension 4, contained in $\mathbb{P}^2 \times \mathbb{P}^5$. A simple calculation shows that for generic choices of a_i, b_i and c_i inside Equation 5, the generic fiber of σ is smooth and the singular fibers have an isolated singularity. The net degenerates along a plane sextic, and recall that the degree 8 complete intersection of the Q_i is (generically) a smooth K3 surface S, that is the base locus of the net.

We are now going to perform the hyperbolic reduction (see [7, Sect. 1] for details) of the quadric bundle with respect to the constant section given by any one of the points of S. The result will be a second quadric firbation, but this time the fibers will be surfaces, and the total space a Fano fourfold.

We observe that $\mathcal{Q}$ is given by a 6×6 symmetric matrix $M_{\mathcal{Q}}$ of linear forms, that define a map $\mathcal{O}_{\mathbb{P}^2}^{\oplus 6} \to \mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 6}$. This means that $\mathcal{Q}$ can be seen as the zero locus of $(\mathcal{O}_{\mathbb{P}^2}^{\oplus 6}, q, \mathcal{O}_{\mathbb{P}^2}(1))$, where by this we mean a line bundle valued quadratic form q on $\mathcal{O}_{\mathbb{P}^2}^{\oplus 6}$, with values in $\mathcal{O}_{\mathbb{P}^2}(1)$. For simplicity, in this section we will denote $E := \mathcal{O}_{\mathbb{P}^2}^{\oplus 6}$. The K3 surface has points over $\mathbb{C}$ hence we can chose the constant section given by one of these over $\mathbb{P}^2$. This gives an inclusion of an isotropic line sub-bundle $N := \mathcal{O}_{\mathbb{P}^2} \subset E$. The hyperbolic reduction of $\mathcal{Q}$ naturally lives in the projectivized bundle $N^{\perp}/N$ which is defined by the following two short exact sequences.

$$0 \to N^{\perp} \to E \to Hom(\mathcal{O}_{\mathbb{P}^2}, \mathcal{O}_{\mathbb{P}^2}(1) \to 0; \tag{10}$$

$$0 \to N \to N^{\perp} \to N^{\perp}/N \to 0. \tag{11}$$

En easy Chern class computation based on the above SES shows that $c_1(N^{\perp}/N) = h$, where h is the hyperplane class of $\mathbb{P}^2$. Let us denote by $\sigma' : \mathcal{Q}' \to \mathbb{P}^2$ the quadric surface bundle obtained by hyperbolic reduction of $\mathcal{Q}$ at the constant section given by projectivizing N.

Lemma 7.1. *The quadric surface fibration $\sigma' : \mathcal{Q}' \to \mathbb{P}^2$ is a smooth Fano variety.*

Proof. Hyperbolic reduction preserves the discriminant and the corank of the singular fibers, hence smoothness holds. Let us set $H_1 = \sigma'^* h$ and $H_2 = \mathcal{O}_{\mathbb{P}(N^{\perp}/N)}(1)_{|\mathcal{Q}'}$. In order to compute the anticanonical bundle, we can use the formula of Proposition 2.13 of [44] and obtain

$$-K_{\mathcal{Q}'} = \frac{12-2-6}{2+2} H_1 + 2H_2.$$

Since H_2 is ample, one easily sees that $-K_{\mathcal{Q}'}$ is ample as well and $\mathcal{Q}'$ is Fano. □

Theorem 7.2. *The Chow motive of the Fano fourfold $\mathcal{Q}'$ has a finite dimensional submotive $h(S)(1) \to h(\mathcal{Q}')$, that is the (twist of the) Chow motive of the degree 8 K3 surface $S := \bigcap_{i=1}^3 Q_i$.*

Proof. Let S be the K3 surface defined by the intersection of the 3 quadrics Q_1, Q_2, Q_3. The quadrics Q_i define a linear system of dimension two $|Q| = \mathbb{P}^2$. This projective plane naturally contains the discriminant curve D of degree 6, that parametrizes singular quadrics, and under our genericity hypotheses the sextic is smooth. The curve D defines another smooth K3 surface T, which is the double cover of $\mathbb{P}^2$ ramified in D.

By the work of Kuznetsov [33] on derived categories of quadric fibrations, we know that there exists a semi-orthogonal decomposition for the bounded derived category of coherent sheaves on $\mathcal{Q}'$:

$$D^{\mathbf{b}}(\mathcal{Q}') = \langle D^{\mathbf{b}}(T,\alpha), \sigma'^{*}(D^{\mathbf{b}}(\mathbb{P}^2)) \otimes \mathcal{O}_{\mathcal{Q}'/\mathbb{P}^2}(1), \sigma'^{*}(D^{\mathbf{b}}(\mathbb{P}^2)) \otimes \mathcal{O}_{\mathcal{Q}'/\mathbb{P}^2}(2)\rangle. \tag{12}$$

Here T is our K3 double cover of $\mathbb{P}^2$, and $\alpha \in Br[2](T)$ the (2-torsion) Brauer class on T that comes from the Brauer-Severi variety on T defined by the relative Hilbert scheme of lines inside the fibers of $\sigma' : \mathcal{Q}' \to \mathbb{P}^2$. Moreover, thanks to the work of Căldăraru [18] , we know that we have a derived equivalence

$$D^{\mathbf{b}}(T,\alpha) \cong D^{\mathbf{b}}(S).$$

The fully faithful functor $D^{\mathbf{b}}(S) \to D^{\mathbf{b}}(\mathcal{Q}')$ is Fourier-Mukai. A straightforward application of Grothendieck-Riemann-Roch shows that this induces a correspondence $\gamma \in A^3(S \times \mathcal{Q}')$ that induces an isomorphism $A_0(S)_{hom} \xrightarrow{\sim} A_1(\mathcal{Q}')_{hom}$. This follows from the fact that the homologically trivial part of S and $\mathcal{Q}'$ are concentrated in codimension two and three respectively. For S, this follows from the reduced Chow-Künneth decomposition of $h(S)$ while for $\mathcal{Q}'$ it follows from the decomposition, proved in Vial [58, Cor. 4.4],

$$h(\mathcal{Q}') \cong \bigoplus_{i=0}^{2} h(\mathbb{P}^2)(i) \oplus (Z, r, 1), \tag{13}$$

because $h(\mathcal{Q}')$ is a quadric bundle over $\mathbb{P}^2$. Here Z is a surface and r a projector. Therefore

$$A^3(X)_{hom} = A^3((Z, r, 1) = A^2((Z, r) = r_*(A_0(Z)_{hom}).$$

From the reduced CK decompositions of the surfaces S and Z, that both have no odd cohomology, we get

$$A^3(h(S)(1)) = A^2(t_2(S)) = A_0(S)_{hom}; \quad A^3((Z, r, 1) = A^2((Z, r) = r_*(A_0(Z)_{hom} \subset A_0(Z)_{hom}.$$

Therefore the correspondence $\gamma \in A^3(S \times \mathcal{Q}')$ induces a map of motives $h(S)(1) \to h(\mathcal{Q}')$ that gives an injection on Chow groups and hence displays $h(S)(1)$ as a submotive of $h(\mathcal{Q}')$. □

7.2. **Singular cubics and 15-nodal K3 surfaces.** The relation between smooth cubic fourfolds and polarized K3 surfaces, proved in [28], has been recently extended to the case of fourfolds with isolated ADE singularities. A.K. Stegmann in [55] constructs the moduli space of cubic fourfolds with a certain combination of isolated ADE singularities as a GIT quotient and compares it to the moduli space of certain quasi-polarized K3 surface of degree 6, proving that the two moduli spaces are isomorphic. Here we consider the motive of a cubic fourfold X with isolated singularities associated to a K3 surfaces of degree 6 with 15 nodes belonging to $\mathcal{G}$. Since we are now working with singular varieties, we need to change the category of motives from $\mathcal{M}_{rat}(\mathbf{C})$ to Voevodsky's triangulated category of motives $\mathbf{DM_Q}(\mathbf{C})$.

Let $X \subset \mathbb{P}^5$ be a cubic fourfold with isolated singularities. Projection from a double point $p \in X$ gives a birational map $\pi_p : X \dashrightarrow \mathbb{P}^4$ which can be factored as

$$\tilde{X}_p \xrightarrow{q_1} X \ ; \ \tilde{X}_p \xrightarrow{q_2} \mathbb{P}^4$$

where q_1 is the blow-up of the singular point p and q_2 is the blow-down of the lines contained in X passing through p. These lines are parametrized by a normal surface S_p of degree 6 in $\mathbb{P}^4$. The surface S_p is a complete intersection of a quadric Q_p and a cubic C_p. The quadric Q_p is completely determined by S_p while the cubic C_p in $\mathbb{P}^4$ containing S_p is uniquely determined modulo those cubics containing the quadric Q_p. Conversely, starting from a $(2,3)$-complete intersection in $\mathbb{P}^4$, the blow up of the sextic surface followed by the contraction of the strict transform of the quadric yields a singular cubic fourfold with a double point, which is the image of the quadric. The type of this singularity depends on the rank of the quadric.

The cubic fourfold X and the complete intersection S_p can both have singularities, and still the birational transformation here above holds true. One of the main results in [55] is that there is a natural correspondence between singularities on the cubic fourfold and on the sextic surface, including the type of singularities.

Suppose that the surface S_p has only isolated singularities. Since the singularities of S_p are simple the minimal resolution of S_p is a K3 surface. If S_p is singular at a point y then either $y = \pi_p(p')$, with $p' \neq p$ a singular point of X, or y is singular for the quadric Q_p. In this second case the cubic C_p cannot be singular at y, because otherwise X would be singular along the line $\overline{py}$ while X has only isolated singularities. Therefore the singularities of $X - \{p\}$ are in 1-1 correspondence with the singularities of S_p not contained in $\operatorname{Sing} Q_p$. Let E_p be the exceptional divisor of the the blow-up $\tilde{X}_p \to X$ at p. Then E_p is isomorphic to Q_p and the singularities of $\tilde{X}_p$ on E_p correspond to the singularities of S_p which are contained in $\operatorname{Sing} Q_p$. If X has only a single A_1 singularity p then the surface S_p is a smooth K3 surface, see [55, 5.1 and 5.2]. Note that, due to a result of Varchenko (see [57, Theorem on the Upper Bound, p. 2781]) the maximal number of isolated singularities which can occur on a cubic fourfold is 15.

Let $\mathbf{DM_Q}(\mathbf{C})$ be Voevodsky's triangulated category of motives (with $\mathbf{Q}$-coefficients). There is a fully faithful embedding

$$F : \mathcal{M}_{rat}(\mathbf{C}) \to \mathbf{DM_Q}(\mathbf{C}). \tag{14}$$

For every complex variety V one can define a motive $M(V) \in \mathbf{DM_Q}(\mathbf{C})$ and every blow-up diagram

$$\begin{array}{ccc} E & \xrightarrow{\bar{j}} & Y = \mathrm{Bl}_Z \\ {\scriptstyle \bar{\sigma}}\big\downarrow & & {\scriptstyle \sigma}\big\downarrow \\ Z & \xrightarrow{j} & X \end{array} \tag{15}$$

induces a distinguished triangle in $\mathbf{DM_Q}(\mathbf{C})$, see [59, (4.1.3)]

$$M(E) \xrightarrow{(\bar{\sigma})_* + \bar{j}_*} M(Z) \oplus M(Y) \xrightarrow{j_* - \sigma_*} M(X) \longrightarrow M(E)[1] \tag{16}$$

where E is the exceptional divisor.

Proposition 7.3. *Let $X \subset \mathbb{P}^5$ be a cubic fourfold with isolated singularities. Assume that for some singular point p of X the associated surface S_p belongs to the family $\mathcal{G}$. Then the motive $M(X) \in \mathbf{DM_Q}(\mathbf{C})$ is Schur-finite and lies in the triangulated tensor category $\mathbf{DM}^{Ab}_{\mathbf{Q}}(\mathbf{C})$ generated by the motives of curves.*

Proof. Since $\tilde{X}_p$ is the blow-up of $\mathbb{P}^4$ along S_p there is a diagram

$$\begin{array}{ccc} E_{S_p} & \longrightarrow & \tilde{X}_p = \mathrm{Bl}_{S_p} \\ \downarrow & & \downarrow \\ S_p & \longrightarrow & \mathbb{P}^4 \end{array}$$

where E_{S_p} is the exceptional divisor of the blow-up $q_2 : \tilde{X}_p \to \mathbb{P}^4$. Therefore from (16) we get a distinguished triangle in $\mathbf{DM_Q}(\mathbf{C})$

$$M(E_{S_p}) \longrightarrow M(S_p) \oplus M(\tilde{X}_p) \longrightarrow M(\mathbb{P}^4) \longrightarrow M(E_{S_p})[1]. \tag{17}$$

The exceptional divisor E_{S_p} is a $\mathbb{P}^1$-bundle over S_p. By 6.1 and 6.4 the motive $h(S_p)$ equals the motive of a smooth K3 surface and is finite-dimensional. Therefore its image in $\mathbf{DM_Q}(\mathbf{C})$ under the functor (14) is finite dimensional. By the projective bundle theorem, that is valid also in $\mathbf{DM_Q}(\mathbf{C})$, see [59, (4.1.11)] , we get

$$M(E_{S_p}) \simeq M(S_p)(1)[2]$$

Therefore the motives $M(E_{S_p})$, $M(S_p)$ and $M(\mathbb{P}^4)$ are finite dimensional in $\mathbf{DM_Q}(\mathbf{C})$. Finite dimensional objects in the triangulated category $\mathbf{DM_Q}(\mathbf{C})$ are also Schur-finite and Schur-finiteness has the two out of three property in $\mathbf{DM_Q}(\mathbf{C})$, see [45, Prop. 5.3]. Therefore from (17) we get that the motive $M(\tilde{X}_p)$ is Schur-finite.
The blow-up $q_1 : \tilde{X}_p \to X$ induces a distinguished triangle in $\mathbf{DM_Q}(\mathbf{C})$ as in (16)

$$M(E_p) \longrightarrow M(\{p\}) \oplus M(\tilde{X}_p) \longrightarrow M(X) \longrightarrow M(E_p)[1].$$

The exceptional divisor $E_p \subset \tilde{X}_p$ is isomorphic to the quadric $Q_p \subset \mathbb{P}^4$. Therefore $M(E_p)$ is Schur-finite in $\mathbf{DM_Q}(\mathbf{C})$. The motives $M(E_p)$, $M(\{p\})$ and $M(\tilde{X}_p)$ are Schur-finite. By the the two out of three property we get that the motive $M(X)$ is Schur-finite. Moreover, since the motives of E_p, $\{p\}$ and $\tilde{X}_p$ are of Abelian type in $\mathcal{M}_{rat}(\mathbf{C})$, the motive $M(X)$ belongs to the triangulated tensor subcategory $\mathbf{DM}^{ab}_{\mathbf{Q}}(\mathbf{C})$ of $\mathbf{DM_Q}(\mathbf{C})$ generated by the motives of curves, see [11, Prop. 1.5.6.]. □

7.3. **A family of cubic fourfolds with associated 15-nodal, degree six K3 surface.** The following construction gives an example of a dimension 4 family of singular cubic fourfolds, obtained from K3 surfaces belonging to the family $\mathcal{G}$ of 15-nodal K3 surfaces. Therefore, by Prop. 7.3 the motive of all these fourfolds in $\mathbf{DM_Q}(\mathbf{C})$ is Schur-finite.

The mere existence of a family of dimension 4 of 15-nodal K3 surfaces of degree 6 is assured by [24, Thm. 8.3], but for the lack of a precise reference of our knowledge, we give here a geometric construction of such surfaces.

Proposition 7.4. *A sextic, 15-nodal complete intersection surface in $\mathbb{P}^4$ is a double cover of $\mathbb{P}^2$ ramified along a degenerate sextic F given by two conics and two lines. Such surfaces form a 4-dimensional family. The sextic F has two everywhere tangent conics.*

Proof. Let us consider such a K3 surface $S \subset \mathbb{P}^4$ and project it onto $\mathbb{P}^2$ from a line joining two nodes. Call them q_1 and q_2. Since both the nodes have multiplicity 2 this exhibits S as a double cover of the plane, ramified along a sextic curve F.

For the generic desingularized K3 surface $\tilde{S}$ in our family, we have that $NS(S) \otimes \mathbf{Q}$ is generated by the classes $H, E_1, \ldots, E_{15}$, where $H^2 = 6$ is the natural polarization and the E_i are the exceptional divisors with $E_i^2 = -2$. Hence the polarization on $\tilde{S}$ that gives the map to $\mathbb{P}^2$ is $|H - E_1 - E_2|$, and the family has dimension 4.

Since the projective model in $\mathbb{P}^4$ has 15 nodes, the ramification sextic cannot be smooth. In fact, F has 13 nodes, that are the images of the nodes $q_3, \ldots, q_{15}$, not contained inside the line center of projection. Hence it is reducible, and it splits as two conics and two lines. The images of the two exceptional divisors E_1 and E_2, exactly as in the well-known case of the double cover of $\mathbb{P}^2$ defined by a Kummer surface, are sent to two conics C_1 and C_2 in $\mathbb{P}^2$, which are everywhere tangent to F. That is, $C_i \cap F = 2\Delta_i$, where the Δ_i are degree 6 divisors. The strict transform in $\tilde{S}$ of each C_i is the union of two (-2)curves that intersect in 6 points. One of these two curves corresponds to (one of the) node(s) from which we project. $\square$

Remark 7.5. *We also observe that a naive parameter count gives* $5+5+2+2-8 = 6$ *for the two lines and two conics up to projectivity. The remaining two conditions to descend to the dimension of the family are given by the tangency conditions.*

Now, the construction of the corresponding family of cubic fourfolds is the same as in the one nodal case (see for example [34, Lemma 5.1]). Take any K3 surface $S \subset \mathbb{P}^4$ in the family $\mathcal{G}$ and consider the map

$$\psi : \mathbb{P}^4 \dashrightarrow \mathbb{P}^5, \tag{18}$$

given by the full linear system of cubics through S. This induces a birational transformation that consists first in blowing-up S, and then contract to one point all the trisecant lines to S. The union of these trisecant lines is exactly the strict transform of the only quadric through S, which is then contracted on a (singular) point of X, giving the birational transformation already described in Sect. 7.2. By this construction and Prop. 7.3, we obtain the following.

Theorem 7.6. *Let* $S \subset \mathbb{P}^4$ *be any K3 surface from the family* $\mathcal{G}$*, and let* X *be a singular cubic fourfold obtained from* S *via the map* ψ *of Eq. 18. Then* X *has Schür finite motive in* $\mathbf{DM}_{\mathbf{Q}}(\mathbf{C})$ *and belong to the subcategory* $\mathbf{DM}_{\mathbf{Q}}^{Ab}$.

Appendix A. Lattice theoretic computations.

In this Appendix, we collect the proofs of the lattice theoretical results contained in Sect. 3, in order to make the body of the paper lighter. The first proof we need is the following.

Proof. (of Lemma 3.4) The proof is divided in multiple cases depending on the value of the discriminants.

<u>Case 1</u>: if $d_1, d_2, d_3 \equiv 0[6]$ such that $d_1 = 6n_1, d_2 = 6n_2$ and $d_3 = 6n_3$, for $n_1, n_2 \geq 2$ and $n_3 = \prod_i p_i^2$ with p_i a prime number.

Let us consider the rank 4 lattice,

$M :=< h^2, \alpha_1, \alpha_2, \alpha_3 >$, with $\alpha_1 = e_1^1 + n_1 e_2^1, \alpha_2 = e_1^2 + n_2 e_2^2$ and $\alpha_3 = \sqrt{n_3} a_1$.

The Gram matrix of M with respect to this basis is:

$$\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2n_1 & 0 & 0 \\ 0 & 0 & 2n_2 & 0 \\ 0 & 0 & 0 & 2n_3 \end{pmatrix}.$$

Therefore, M is a positive definite saturated sublattice of L. Moreover, for any nonzero $v = x_1 h^2 + x_2 \alpha_1 + x_3 \alpha_2 + x_4 \alpha_3$, where $x_1, .., x_4$ are integers not all zeros, we have

$$(v, v) = 3x_1^2 + 2n_1 x_2^2 + 2n_2 x_3^2 + 2n_3 x_4^2 \geq 3,$$

$\forall n_1, n_2 \geq 2, n_3$ as required.

We denote by $\mathcal{C}_M \subset \mathcal{C}$ the locus of smooth cubic fourfold such that there is a primitive embedding $M \subset A(X)$ preserving h^2. The locus $\mathcal{C}_M$ is nonempty of codimension 3 by [63, Proposition 2.3]. Let us consider now the sublattices

$$K_{d_1} :=< h^2, \alpha_1 >\subset M$$
$$K_{d_2} :=< h^2, \alpha_2 >\subset M$$
$$K_{d_3} :=< h^2, \alpha_3 >\subset M$$

of discriminant respectively d_1, d_2 and d_3. By [63, Lemma 2.4, Proposition 2.3], we have that $\emptyset \neq \mathcal{C}_M \subset \mathcal{C}_{d_1} \cap \mathcal{C}_{d_2} \cap \mathcal{C}_{d_3}$ as required.

<u>Case 2</u>: if $d_1, d_2 \equiv 0[6], d_3 \equiv 2[6]$ such that $d_1 = 6n_1, d_2 = 6n_2$ and $d_3 = 6n_3 + 2$ for $n_1, n_2 \geq 2$, and $n_3 = \prod_i p_i^2$ with p_i a prime number.

This time we consider the positive definite saturated rank 4 sublattice of L:

$$M :=< h^2, \alpha_1, \alpha_2, \alpha_3 + (0,0,1) > .$$

The Gram matrix of M is:

$$\begin{pmatrix} 3 & 0 & 0 & 1 \\ 0 & 2n_1 & 0 & 0 \\ 0 & 0 & 2n_2 & 0 \\ 1 & 0 & 0 & 2n_3+1 \end{pmatrix}.$$

Additionally, for any nonzero $v = x_1 h^2 + x_2 \alpha_1 + x_3 \alpha_2 + x_4(\alpha_3 + (0,0,1))$ we have that:

$$\begin{aligned} (v, v) = & \quad 3x_1^2 + 2n_1 x_2^2 + 2n_2 x_3^2 + (2n_3+1)x_4^2 + 2x_1 x_4 \\ = & \quad 2x_1^2 + 2n_1 x_2^2 + 2n_2 x_3^2 + 2n_3 x_4^2 + (x_1 + x_4)^2 \geq 3 \end{aligned}$$

$\forall n_1, n_2 \geq 2$ and $n_3 = \prod_i p_i^2$. Then, applying [63, Proposition 2.3], we have that $\mathcal{C}_M$ is nonempty of codimension 3. Now we define the following saturated rank 2 sublattices of M:

$$K_{d_1} :=< h^2, \alpha_1 > \text{ with Gram matrix } \begin{pmatrix} 3 & 0 \\ 0 & 2n_1 \end{pmatrix}$$

$$K_{d_2} :=< h^2, \alpha_2 > \text{ with Gram matrix } \begin{pmatrix} 3 & 0 \\ 0 & 2n_2 \end{pmatrix}$$

$$K_{d_3} := < h^2, \alpha_3 + (0,0,1) > \text{ with Gram matrix } \begin{pmatrix} 3 & 1 \\ 1 & 2n_3+1 \end{pmatrix}$$

Again, by [63, Lemma 2.4] and [63, Proposition 2.3], we have that $\mathcal{C}_M \subset \mathcal{C}_{K_{d_1}} = \mathcal{C}_{d_1}$, $\mathcal{C}_M \subset \mathcal{C}_{K_{d_2}} = \mathcal{C}_{d_2}$ and $\mathcal{C}_M \subset \mathcal{C}_{K_{d_3}} = \mathcal{C}_{d_3}$. Consequently, $\emptyset \neq \mathcal{C}_M \subset \mathcal{C}_{d_1} \cap \mathcal{C}_{d_2} \cap \mathcal{C}_{d_3}$ in this case as well.

Case 3: if $d_1 \equiv 0[6]$ and $d_2, d_3 \equiv 2[6]$ such that $d_1 = 6n_1, d_2 = 6n_2 + 2$ and $d_3 = 6n_3 + 2$ for $n_1 \geq 2, n_2 \geq 1$ and $n_3 = \prod_i p_i^2$ with p_i a prime number.

Let us consider the rank 4 lattice,

$$M := < h^2, \alpha_1, \alpha_2 + (0,1,0), \alpha_3 + (0,0,1) >$$

The Gram matrix of M is:

$$\begin{pmatrix} 3 & 0 & 1 & 1 \\ 0 & 2n_1 & 0 & 0 \\ 1 & 0 & 2n_2+1 & 0 \\ 1 & 0 & 0 & 2n_3+1 \end{pmatrix}.$$

The lattice M is a positive definite saturated sublattice of L. Moreover, for any nonzero $v = x_1 h^2 + x_2\alpha_1 + x_3(\alpha_2 + (0,1,0)) + x_4(\alpha_3 + (0,0,1)) \in M$,

$$\begin{aligned} (v,v) = &\ 3x_1^2 + 2n_1x_2^2 + (2n_2+1)x_3^2 + (2n_3+1)x_4^2 + 2x_1x_3 + 2x_1x_4 \\ = &\ x_1^2 + 2n_1x_2^2 + 2n_2x_3^2 + 2n_3x_4^2 + (x_1+x_3)^2 + (x_1+x_4)^2 \geq 3 \end{aligned}$$

$\forall n_1 \geq 2, n_2 \geq 1, n_3$ as required. Then $\mathcal{C}_M$ is nonempty of codimension 3 by [63, Proposition 2.3]. Now let us consider the saturated rank 2 sublattices of M:

$$\begin{aligned} K_{d_1} &:= < h^2, \alpha_1 > \text{ of Gram matrix } \begin{pmatrix} 3 & 0 \\ 0 & 2n_1 \end{pmatrix} \\ K_{d_2} &:= < h^2, \alpha_2 + (0,1,0) > \text{ of Gram matrix } \begin{pmatrix} 3 & 1 \\ 1 & 2n_2+1 \end{pmatrix} \\ K_{d_3} &:= < h^2, \alpha_3 + (0,0,1) > \text{ of Gram matrix } \begin{pmatrix} 3 & 1 \\ 1 & 2n_3+1 \end{pmatrix} \end{aligned}$$

It is straightforward to see that $\emptyset \neq \mathcal{C}_M \subset \mathcal{C}_{d_1} \cap \mathcal{C}_{d_2} \cap \mathcal{C}_{d_3}$ also in this case.

Case 4: if $d_1, d_2, d_3 \equiv 2[6]$ such that $d_1 = 6n_1 + 2, d_2 = 6n_2 + 2$ and $d_3 = 6n_3 + 2$ for $n_1, n_2 \geq 1$ and $n_3 = \prod_i p_i^2$ with p_i a prime number.

Let us define the lattice M as follows

$$M := < h^2, \alpha_1 + (1,0,0), \alpha_2 + (0,1,0), \alpha_3 + (0,0,1) >$$

The Gram matrix of M with respect to this basis is:

$$\begin{pmatrix} 3 & 1 & 1 & 1 \\ 1 & 2n_1+1 & 0 & 0 \\ 1 & 0 & 2n_2+1 & 0 \\ 1 & 0 & 0 & 2n_3+1 \end{pmatrix},$$

and M is a rank 4 positive definite saturated sublattice of L. Furthermore, for any nonzero $v = x_1h^2 + x_2(\alpha_1 + (1,0,0)) + x_3(\alpha_2 + (0,1,0)) + x_4(\alpha_3 + (0,0,1)) \in M$, we have that:

$$\begin{aligned}(v,v) = &\quad 3x_1^2 + (2n_1+1)x_2^2 + (2n_2+1)x_3^2 + (2n_3+1)x_4^2 + 2x_1x_2 + 2x_1x_3 + 2x_1x_4 \\ = &\qquad\qquad 2n_1x_2^2 + 2n_2x_3^2 + 2n_3x_4^2 + (x_1+x_2)^2(x_1+x_3)^2 + (x_1+x_4)^2 \geq 3\end{aligned}$$

$\forall n_1, n_2 \geq 1, n_3$ as required. As usual, we consider now the rank 2 saturated sublattices of M:

$$\begin{aligned}K_{d_1} &:=< h^2, \alpha_1 + (1,0.0) > \\ K_{d_2} &:=< h^2, \alpha_2 + (0,1,0) > \\ K_{d_3} &:=< h^2, \alpha_3 + (0,0,1) >\end{aligned}$$

Applying [63, Lemma 2.4] and [63, Proposition 2.3], we obtain that $\emptyset \neq \mathcal{C}_M \subset \mathcal{C}_{d_1} \cap \mathcal{C}_{d_2} \cap \mathcal{C}_{d_3}$ also in this case.

□

Now we continue with the proof of Prop. 3.5.

*Proof. (of Prop. 3.5)*We will prove that $\forall n \geq 4$, $\bigcap_{k=1}^{n} \mathcal{C}_{d_k} \neq \emptyset$.

Since for $n \leq 3$, this formula has already been proved (see Proposition 3.1 and Lemma 3.4), we proceed to consider the case $n = 4$.

<u>For $n = 4$:</u> $d_1 = 6n_1, d_2 = 6n_2, d_3 = 6n_3, d_4 = 6n_4$ for $n_1, n_2 \geq 2$ and $n_3, n_4 = \prod_i p_i^2$ with p_i a prime number.

Let us consider the rank 5 lattice,

$M :=< h^2, \alpha_1, \alpha_2, \alpha_3, \alpha_4 >$, with $\alpha_1 = e_1^1 + n_1e_2^1, \alpha_2 = e_1^2 + n_2e_2^2$, $\alpha_3 = \sqrt{n_3}a_1$ and $\alpha_4 = \sqrt{n_4}a_2$.

The lattice M is a positive definite saturated sublattice of L with the following Gram matrix:

$$\begin{pmatrix} 3 & 0 & 0 & 0 & 0 \\ 0 & 2n_1 & 0 & 0 & 0 \\ 0 & 0 & 2n_2 & 0 & 0 \\ 0 & 0 & 0 & 2n_3 & \sqrt{n_3n_4} \\ 0 & 0 & 0 & \sqrt{n_3n_4} & 2n_4 \end{pmatrix}.$$

Furthermore, for any nonzero $v = x_1h^2 + x_2\alpha_1 + x_3\alpha_3 + x_4\alpha_3 + x_5\alpha_4 \in M$, we have that

$$(v,v) = 3x_1^2 + 2n_1x_2^2 + 2n_2x_3^2 + n_3x_4^2 + n_4x_5^2 + (\sqrt{n_3}x_4 + \sqrt{n_4}x_5)^2 \geq 3$$

$\forall n_1, n_2 \geq 2$, $n_3, n_4 = \prod_i p_i^2$ with p_i a prime number (thus $n_3, n_4 \geq 4$). The locus $\mathcal{C}_M$ is then nonempty of codimension 4. Now let us consider the rank 2 saturated sublattices of M:

$$K_{d_i} :=< h^2, \alpha_i >, i = 1, \ldots, 4,$$

each of discriminant d_i. By [63, Lemma 2.4] and [63, Proposition 2.3], We have that $\emptyset \neq \mathcal{C}_M \subset \mathcal{C}_{d_1} \cap \mathcal{C}_{d_2} \cap \mathcal{C}_{d_3} \cap \mathcal{C}_{d_4}$ with the required conditions on d_i.

Then the same method applies to check that $\emptyset \neq \mathcal{C}_M \subset \bigcap_{k=1}^{n} \mathcal{C}_{d_k}$, for $5 \leq n \leq 19$, hence we refrain to give details. Nevertheless, let us go through the final, $n = 20$, step.

<u>For n=20</u>: if $d_i = 6n_i,\ i = 1, .., 20$, with $n_1, n_2 \geq 2$ and $n_3, .., n_{20} = \prod_i p_i^2$ with p_i a prime number. We need to consider the rank 21 lattice M generated by:

$< h^2, \alpha_1, \alpha_2, \sqrt{n_3}a_1, \sqrt{n_4}a_2, \sqrt{n_5}t_1^1, \sqrt{n_6}t_3^1, \sqrt{n_7}t_6^1, \sqrt{n_8}t_1^2, \sqrt{n_9}t_3^2, \sqrt{n_{10}}t_6^2, \sqrt{n_{11}}t_2^1, \sqrt{n_{12}}t_2^2, \sqrt{n_{13}}t_4^1, \sqrt{n_{14}}t_4^2, \sqrt{n_{15}}t_7^1, \sqrt{n_{16}}t_7^2, \sqrt{n_{17}}t_8^1, \sqrt{n_{18}}t_8^2, \sqrt{n_{19}}t_5^1, \sqrt{n_{20}}t_5^2 >$,

with the following Gram matrix:

$$\left(\begin{array}{c|c} A & B \\ \hline B^T & C \end{array}\right)$$

such that

$$A = \begin{pmatrix}
3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2n_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2n_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2n_3 & \sqrt{n_3n_4} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \sqrt{n_3n_4} & 2n_4 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2n_5 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2n_6 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 2n_7 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2n_8 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2n_9 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2n_{10}
\end{pmatrix}$$

$$B = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\sqrt{n_5n_{11}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\sqrt{n_6n_{11}} & 0 & -\sqrt{n_6n_{13}} & 0 & 0 & 0 & 0 & 0 & -\sqrt{n_6n_{19}} & 0 \\
0 & 0 & 0 & 0 & -\sqrt{n_7n_{15}} & 0 & 0 & 0 & -\sqrt{n_7n_{19}} & 0 \\
0 & -\sqrt{n_8n_{12}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\sqrt{n_9n_{12}} & 0 & -\sqrt{n_9n_{14}} & 0 & 0 & 0 & 0 & 0 & -\sqrt{n_9n_{20}} \\
0 & 0 & 0 & 0 & 0 & -\sqrt{n_{10}n_{16}} & 0 & 0 & 0 & -\sqrt{n_{10}n_{20}}
\end{pmatrix}$$

$$C=\begin{pmatrix}
2n_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 2n_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 2n_{13} & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 2n_{14} & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 2n_{15} & 0 & -\sqrt{n_{15}n_{17}} & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 2n_{16} & 0 & -\sqrt{n_{16}n_{18}} & 0 & 0\\
0 & 0 & 0 & 0 & -\sqrt{n_{15}n_{17}} & 0 & 2n_{17} & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & -\sqrt{n_{16}n_{18}} & 0 & 2n_{18} & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2n_{19} & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2n_{20}
\end{pmatrix}$$

The lattice $M \subset L$ is a positive definite saturated lattice containing h^2. Moreover, for all $v \in M$, we have that

$$(v, v) \neq 2.$$

for $n_1, n_2 \geq 2$, $n_3, .., n_{20}$ as required. Then, by [63, Proposition 2.3, Lemma 2.4], the locus $\mathcal{C}_M$ is nonempty of codimension 20.
Finally, we consider the rank 2 saturated sublattices $K_{d_i} \subset M$ generated by h^2 and another element from the basis of M: $\alpha_1, \alpha_2, \sqrt{n_3}a_1, \sqrt{n_4}a_2, \sqrt{n_5}t_1^1, \sqrt{n_6}t_3^1, \sqrt{n_7}t_6^1, \sqrt{n_8}t_1^2, \sqrt{n_9}t_3^2, \sqrt{n_{10}}t_6^2, \sqrt{n_{11}}t_2^1$, $\sqrt{n_{12}}t_2^2, \sqrt{n_{13}}t_4^1, \sqrt{n_{14}}t_4^2, \sqrt{n_{15}}t_7^1, \sqrt{n_{16}}t_7^2, \sqrt{n_{17}}t_8^1, \sqrt{n_{18}}t_8^2, \sqrt{n_{19}}t_5^1, \sqrt{n_{20}}t_5^2$. The lattice K_{d_i} is of discriminant d_i, for all i. Applying [63, Propsition 2.3, Lemma 2.4] again, we obtain $\emptyset \neq \mathcal{C}_M \subset \mathcal{C}_{d_1} \cap .. \cap \mathcal{C}_{d_{20}}$ with the required conditions on d_i. □

Finally we give the proof of our first main Thm. 3.6.

Proof. (of Thm. 3.6) Let us consider $\bigcap_{k=1}^{n} \mathcal{C}_{d_k}$.
For $n = 2$: see [63, Theorem 3.1].
For $n = 3$: see Lemma 3.4.
For $1 \leq n \leq 20$: for $d_k \equiv 0$ [6], $d_3, .., d_{20} = 6\prod_i p_i^2$, see Proposition 3.5.
In the following, we will describe in detail the case $n = 4$. For higher values of n, it will be clear that one needs to use exactly the same strategy of $n = 4$, hence for the sake of shortness, we will omit most of them.

For $n = 4$:

Case 1: if all discriminants $d_1, d_2, d_3, d_4 \equiv 0[6]$. This case is proved in Proposition 3.5.

Case 2: if $d_1 = 6n_1, d_2 = 6n_2, d_3 = 6n_3, d_4 = 6n_4 + 2$ such that $n_1, n_2 \geq 2$ and $n_3, n_4 = \prod_i p_i^2$ for certain prime numbers p_i.
We consider the rank 5 lattice $M :=< h^2, \alpha_1, \alpha_2, \alpha_3, \alpha_4 + (0, 0, 1) >$, with $\alpha_1 = e_1^1 + n_1 e_2^1, \alpha_2 = e_1^2 + n_2 e_2^2$, $\alpha_3 = \sqrt{n_3}a_1$ and $\alpha_4 = \sqrt{n_4}a_2$.

The Gram matrix of M with respect to this basis is:

$$\begin{pmatrix} 3 & 0 & 0 & 0 & 1 \\ 0 & 2n_1 & 0 & 0 & 0 \\ 0 & 0 & 2n_2 & 0 & 0 \\ 0 & 0 & 0 & 2n_3 & \sqrt{n_3n_4} \\ 1 & 0 & 0 & \sqrt{n_3n_4} & 2n_4+1 \end{pmatrix},$$

and M is a positive definite saturated sublattice of L. For all $v \in M$ $v = x_1h^2 + x_2\alpha_1 + x_3\alpha_2 + x_4\alpha_3 + x_5(\alpha_4 + (0,0,1)) \in M$, we have that

$$(v,v) = 2x_1^2 + 2n_1x_2^2 + 2n_2x_3^2 + n_3x_4^2 + n_4x_5^2 + (\sqrt{n_3}x_4 + \sqrt{n_4}x_5)^2 + (x_1+x_5)^2 \neq 2,$$

for all $n_1, n_2 \geq 2$, $n_3, n_4 = \prod_i p_i^2$ with p_i a prime number (thus $n_3, n_4 \geq 4$). Then $\mathcal{C}_M$ is nonempty of codimension 4. Moreover, let us consider

$$K_{d_i} :=< h^2, \alpha_i >, i : 1, .., 4.$$

Hence, by [63, Lemma 2.4, Proposition 2.3], we have that $\emptyset \neq \mathcal{C}_M \subset \mathcal{C}_{d_1} \cap \mathcal{C}_{d_2} \cap \mathcal{C}_{d_3} \cap \mathcal{C}_{d_4}$ with the required conditions on d_i.

Case 3: if $d_1 = 6n_1, d_2 = 6n_2, d_3 = 6n_3+2, d_4 = 6n_4+2$ such that $n_1, n_2 \geq 2$ and $n_3, n_4 = \prod_i p_i^2$ with p_i a prime number.

We consider the rank 5 lattice,

$$M :=< h^2, \alpha_1, \alpha_2, \alpha_3 + (0,1,0), \alpha_4 + (0,0,1) >,$$

with $\alpha_1 = e_1^1 + n_1e_2^1, \alpha_2 = e_1^2 + n_2e_2^2$, $\alpha_3 = \sqrt{n_3}a_1$ and $\alpha_4 = \sqrt{n_4}a_2$.

The Gram matrix of M is

$$\begin{pmatrix} 3 & 0 & 0 & 1 & 1 \\ 0 & 2n_1 & 0 & 0 & 0 \\ 0 & 0 & 2n_2 & 0 & 0 \\ 1 & 0 & 0 & 2n_3+1 & \sqrt{n_3n_4} \\ 1 & 0 & 0 & \sqrt{n_3n_4} & 2n_4+1 \end{pmatrix}$$

and M is a positive definite saturated sublattice of L. In addition, for all $v \in M$ $v = x_1h^2 + x_2\alpha_1 + x_3\alpha_2 + x_4(\alpha_3 + (0,1,0)) + x_5(\alpha_4 + (0,0,1)) \in M$, we have that

$$(v,v) = x_1^2 + 2n_1x_2^2 + 2n_2x_3^2 + n_3x_4^2 + n_4x_5^2 + (\sqrt{n_3}x_4 + \sqrt{n_4}x_5)^2 + (x_1+x_5)^2 + (x_1+x_4)^2 \neq 2,$$

for all $n_1, n_2 \geq 2$, $n_3, n_4 = \prod_i p_i^2$ with p_i a prime number. Then $\mathcal{C}_M$ is nonempty of codimension 4 by [63, Proposition 2.3]. Moreover, if we consider

$$K_{d_i} :=< h^2, \alpha_i >, i : 1, .., 4,$$

then, by [63, Lemma 2.4, Proposition 2.3], we have that $\emptyset \neq \mathcal{C}_M \subset \mathcal{C}_{d_1} \cap \mathcal{C}_{d_2} \cap \mathcal{C}_{d_3} \cap \mathcal{C}_{d_4}$ with the required conditions on d_i.

Case 4: if $d_1 = 6n_1, d_2 = 6n_2 + 2, d_3 = 6n_3 + 2, d_4 = 6n_4 + 2$ such that $n_1 \geq 2$, $n_2 \geq 1$ and $n_3, n_4 = \prod_i p_i^2$ with p_i a prime number.

We consider the rank 5 lattice,

$$M := < h^2, \alpha_1, \alpha_2 + (1,0,0), \alpha_3 + (0,1,0), \alpha_4 + (0,0,1) > .$$

The Gram matrix of M is:

$$\begin{pmatrix} 3 & 0 & 1 & 1 & 1 \\ 0 & 2n_1 & 0 & 0 & 0 \\ 1 & 0 & 2n_2+1 & 0 & 0 \\ 1 & 0 & 0 & 2n_3+1 & \sqrt{n_3 n_4} \\ 1 & 0 & 0 & \sqrt{n_3 n_4} & 2n_4+1 \end{pmatrix}.$$

Hence, M is a positive definite saturated sublattice of L. Furthermore, for all $v \in M$, $v = x_1 h^2 + x_2\alpha_1 + x_3(\alpha_2 + (1,0,0)) + x_4(\alpha_3 + (0,1,0)) + x_5(\alpha_4 + (0,0,1)) \in M$, we have that

$$(v,v) = 2n_1x_2^2 + 2n_2x_3^2 + n_3x_4^2 + n_4x_5^2 + (\sqrt{n_3}x_4 + \sqrt{n_4}x_5)^2 + (x_1+x_5)^2 + (x_1+x_4)^2 + (x_1+x_3)^2 \neq 2$$

for all $n_1 \geq 2$, $n_2 \geq 1$ and $n_3, n_4 = \prod_i p_i^2$ with p_i a prime number (thus $n_3, n_4 \geq 4$). Then $\mathcal{C}_M$ is nonempty of codimension 4. Moreover, let us define

$$K_{d_i} := < h^2, \alpha_i >, i : 1, .., 4.$$

We have then that $\emptyset \neq \mathcal{C}_M \subset \mathcal{C}_{d_1} \cap \mathcal{C}_{d_2} \cap \mathcal{C}_{d_3} \cap \mathcal{C}_{d_4}$ with the required conditions on d_i.

Case 5: if $d_1 = 6n_1 + 2, d_2 = 6n_2 + 2, d_3 = 6n_3 + 2, d_4 = 6n_4 + 2$ such that $n_1, n_2 \geq 1$ and $n_3, n_4 = \prod_i p_i^2$ with p_i a prime number.

We need to consider the rank 5 lattice,

$$M := < h^2, \alpha_1 + (1,0,0), \alpha_2 + (1,0,0), \alpha_3 + (0,1,0), \alpha_4 + (0,0,1) >$$

with the following Gram matrix:

$$\begin{pmatrix} 3 & 1 & 1 & 1 & 1 \\ 1 & 2n_1+1 & 0 & 0 & 0 \\ 1 & 0 & 2n_2+1 & 0 & 0 \\ 1 & 0 & 0 & 2n_3+1 & \sqrt{n_3 n_4} \\ 1 & 0 & 0 & \sqrt{n_3 n_4} & 2n_4+1 \end{pmatrix}.$$

The lattice M is a positive definite saturated sublattice of L. For all $v \in M$, $v = x_1h^2 + x_2(\alpha_1 + (1,0,0)) + x_3(\alpha_2 + (1,0,0)) + x_4(\alpha_3 + (0,1,0)) + x_5(\alpha_4 + (0,0,1)) \in M$, we have that

$$(v,v) = 2n_1x_2^2 + 2n_2x_3^2 + n_3x_4^2 + n_4x_5^2 + (\sqrt{n_3}x_4 + \sqrt{n_4}x_5)^2 + (x_1+x_5)^2 + (x_1+x_4)^2 + (x_1+x_3)^2 + (x_1+x_2)^2 - x_1^2 \neq 2,$$

for all $n_1, n_2 \geq 1$, $n_3, n_4 = \prod_i p_i^2$ with p_i a prime number. Hence $\mathcal{C}_M$ is nonempty of codimension 4. As usual, we set the rank 2 saturated sublattices

$$K_{d_i} := < h^2, \alpha_i >, i : 1, .., 4.$$

Applying [63, Lemma 2.4], we have that $\emptyset \neq \mathcal{C}_M \subset \mathcal{C}_{d_1} \cap \mathcal{C}_{d_2} \cap \mathcal{C}_{d_3} \cap \mathcal{C}_{d_4}$ with the required conditions on d_i.

For $5 \leq n \leq 19$: In the same way, one can continue to consider, for each intersection of n Hassett divisors, certain positive definite saturated rank $n+1$ lattice E containing h^2 and check that for a nonzero $v \in E$, $(v.v) \geq 3$ or equivalently for all $v \in E$, $(v, v) \neq 2$. This allows us to conclude the existence of the required family of cubics inside the intersection (by [63, Lemma 2.4, Proposition 2.3]).

Finally, in order to study the intersection of 20 Hassett divisors $\mathcal{C}_{d_k}$ such that all $d_k \equiv 2[6]$, We consider, for example, the rank 21 lattice M, generated by

$< h^2, \alpha_1 + (0,1,0), \alpha_2 + (1,0,0), \sqrt{n_3}a_1 + (1,0,0), \sqrt{n_4}a_2 + (0,1,0), \sqrt{n_5}t_1^1 + (0,0,1), \sqrt{n_6}t_3^1 + (0,0,1), \sqrt{n_7}t_6^1 + (1,0,0), \sqrt{n_8}t_1^2 + (0,1,0), \sqrt{n_9}t_3^2 + (0,0,1), \sqrt{n_{10}}t_6^2 + (0,0,1), \sqrt{n_{11}}t_2^1 + (0,0,1), \sqrt{n_{12}}t_2^2 + (0,0,1), \sqrt{n_{13}}t_4^1 + (0,0,1), \sqrt{n_{14}}t_4^2 + (1,0,0), \sqrt{n_{15}}t_7^1 + (0,1,0), \sqrt{n_{16}}t_7^2 + (0,1,0), \sqrt{n_{17}}t_8^1 + (0,1,0), \sqrt{n_{18}}t_8^2 + (0,1,0), \sqrt{n_{19}}t_5^1 + (0,1,0), \sqrt{n_{20}}t_5^2 + (0,1,0) >.$

The Gram matrix of M is:

$$\left(\begin{array}{c|c} A & B \\ \hline B^T & C \end{array}\right)$$

with

$$A = \begin{pmatrix} 3 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2n_1+1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 2n_2+1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 2n_3+1 & \sqrt{n_3 n_4} & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & \sqrt{n_3 n_4} & 2n_4+1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 2n_5+1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 2n_6+1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 2n_7+1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 2n_8+1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 2n_9+1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 2n_{10}+1 \end{pmatrix}$$

$$B=\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1\\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1\\
1-\sqrt{n_5n_{11}} & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1-\sqrt{n_6n_{11}} & 1 & 1-\sqrt{n_6n_{13}} & 0 & 0 & 0 & 0 & 0 & -\sqrt{n_6n_{19}} & 0\\
0 & 0 & 0 & 1 & -\sqrt{n_7n_{15}} & 0 & 0 & 0 & -\sqrt{n_7n_{19}} & 0\\
0 & -\sqrt{n_8n_{12}} & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1\\
1 & 1-\sqrt{n_9n_{12}} & 1 & -\sqrt{n_9n_{14}} & 0 & 0 & 0 & 0 & 0 & -\sqrt{n_9n_{20}}\\
1 & 1 & 1 & 0 & 0 & -\sqrt{n_{10}n_{16}} & 0 & 0 & 0 & -\sqrt{n_{10}n_{20}}
\end{pmatrix}$$

$$C=\begin{pmatrix}
2n_{11}+1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 2n_{12}+1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 1 & 2n_{13}+1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 2n_{14}+1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 2n_{15}+1 & 0 & -\sqrt{n_{15}n_{17}} & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 2n_{16}+1 & 0 & -\sqrt{n_{16}n_{18}} & 0 & 0\\
0 & 0 & 0 & 0 & -\sqrt{n_{15}n_{17}} & 0 & 2n_{17}+1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & -\sqrt{n_{16}n_{18}} & 0 & 2n_{18}+1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2n_{19}+1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2n_{20}+1
\end{pmatrix}.$$

The lattice M is a positive definite saturated sublattice of L. For all $v\in M, (v,v)\neq 2\ \forall n_1,n_2\geq 1$, $n_k=\prod_i p_i^2$ with p_i a prime number for $k=3,..,20$. Then, we consider rank 2 saturated sublattices $h^2\in K_{d_i}$ of discriminant d_i in the same way as we did in the preceding cases. Therefore, by [63, Lemma 2.4, Proposition 2.3], $\emptyset\neq\mathcal{C}_M\subset\bigcap_{k=1}^{20}\mathcal{C}_{d_k}\neq\emptyset$ for all d_k satisfying $(*)$ and d_k satisfying $(**)$ for $k\geq 3$. In this case $\mathcal{C}_M$ is a dimension zero subvariety of $\mathcal{C}$. □

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