On Random Extended Intervals and their ARMA Processes

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Babel Raïssa GUEMDJO KAMDEM[*], Jules SADEFO KAMDEM[†], Carlos OUGOUYANDJOU[*]

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Abstract

This work introduces and characterizes the so called "random extended intervals", these are random intervals for which the left bound may be higher than the right one. To carry out this study, we introduce on the set of random extended intervals a structure of metric space relevant to study extended interval-valued ARMA time series. This is done by extending the Hausdorff metric on extended intervals and defining a family of metrics d_γ relevant for the set of random extended intervals and which do not have some disadvantages of the Hausdorff metric. We show that there exists a unique metric d_γ for which $\gamma(t)dt$ is what we have called "adapted measure" and we use this metric to define variability for random extended intervals.

Keywords: Uncertainty Modeling, Stochastic Processes, Random Extended Interval, Hausdorff Metrics , Time series, ARMA

1 Introduction

Intervals analysis (see Bauch (1992); Moore (1966); Jaulin et al. (2001); Alefeld and Herzberger (2012)) initially developed from the 1960s to take into account in a rigorous way, different types of uncertainties (rounding errors due to finite precision calculations, measurement uncertainties, linearization errors) makes it possible to build supersets of the domain of variation of a real function. Coupled with the usual theorems of existence, for example, the Brouwer or Miranda theorems, the interval theory also makes it possible to rigorously prove the existence of solutions for a system of equations (see Goldsztejn et al. (2005)). With interval analysis, it was now possible to modeling interval data.

In recent years, more precisely since the end of 1980s years, interval modeling has caught the attention of a growing number of researchers. The advantage of an interval-valued time series over a point-valued time series lies in that it contains both the trend (or level) information and volatility information (e.g., the range between the boundaries), while some informational loss is encountered when one uses a conventional point-valued data set, e.g., the closing prices of a stock collected at a specific time point within each time period, since it fails to record the valuable intraday information. Higher-frequency point-valued observations could result in hardly discriminating information from noises. A solution is to analyze the information in an interval format by collecting the maximum and minimum prices in a day, which avoids undesirable noises in the intraday data and contains more information than point-valued observations Sun et al. (2018). There is an increasing literature using interval financial time series for markets risk analysis. A part of this literature is based on fuzzy approach (see for instance Mbairadjim et al. (2014b), Mbairadjim et al. (2014a) and Sadefo Kamdem et al. (2012) and some references therein).

[*]Institut de Mathématiques et de Science Physique, UAC, Bénin
[†]Université de Montpellier (MRE EA 7491)

A huge progress in the field of interval-valued time series has been done by Billard and Diday Billard and Diday (2000, 2003) who first proposed a linear regression model for the center points of 37 interval-valued data. They have been followed by other authors Maia et al. (2008); Hsu and Wu (2008); Wang and Li (2011); González-Rivera and Lin (2013); Wang et al. (2016). To study interval data, all those references apply point-valued techniques on the center, the left bound or the right bound. By so doing, they may not efficiently make use of the information contained in interval data. In 2016, Han et al Han et al. (2016) developed a minimum-distance estimator to match the interval model predictor with the observed interval data as much as possible. They proposed a parsimonious autoregressive model for a vector of interval-valued time series processes with exogenous explanatory interval variables in which an interval observation is considered as a set of ordered numbers. It is shown that their model can efficiently utilize the information contained in interval data, and thus provides more efficient inferences than point-based data and models Han et al. (2015). Despite all those good items, the classical theory of interval modeling has some inconveniences. We can enumerate two which are address in another work and in the present paper, respectively.

Firstly, the set of random intervals (or more generally random sets) is not a vector space. Indeed, the set of intervals is not an abelian group for the classical addition of intervals. So, all the useful theorems obtain through orthogonal projection as Wold decomposition Theorem cannot be extended to interval-valued processes. Secondly, in time series, interval-valued data does not take into account some specifications or details of the study period, as for instance in the financial markets where a movement in stock prices during a given trading period is an observation of bounded intervals by maximum and minimum daily prices (see Han et al. (2016)). One can use two concepts to address each of those both inconveniences. One can consider the set of random intervals as a "pseudovector space" where vectors do not necessarily have opposites. This concept of pseudovector space has been developed in Sadefo Kamdem et al. (2020) to address the first inconvenience stated above. The second inconvenience can be address by working with "extended intervals" instead of classical intervals; as in the present paper.

Indeed, it may be often more relevant to consider extended intervals formed by the opening and closing prices, regarding stock prices. Also, for the daily temperature in meteorology, instead of taking the max and min, it would be better in some cases to take the morning and evening temperature, as well as for the systolic and diastolic blood pressures in medicine. For this last example of blood pressure, when plotting the blood pressure of somebody as extended intervals of morning and evening records, one can see easily days where the morning blood pressure was higher than the evening one, which can indicate illness or emotional issue.

Therefore, given the constraints imposed by classical interval theory and its application on time series, our approach is based on the concept of extended or generalized intervals for which the left bound is not necessarily less than the right one. This generalization makes our modeling approach relevant for time series analysis. This generalization guarantees the completeness of interval space and consistency between interval operations. Extended intervals are also used for time series analysis in Han et al. (2012) but their approach does not highlight the advantages of generalized interval-valued variables.

Our contribution is therefore both theoretical and empirical. In other words, we have conceptualized and redefined some of the specific characteristics of the set of extended intervals. More precisely, we define on the set of extended intervals, a topology which generalizes the natural topology on the set of classical interval, unlike the topology introduces by Ortolf and Kaucher Ortolf (1969); Kaucher (1973) on generalized intervals, and which restricted on classical interval is completely different from their natural topology.

The rest of the work is organized as follows: The main purpose of Section 2 is to fix notations, give a consistent definition of extended intervals, and introduce operators such as inclusion, intersection, and union. In Section 3 we introduce a suitable class of distances on the set of random extended intervals, that solves a disadvantage of the Hausdorff. We use this new

distance to define variance and covariance of random extended intervals and we show that they share some useful properties with point-valued random variables, (see propositions 3.2 and 3.3). Section 4 is concerned with stationary extended interval-valued time series and ARMA model are investigated. In Section 5, we prove the Wold decomposition version of extended interval-valued time series. Section 6 is about the numerical study and we have used there the software R. We make a simulation of an I-AR(1) process and illustrate the interpretation of a plot of extended intervals on a few data on blood pressure. We also do empirical analysis and forecasting of the French CAC 40 market index from June 1st to July 26, 2019.

2 Extended intervals

In this section, we first recall some concepts related to standard intervals. Next, we define what we mean by "extended interval" and we introduce a new topology on the union $\mathbb{R} \cup \mathbb{R}_{\leftarrow}$ of real numbers with decreasing real numbers, that generalizes the topology of $\mathbb{R}$.

Let $K_{kc}(\mathbb{R})$ be the set of nonempty compact (and convex) intervals. For $A = [a_1, a_2], B = [b_1, b_2] \in K_{kc}(\mathbb{R})$ and $\lambda \in \mathbb{R}$, we recall the operations

$$A + B = [a_1 + b_1, a_2 + b_2] \tag{1}$$

$$\lambda A = \begin{cases} [\lambda a_1, \lambda a_2] & \text{if } \lambda \geq 0 \\ [\lambda a_2, \lambda a_1] & \text{if } \lambda \leq 0 \end{cases}. \tag{2}$$

It is noteworthy that $K_{kc}(\mathbb{R})$ is closed under those operations, but it is not a vector space, since $A + (-1)A$ is not necessarily $\{0\}$, unless $A = \{0\}$. The Hausdorff distance d_H is defined for closed intervals $[a_1, a_2]$ and $[b_1, b_2]$ by

$$d_H([a_1, a_2], [b_1, b_2]) = max(|b_1 - a_1|, |b_2 - a_2|).$$

It is well-known that $(K_{kc}(\mathbb{R}), d_H)$ is a complete metric space (see Yang and Li (2005) for details). For $A \in K_{kc}(\mathbb{R})$, the support function of A is the function $s(\cdot, A) : \mathbb{R} \to \mathbb{R}$ defined by

$$s(x, A) = \sup\{ax \ ; \ a \in A\}. \tag{3}$$

Equivalently, if we set $A = [a_1, a_2]$,

$$s(x, A) = \max(xa_1, xa_2).$$

Keep in mind that $s(x, A)$ returns x times the left bound of A when x is negative and x times the right bound of A when x is positive. The precede remark will be used to extend the support function on extended closed intervals.

It is clear that the daily temperature of a given region is not a single real value, but a continuous function from day hours to the set of real numbers. In classical interval analysis Maia et al. (2008); Kearfott (1996); Moore (1966) the daily temperature is the interval bounded from below and above by the minimum and maximum daily temperatures. This approach is quite precise, but we can still do better. In fact, most often, from midnight (i.e. from 00:00), the temperature is decreasing until the sun begins to rise. When sun is rising, the temperature increases until the sun is at its zenith and temperature begins decreasing again, and so on. (The reader can visit the page hourly weather of the web site weather.com.) In such a situation, we may consider that the temperature of each day is a union of what we will call extended intervals, each of them being either increasing or decreasing. This model gives more information. Decreasing interval indicates that the temperature is more or less decreasing during the considered period.

Definition 2.1. *Let $\underline{A}, \overline{A} \in \mathbb{R} \cup \{\pm\infty\}$. An **extended interval** is the range A of real numbers between $\underline{A}$ and $\overline{A}$, and runs through from $\underline{A}$ to $\overline{A}$.*

A difference with standard intervals is that, for extended intervals we do not impose that $\underline{A} \leq \overline{A}$. But, the running direction is important. We say that A is an **increasing extended interval** or a **proper interval** when $\underline{A} < \overline{A}$, is an **decreasing extended interval** or an **improper interval** when $\underline{A} > \overline{A}$ and is a **degenerate interval** when $\underline{A} = \overline{A}$. When $\underline{A}$ and $\overline{A}$ are in A, we say that A is an extended closed interval and denote it by $A = \lfloor \underline{A}, \overline{A} \rfloor$. We also have extended open intervals $\rfloor \underline{A}, \overline{A} \lfloor$, $\mathbb{R} = \rfloor -\infty, \infty \lfloor$ and $\mathbb{R}_{\leftarrow} := \rfloor \infty, -\infty \lfloor$.

Any non-degenerate extended interval A represents the classical interval from $\min(\underline{A}, \overline{A})$ to $\max(\underline{A}, \overline{A})$ ran in the increasing direction (for increasing extended interval) or ran in the decreasing direction (for decreasing extended interval). We call $\underline{A}$ the left bound and $\overline{A}$ the right bound of the extended interval A. To extend the natural topology of $\mathbb{R}$ to $\mathbb{R}_{\rightleftarrows} := \mathbb{R} \cup \mathbb{R}_{\leftarrow}$ we need to define properly intersection for extended intervals. A preliminary observation is that in $\mathbb{R}_{\rightleftarrows}$ we have that $([0,1] \cap [1,2])^c = \{1\}^c = (\mathbb{R} \cup \mathbb{R}_{\leftarrow}) \setminus \{1\}$ and $[0,1]^c \cup [1,2]^c = (-\infty, 1) \cup (2, \infty) \cup \mathbb{R}_{\leftarrow}$; where A^c represent the complementary of A in $\mathbb{R}_{\rightleftarrows}$. So, for the Moivre's formula $(A \cap B)^c = A^c \cup B^c$ to hold, we must precise if a given degenerate interval is increasing or decreasing. For any $a \in \mathbb{R}$, we denote by $\{a\}$ the increasing degenerate extended interval made by a and by $\{a\}_{\leftarrow}$ the decreasing one.

We will write $A \subseteq B$ when $\underline{B} \leq \underline{A} \leq \overline{A} \leq \overline{B}$ or $\underline{B} \geq \underline{A} \geq \overline{A} \geq \overline{B}$. This definition of inclusion takes into account the running direction. For instance, $\lfloor 1, 2 \rfloor \not\subseteq \lfloor 3, 1 \rfloor$, although all elements of $\lfloor 1, 2 \rfloor$ are elements of $\lfloor 3, 1 \rfloor$. The only obstruction for the inclusion to hold in this example is the difference in the running direction between both intervals. In addition, we agree that $\{2\}_{\leftarrow} \not\subseteq \lfloor 1, 3 \rfloor$ and $\{2\} \not\subseteq \lfloor 3, 1 \rfloor$ but $\{2\}_{\leftarrow} \subseteq \lfloor 3, 1 \rfloor$ and $\{2\} \subseteq \lfloor 1, 3 \rfloor$. We extend now the definition of intersection to extended intervals in such a way that the elementary properties $A \cap A = A$, $A \cap \emptyset = \emptyset$, $A \cap \mathbb{R}_{\rightleftarrows} = A$ and $(A \subseteq B \implies A \cap B = A)$ still hold.

Definition 2.2. *If A and B are running in opposite directions then $A \cap B = \emptyset$. Otherwise, the intersection $A \cap B$ is the biggest extended interval C such that $C \subseteq A$ and $C \subseteq B$.*

Example 1. $\lfloor 2, 1 \rfloor \cap \lfloor 3, 1 \rfloor = \lfloor 2, 1 \rfloor$, $\lfloor 3, 1 \rfloor \cap \lfloor 4, 2 \rfloor = \lfloor 3, 2 \rfloor$, $\lfloor 1, 2 \rfloor \cap \lfloor 3, 1 \rfloor = \emptyset$, $\lfloor 1, 2 \rfloor \cap \lfloor 2, 1 \rfloor = \emptyset$.

Definition 2.3. *The natural topology of $\mathbb{R}_{\rightleftarrows}$ is the topology generated by the set of extended open intervals.*

The topology induced on $\mathbb{R}$ by the one of $\mathbb{R}_{\rightleftarrows}$ coincide with the natural topology of $\mathbb{R}$. We denote by $\mathcal{K}(\mathbb{R})$ the set of all extended closed intervals except decreasing degenerate extended intervals. That means, all degenerate intervals in $\mathcal{K}(\mathbb{R})$ are increasing. We extend Hausdorff distance on $\mathcal{K}(\mathbb{R})$ as

$$d_H(A, B) = max(|\underline{A} - \underline{B}|, |\overline{A} - \overline{B}|). \tag{4}$$

Example 2. *In $\mathcal{K}(\mathbb{R})$, the extended closed interval $\lfloor \underline{A}, \overline{A} \rfloor$ and $\lfloor \overline{A}, \underline{A} \rfloor$ are different, unless $\underline{A} = \overline{A}$, and $d_H(\lfloor \underline{A}, \overline{A} \rfloor, \lfloor \overline{A}, \underline{A} \rfloor) = |\overline{A} - \underline{A}|$. This distance can be viewed as the effort needed to turn $\lfloor \underline{A}, \overline{A} \rfloor$ into $\lfloor \overline{A}, \underline{A} \rfloor$.*

Theorem 1. *$(\mathcal{K}(\mathbb{R}), d_H)$ is a complete metric space.*

Proof. Assume that $(A_n = \lfloor \underline{A}_n, \overline{A}_n \rfloor)_n$ is a Cauchy sequence. Then $(\underline{A}_n)_n$ and $(\overline{A}_n)_n$ are Cauchy sequences in $\mathbb{R}$ and so converge, say to $\underline{A}$ and $\overline{A}$ respectively. In fact $d_H(A_p, A_q) = \max(|\underline{A}_q - \underline{A}_p|, |\overline{A}_q - \overline{A}_p|)$ goes to 0 as p, q go to infinity implies that $|\underline{A}_q - \underline{A}_p|$ and $|\overline{A}_q - \overline{A}_p|)$ goes to 0 as p, q go to infinity. Finally, $(A_n)_n$ converges to $A = \lfloor \underline{A}, \overline{A} \rfloor$ since $d_H(A_n, A) = \max(|\underline{A}_n - \underline{A}|, |\overline{A}_n - \overline{A}|)$. □

We endow $\mathcal{K}(\mathbb{R})$ with the topology induced by the Hausdorff distance d_H. We extend multiplication (2) on extended intervals in such a way that multiplication of an increasing extended interval by a negative number gives a decreasing extended interval and vice versa. This ensure the consistency of the extensions on $\mathcal{K}(\mathbb{R})$ of the internal composition laws (1)-(2):

$$\lambda \times A = \lfloor \lambda \underline{A}, \lambda \overline{A} \rfloor, \quad A - B = \lfloor \underline{A} - \underline{B}, \overline{A} - \overline{B} \rfloor, \qquad \forall \lambda \in \mathbb{R}. \tag{5}$$

The operator $-$ can be seen as an extension the difference of Hukuhara defines for standard intervals by $A - B = [\min(\underline{A} - \underline{B}, \overline{A} - \overline{B}), \max(\underline{A} - \underline{B}, \overline{A} - \overline{B})]$. It is nothing to see that $(\mathcal{K}(\mathbb{R}), +, \cdot)$ is a vector space and $0 := [0, 0]$ is the zero vector.

For extended closed intervals A and B the support function reads

$$s_A(u) = \begin{cases} \sup\{ux;\ x \in A\} & \text{if } \underline{A} \leq \overline{A} \\ \inf\{ux;\ x \in A\} & \text{if } \overline{A} < \underline{A} \end{cases}. \tag{6}$$

For instance, $s_A(-1) = -\underline{A}$ and $s_A(1) = \overline{A}$. Hence the support function from the vector space of extended closed intervals to the vector space $\mathbb{R}^{\{-1,1\}}$ of maps from $\{-1, 1\}$ to $\mathbb{R}$, is linear. That is for all extended closed intervals A, B,

$$\begin{aligned} s_{A+B} &= s_A + s_B \\ s_{\lambda A} &= \lambda s_A, \qquad \forall \lambda \in \mathbb{R} \\ s_{A-B} &= s_A - s_B. \end{aligned}$$

For any extended interval A, we call vector of s_A the column vector $S_A = (-s_A(-1), s_A(1))'$.

3 Extended interval-valued random variables

Let $(\Omega, \mathscr{A}, P)$ be a probability space. For any $A \in \mathcal{K}(\mathbb{R})$, we set

$$hits(A) = \{B \in \mathcal{K}(\mathbb{R}); A \cap B \neq \emptyset\}$$

the set of extended closed intervals that hit A. We endow the set $\mathcal{K}(\mathbb{R})$ of extended closed intervals with the $\sigma-$algebra $\mathfrak{B}(\mathcal{K}(\mathbb{R}))$ generated by $\{hits(A);\ A \in \mathcal{K}(\mathbb{R})\}$. For simplicity, we denote $X^{-1}(hits(A)) := \{\omega \in \Omega;\ X(\omega) \cap A \neq \emptyset\}$ by $X^{-1}(A)$ and call it the inverse image of A by X. This inverse image $X^{-1}(A)$ is the collection of $\omega \in \Omega$ such that $X(\omega)$ hits A.

Definition 3.1. *A random extended interval on a probability space $(\Omega, \mathscr{A}, P)$ is a map $X : \Omega \to \mathcal{K}(\mathbb{R})$ such that for any $A \in \mathcal{K}(\mathbb{R})$, $X^{-1}(A) \in \mathscr{A}$.*

So, a random extended interval is a measurable map $X : \Omega \to \mathcal{K}(\mathbb{R})$ from the underlying probability space to $\mathcal{K}(\mathbb{R})$ endowed with the $\sigma-$algebra $\mathfrak{B}(\mathcal{K}(\mathbb{R}))$. We denote by $\mathcal{U}[\Omega, \mathcal{K}(\mathbb{R})]$ the set of random extended intervals. $\mathcal{U}[\Omega, \mathcal{K}(\mathbb{R})]$ inherit from the vector space structure of $\mathcal{K}(\mathbb{R})$. The distribution of $X \in \mathcal{U}[\Omega, \mathcal{K}(\mathbb{R})]$ is the map $P_X : \mathfrak{B}(\mathcal{K}(\mathbb{R})) \to [0, 1]$ defined on $\mathcal{O} \in \mathfrak{B}(\mathcal{K}(\mathbb{R}))$ by

$$P_X(\mathcal{O}) := P(X \in \mathcal{O}).$$

Definition 3.2. *A map $f : \Omega \to \mathbb{R}$ is called a **selection map** for a random extended interval X when $f(\omega) \in X(\omega)$ for almost every $\omega \in \Omega$.*

Selection maps for $X = \lfloor \underline{X}, \overline{X} \rfloor$ are then maps leaving between $\underline{X}$ and $\overline{X}$. For instance, $\underline{X}$ and $\overline{X}$ are selection maps for X. The expectation of X is the set of expectations of measurable selection maps for X. More precisely,

Definition 3.3. *The **expectation** of a random extended interval X on a probability space $(\Omega, \mathscr{A}, P)$ is the extended interval*

$$E[X] = \lfloor E[\underline{X}], E[\overline{X}] \rfloor. \tag{7}$$

The expectation of X is the classical interval $\{E[f];\ f \in L^1(\Omega)$ and f is a selection map for $X\}$ together with the running direction coming from X.

Proposition 3.1. *For any $X, Y \in \mathcal{U}[\Omega, \mathcal{K}(\mathbb{R})]$ and $\lambda \in \mathbb{R}$, $E[X + \lambda Y] = E[X] + \lambda E[Y]$.*

We denote by $\mathcal{S}_X = \{f \in L^1(\Omega)$ and f is a selection map for $X\}$ the set of integrable selection maps for X and $\mathcal{S}_X(\mathscr{A}_0) = \{f \in L^1(\Omega, \mathscr{A}_0)$ and f is a selection map for $X\}$ the set of $(\Omega, \mathscr{A}_0)-$integrable selection maps for X, being $\mathscr{A}_0$ a sub$-\sigma-$field of $\mathscr{A}$.

3.1 A new adapted distance D_γ

To quantify the variability of X, that is the dispersion of X around its expectation, we need a suitable distance measure on random extended intervals. The first distance that could come to mind is the Hausdorff distance. But, a disadvantage of the Hausdorff distance is for instance that $d_H([0,2],[5,6]) = 5 = d_H([0,2],[5,7])$, while intuitively the distance between $[0,2]$ and $[5,6]$ should be less than the distance between $[0,2]$ and $[5,7]$.

In Bertoluzza et al. (1995) the authors defined the squared distance $d^2_\gamma(A,B)$ between two standard intervals as follow. For any interval $A = [\underline{A}, \overline{A}]$, we consider the one-to-one map $\nabla_A : [0,1] \to A$, $t \mapsto t\underline{A} + (1-t)\overline{A}$. Then the squared distance $d^2_\gamma(A,B)$ is given by

$$d^2_\gamma(A,B) = \int_0^1 (\nabla_A(t) - \nabla_B(t))^2 \gamma(t)dt = \int_0^1 \left(t(\underline{A} - \underline{B}) + (1-t)(\overline{A} - \overline{B})\right)^2 \gamma(t)dt, \tag{8}$$

where $\gamma(t)dt$ is a Borel measure on $[0,1]$ such that:

$$\gamma(t) \geq 0 \text{ for every } t \in [0,1]; \tag{9a}$$

$$\int_0^1 \gamma(t)dt = 1; \tag{9b}$$

$$\gamma(t) = \gamma(1-t); \tag{9c}$$

$$\gamma(0) > 0 \tag{9d}$$

We extend d_γ on extended intervals with the same formula (8) and assumptions (9a)-(9d). If $d^2_\gamma(A,B) = 0$ then $\nabla_A(t) = \nabla_B(t)$ for almost every $t \in [0,1]$, which implies that $\underline{A} = \underline{B}$ and $\overline{A} = \overline{B}$; thus $A = B$. For triangular inequality, we first write

$$(\nabla_A(t) - \nabla_C(t))^2 = (\nabla_A(t) - \nabla_B(t))^2 + (\nabla_B(t) - \nabla_C(t)) + 2(\nabla_A(t) - \nabla_B(t))(\nabla_B(t) - \nabla_C(t)).$$

Hence,

$$d^2_\gamma(A,C) = d^2_\gamma(A,B) + d^2_\gamma(B,C) + 2\int_0^1 (\nabla_A(t) - \nabla_B(t))(\nabla_B(t) - \nabla_C(t))\gamma(t)dt. \tag{10}$$

From here, using Hölder's inequality, one gets the triangular inequality. Thus, d_γ is a distance on the set $\mathcal{K}(\mathbb{R})$ of extended intervals. The two extended intervals $A = \lfloor \underline{A}, \overline{A} \rfloor$ and $\tilde{A} = \lfloor \overline{A}, \underline{A} \rfloor$ represent the same standard interval but are different in $\mathcal{K}(\mathbb{R})$, and $d_\gamma(A, \tilde{A}) = |\underline{A} - \overline{A}| cst$ (with $cst = \left(\int_0^1 (2t-1)^2\gamma(t)dt\right)^{1/2} \neq 0$) vanishes if and only if $\underline{A} = \overline{A}$. This distance can be seen as the effort needs to turn $\tilde{A}$ into A.

Conditions (9a)-(9b) are required if we want the distance d_γ on degenerate intervals $[a,a]$ and $[b,b]$ gives the usual distance $|b-a|$. In other hand, the distance d_γ is suitable for intervals since it doesn't share some disadvantages of the Hausdorff distance, see Bertoluzza et al. (1995) for more details.

The norm of an interval A is the distance between A and 0: $\|A\| = d_\gamma(A,0)$. Condition (9c) means that there is no preferable position between left and right bounds. More precisely, this condition implies that $\|\lfloor a,0 \rfloor\| = \|\lfloor 0,a \rfloor\| = |a| \left(\int_0^1 t^2\gamma(t)dt\right)^{1/2}$. The previous observation justifies the following definition.

Definition 3.4. *We say that $\gamma(t)dt$ is an adapted measure if in addition of conditions (9a)-(9d) one has*

$$\int_0^1 t^2\gamma(t)dt = 1 \tag{9f}$$

Example 3. *One can check that with*

$$\gamma(t) = t(1-t)\left(480 - \frac{10240}{3\pi}\sqrt{t(1-t)}\right) + 1,$$

$\gamma(t)dt$ is an adapted measure. We will refer to this as the standard adapted measure. It has been used in the software R to check Lemma 3.1.

Generally for any $c \in (0, \infty)$,

$$\gamma_c(t) = t(1-t)\left(a + b\sqrt{t(1-t)}\right) + c,$$

defined an adapted measure for $a = -30c + 510$ and $b = \frac{512(c-21)}{3\pi}$.

The d_γ distance can be written again as

$$\begin{aligned} d_\gamma^2(A,B) &= (s_A(-1) - s_B(-1))^2 K(-1,-1) + (s_A(1) - s_B(1))^2 K(1,1) \\ &\quad - 2(s_A(-1) - s_B(-1))(s_A(1) - s_B(1))K(-1,1) \\ &= \begin{pmatrix} -s_A(-1) + s_B(-1) \\ s_A(1) - s_B(1) \end{pmatrix}' \begin{pmatrix} K(-1,-1) & K(-1,1) \\ K(1,-1) & K(1,1) \end{pmatrix} \begin{pmatrix} -s_A(-1) + s_B(-1) \\ s_A(1) - s_B(1) \end{pmatrix} \\ d_\gamma^2(A,B) &= S'_{A-B}\mathcal{K}_\gamma S_{A-B} \end{aligned} \tag{11}$$

where $\mathcal{K}_\gamma = (K(i,j))_{i,j=-1,1}$ is the symmetric matrix given by

$$\begin{cases} K(-1,-1) = \int_0^1 t^2\gamma(t)dt \\ K(1,1) = \int_0^1 (1-t)^2\gamma(t)dt \\ K(-1,1) = K(1,-1) = \int_0^1 t(1-t)\gamma(t)dt \end{cases}. \tag{12}$$

We will often denote $\langle S_{A-B}, S_{A-B}\rangle_\gamma := d_\gamma^2(A,B)$. Since $\mathcal{K}_\gamma$ is symmetric and positive definite, it induced a scalar product on $\mathbb{R}^2$. We use some properties of this inner product in order to perform the proofs of Lemma 3.2 and Theorem 2. The following lemma shows that there exists a unique distance d_γ with $\gamma(t)dt$ an adapted measure. This lemma is also useful for numerical simulations.

Lemma 3.1. *All adapted measure induce the same metric given by*

$$\mathcal{K}_\gamma = \begin{pmatrix} 1 & -1/2 \\ -1/2 & 1 \end{pmatrix} \quad \text{and} \quad d_\gamma^2(A,B) = (\underline{A} - \underline{B})^2 + (\overline{A} - \overline{B})^2 - (\underline{A} - \underline{B})(\overline{A} - \overline{B}).$$

Proof. If $\gamma(t)dt$ is an adapted measure then $K(1,1) = K(-1,-1) = \int_0^1 t^2\gamma(t)dt = 1$. Using conditions (9a)-(9d) one shows that $K(-1,1) = K(1,-1) = -1/2$. □

Let X and Y be two random intervals. For any $\omega \in \Omega$, $X(\omega)$ and $Y(\omega)$ are two extended intervals and one can compute the distance $d_\gamma(X(\omega), Y(\omega))$. We defined a new distance on random extended intervals by taking the squared root of the mean of squared distance $d_\gamma^2(X(\omega), Y(\omega))$ in $(\Omega, \mathscr{A}, P)$.

Definition 3.5. *The D_γ distance is defined for two random extended intervals X, Y by*

$$D_\gamma(X,Y) = \left(E[d_\gamma^2(X,Y)]\right)^{1/2} = \sqrt{\int_\Omega \int_0^1 \left(\nabla_{X(\omega)}(t) - \nabla_{Y(\omega)}(t)\right)^2 \gamma(t)dt\, dP(\omega)},$$

provided the integral converges.

We denote by $\mathcal{L}^2[\Omega, \mathcal{K}(\mathbb{R})]$ the set of random extended intervals X such that $E\|X\|_\gamma^2 := E(d_\gamma^2(X,0)) = D_\gamma^2(X,0) < \infty$.

Lemma 3.2. *$\mathcal{L}^2[\Omega, \mathcal{K}(\mathbb{R})]$ is a vector space under laws (1)-(2).*

Proof. It is enough to show that $\mathcal{L}^2[\Omega, \mathcal{K}(\mathbb{R})]$ is a sub-vector space of $\mathcal{U}[\Omega, \mathcal{K}(\mathbb{R})]$. Let $X, Y \in \mathcal{L}^2[\Omega, \mathcal{K}(\mathbb{R})]$ and $\lambda \in \mathbb{R}$. then $D_\gamma(\lambda X, 0) = |\lambda| D_\gamma(X,0)$ and

$$\begin{aligned} D_\gamma^2(X+Y,0) &= E\left[S'_{X+Y}\mathcal{K}_\gamma S_{X+Y}\right] \\ &= E\left[(S_X+S_Y)'\mathcal{K}_\gamma(S_X+S_Y)\right] \\ &= D_\gamma^2(X,0) + D_\gamma^2(Y,0) + 2E\left[S'_X\mathcal{K}_\gamma S_Y\right] \\ &\leq 2D_\gamma^2(X,0) + 2D_\gamma^2(Y,0). \end{aligned}$$

Last inequality come from the fact that using Cauchy-Schwarz inequality,

$$2S'_X\mathcal{K}_\gamma S_Y = 2\langle S_X, S_Y\rangle_\gamma \leq 2\sqrt{\langle S_X, S_X\rangle_\gamma}\sqrt{\langle S_Y, S_Y\rangle_\gamma} \leq \langle S_X, S_X\rangle_\gamma + \langle S_Y, S_Y\rangle_\gamma$$

□

It is nothing to see that for any $X, Y \in \mathcal{L}^2[\Omega, \mathcal{K}(\mathbb{R})]$, $0 \leq D_\gamma(X,Y) < \infty$ and the triangular inequality for D_γ follows from the one of $d\gamma$. However, D_γ is not a metric on $\mathcal{L}^2[\Omega, \mathcal{K}(\mathbb{R})]$ since $D_\gamma(X,Y) = 0$ does not implies the strict equality $X = Y$; but that there are equal almost everywhere. We denote by $L^2[\Omega, \mathcal{K}(\mathbb{R})]$ the quotient set of $\mathcal{L}^2[\Omega, \mathcal{K}(\mathbb{R})]$ under the equivalent relation "being equal almost everywhere". Then, D_γ is a metric on $L^2[\Omega, \mathcal{K}(\mathbb{R})]$. We will keep denoting any class in $L^2[\Omega, \mathcal{K}(\mathbb{R})]$ by a representative $X \in \mathcal{L}^2[\Omega, \mathcal{K}(\mathbb{R})]$.

Theorem 2. *$(\mathcal{K}(\mathbb{R}), d_\gamma)$ and $\left(L^2[\Omega, \mathcal{K}(\mathbb{R})], D_\gamma\right)$ are complete metric spaces.*

Proof. Assume that $(A_n = \lfloor \underline{A}_n, \overline{A}_n \rfloor)_n$ is a $d_\gamma-$Cauchy sequence in $\mathcal{K}(\mathbb{R})$. Then $(\underline{A}_n, \overline{A}_n)'_n$ is a Cauchy sequence in $\mathbb{R}^2$ and so converges, say to $(\underline{A}, \overline{A})'$. In fact $d_\gamma(A_p, A_q) = S'_{A_p - A_q}\mathcal{K}_\gamma S_{A_p - A_q}$ goes to 0 as p, q go to infinity implies that $S_{A_p - A_q} = (-\underline{A}_p + \underline{A}_q, \overline{A}_p - \overline{A}_q)'$ goes to 0 as p, q go to infinity. Also, $(A_n)_n$ converges to $A = \lfloor \underline{A}, \overline{A} \rfloor$ since $d_\gamma(A_n, A) = S'_{A_n - A}\mathcal{K}_\gamma S_{A_n - A}$. Hence $(\mathcal{K}(\mathbb{R}), d_\gamma)$ is a Complete metric space. Now, assume that $(X_n = \lfloor \underline{X}_n, \overline{X}_n \rfloor)_n$ is a $D_\gamma-$Cauchy sequence in $L^2[\Omega, \mathcal{K}(\mathbb{R})]$. Then from Fatou's Lemma and Definition 3.5,

$$E[\liminf_{p,q\to\infty} d_\gamma^2(X_p(\omega), X_q(\omega))] \leq \liminf_{p,q\to\infty} E[d_\gamma^2(X_p(\omega), X_q(\omega))] = 0.$$

Hence $E[\liminf_{p,q\to\infty} d_\gamma^2(X_p(\omega), X_q(\omega))] = 0$, which implies that for almost every $\omega \in \Omega$, $\liminf_{p,q\to\infty} d_\gamma^2(X_p(\omega), X_q(\omega)) = 0$. Hence there exists a subsequence $(X_{n_k}(\omega))$ which is a Cauchy sequence in the complete metric space $(\mathcal{K}(\mathbb{R}), d_\gamma)$. So, for almost every ω, $(X_{n_k}(\omega))_k$ d_γ-converges to $X(\omega) = \lfloor \underline{X}(\omega), \overline{X}(\omega) \rfloor$, say; setting $X(\omega)$ to be 0 for the remaining ω, one obtains an random extended interval X. As $\lim_{k\to\infty} d_\gamma^2(X_{n_k}, X) = 0$, we also have that $\lim_{k\to\infty} d_\gamma^2(X_n, X_{n_k}) = d_\gamma^2(X_n, X)$ for any n. Using Fatou's lemma again,

$$\lim_{n\to\infty} E[d_\gamma^2(X_n, X)] = \lim_{n\to\infty} E[\liminf_{k\to\infty} d_\gamma^2(X_n, X_{n_k})] \leq \lim_{n\to\infty} \liminf_{k\to\infty} E[d_\gamma^2(X_n, X_{n_k})] = 0,$$

since $\lim_{p,q\to\infty} E[d_\gamma^2(X_p(\omega), X_q(\omega))] = 0$ implies that $\lim_{n,k\to\infty} E[d_\gamma^2(X_n, X_{n_k})] = 0$. □

Definition 3.6. *We say that a sequence (X_n) of random extended intervals converges to X in probability under the metric d_γ when $(d_\gamma^2(X_n, X))$ converges to 0 in probability, that is*

$$\forall \varepsilon > 0, \qquad \lim_{n\to\infty} P(d_\gamma^2(X_n, X) \geq \varepsilon) = 0.$$

Theorem 3. *A sequence (X_n) such that $\sup_n E\|X_n\| < \infty$, converges to X in $(L^2[\Omega, \mathcal{K}(\mathbb{R})], D_\gamma)$ if and only if (X_n) converges to X in probability under the metric d_γ.*

Proof. Let's assume that (X_n) converges to X, that is $(D^2_\gamma(X_n, X) = E[d^2_\gamma(X_n, X)])$ converges to 0. That means that $(d_\gamma(X_n, X))$ converges to 0 in norm L^2 in $(\Omega, \mathscr{A}, P)$, which implies that $(d^2_\gamma(X_n, X))$ converges to 0 in probability. Conversely, assume that (X_n) converges to X in probability under the metric d_γ. So, the inequality $|d_\gamma(X_n, 0) - d_\gamma(X, 0)| \leq d_\gamma(X_n, X)$ implies that $(\|X_n\|)$ converges to $\|X\|$ in probability. By Fatou's Lemma,

$$E\|X\| \leq \liminf_{n\to\infty} E\|X_n\| \leq \sup_n E\|X_n\| < \infty.$$

The inequality

$$d^2_\gamma(X_n, X) \leq 2\|X_n\|^2 + 2\|X\|^2$$

implies that $(d_\gamma(X_n, X))$ is uniformly integrable. Finally, the dominated convergence theorem implies that $(D_\gamma(X_n, X))$ converges to 0. □

Corollary 3.1. *Let (X_n) be a sequence of random extended intervals such that $\sup_n E\|X_n\| < \infty$ and (λ_n) a family of nonnegative real numbers such that $\sum \lambda_n^2 < \infty$. Then $(S_n = \sum_{i=0}^n \lambda_i X_i)$ converges in probability under the metric d_γ.*

Definition 3.7. *The covariance of two random extended intervals X, Y is the real*

$$Cov(X, Y) = E\langle S_{X-E[X]}, S_{Y-E[Y]}\rangle_\gamma = \int_\Omega \int_0^1 \left(\nabla_{X(\omega)}(t) - \nabla_{E[X]}(t)\right)\left(\nabla_{Y(\omega)}(t) - \nabla_{E[Y]}(t)\right)\gamma(t)dt\, dP(\omega). \tag{13}$$

The variance of X is the real

$$Var(X) = Cov(X, X) = E\langle S_{X-E[X]}, S_{X-E[X]}\rangle_\gamma = D^2_\gamma(X, E[X]). \tag{14}$$

The next proposition is the extended interval version of Theorem 4.1 in Yang and Li (2005).

Proposition 3.2. *For all random extended intervals X, Y, Z the following hold:*

① *$Var(C) = 0$, for every constant interval C;*

② *$Var(X + Y) = Var(X) + 2Cov(X, Y) + Var(Y)$;*

③ *$Cov(X, Y) = Cov(Y, X)$;*

④ *$Cov(X + Y, Z) = Cov(X, Z) + Cov(Y, Z)$;*

⑤ *$Cov(\lambda X, Y) = \lambda Cov(X, Y)$;*

⑥ *$Var(\lambda X) = \lambda^2 Var(X)$, for every $\lambda \in \mathbb{R}$;*

⑦ *$P(d_\gamma(X, E[X]) \geq \varepsilon) \leq Var(X)/\varepsilon^2$ for every $\varepsilon > 0$ (Chebyshev inequality).*

Proof. For any constant extended interval C one has $E[C] = C$ and $Var(C) = 0$ follows. Using the linearity of S and the form (11) of the metric d_γ one proves items ②-⑥. Chebyshev inequality follows from the fact that $P(d_\gamma(X, E[X]) \geq \varepsilon) \leq E[d_\gamma(X, E[X])^2]/\varepsilon^2$. □

In the particular case of adapted measures, we have the following results, which are very useful in numerical simulations.

Proposition 3.3. *If $\gamma(t)dt$ is an adapted measure, a, b are random variables and X and random extended interval then*

① $Var(\lfloor a,0\rfloor) = Var(\lfloor 0,a\rfloor) = Var(a)$;

② $Var(\lfloor a,a\rfloor) = Var(a)$;

③ $Cov(\lfloor a,0\rfloor, \lfloor 0,b\rfloor) = -\frac{1}{2}Cov(a,b)$;

④ $Var(X) = Var(\underline{X}) - Cov(\underline{X},\overline{X}) + Var(\overline{X})$;

⑤ $Cov(X,Y) = Cov(\underline{X},\underline{Y}) + Cov(\overline{X},\overline{Y}) - \frac{1}{2}Cov(\underline{X},\overline{Y}) - \frac{1}{2}Cov(\underline{Y},\overline{X})$;

⑥ $E\|X\|^2 = E[\underline{X}^2] + E[\overline{X}^2] - E[\underline{X}\overline{X}]$.

The item ⑤ of above proposition is similar to the one obtain for classical intervals in Example 4.1 in Yang and Li (2005), but the two last terms $-\frac{1}{2}Cov(\underline{X},\overline{Y}) - \frac{1}{2}Cov(\underline{Y},\overline{X})$ are not present in the formula of Yang. This difference can be explained by the fact that for our distance d_γ, there is no preference between the left and the right bound, which is not the case for the distance d_p used in Yang and Li (2005). From the formula of Yang, if the left bounds of X, Y are independent and their right bounds are also independent then $Cov(X,Y) = 0$, which is not the case for our formula ⑤ above.

Let $\mathcal{L}^2[\Omega,\mathcal{K}(\mathbb{R})]_0 = \{X \in \mathcal{U}[\Omega,\mathcal{K}(\mathbb{R})]; E[X] = 0 \text{ and } E[\|X\|_\gamma^2] < \infty\}$, that is the sub-vector space of $\mathcal{L}^2[\Omega,\mathcal{K}(\mathbb{R})]$ made by random extended interval with mean zero. For an random extended interval $X \in \mathcal{L}^2[\Omega,\mathcal{K}(\mathbb{R})]_0$, $Cov(X,X) = 0$ means that $X = E[X] = 0$ almost everywhere. Hence formula (13) cannot define a scalar product on $\mathcal{L}^2[\Omega,\mathcal{K}(\mathbb{R})]_0$. We denote by $L^2[\Omega,\mathcal{K}(\mathbb{R})]_0$ the set of classes of zero mean random extended interval equals almost everywhere. We will keep denoting any class in $L^2[\Omega,\mathcal{K}(\mathbb{R})]_0$ by a representative $X \in \mathcal{L}^2[\Omega,\mathcal{K}(\mathbb{R})]_0$. $L^2[\Omega,\mathcal{K}(\mathbb{R})]_0$ inherits from the structure of vector space of $\mathcal{L}^2[\Omega,\mathcal{K}(\mathbb{R})]_0$ and for $X, Y \in L^2[\Omega,\mathcal{K}(\mathbb{R})]_0$, the formula (13) reads

$$Cov(X,Y) = E\langle S_X, S_Y\rangle_\gamma = \int_\Omega \int_0^1 \nabla_{X(\omega)}\nabla_{Y(\omega)}\gamma(t)dtdP(\omega) \tag{15}$$

and is a scalar product on $L^2[\Omega,\mathcal{K}(\mathbb{R})]_0$.

Theorem 4. *$(L^2[\Omega,\mathcal{K}(\mathbb{R})]_0, Cov)$ is a Hilbert space.*

Proof. From what is written above, Cov is a scalar product on $L^2[\Omega,\mathcal{K}(\mathbb{R})]_0$. For the completeness, use fact that $\langle,\rangle_\gamma$ defined a scalar product on $\mathbb{R}^2$. □

Example 4. *Let N and E be the normal distribution of parameter $(0,1)$ and the exponential distribution of parameter 0.3 respectively. The probability densities of N and E are respectively given by*

$$f(\omega) = (1/\sqrt{2\pi})\exp(-0.5\omega^2)$$
$$g(\omega) = 0.3\exp(-0.3\omega).$$

Let's consider the random extended interval

$$X = \lfloor f(\omega), g(\omega)\rfloor. \tag{16}$$

We may write $X \rightsquigarrow \mathcal{N}E(0,1,0.3)$ to say that the left bound of X follows the standard normal distribution and its right bound follows the exponential distribution with parameter 0.3.

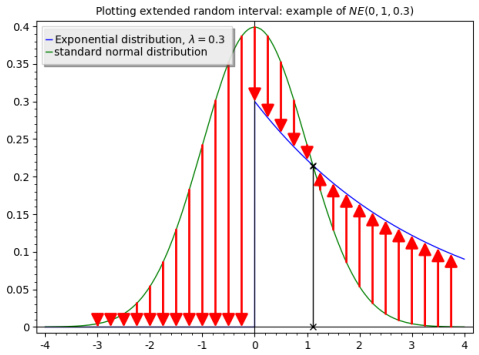


Figure 1: We represent extended intervals with arrows. Arrow point up for increasing extended intervals and down for decreasing extended intervals.

4 Stationary extended interval time series

Let $(X_t)_{t\in\mathbb{Z}}$ be an extended interval time series; that is for any integer t, X_t is an random extended interval. We denote by A_t the expectation of X_t and by $C_t(j) = Cov(X_t, X_{t-j})$ the auto-covariance function.

Definition 4.1. *We say that an extended interval time series (X_t) is stationary when neither A_t nor $C_t(j)$ depends on t. In this case, we just denote them A and $C(j)$ respectively.*

For any $n \in \mathbb{Z}^+$, the auto-covariance matrix is given by

$$\mathbf{C}_n = (C(i-j))_{1\leq i,j\leq n} = \begin{pmatrix} C(0) & C(1) & \cdots & C(n-1) \\ C(1) & C(0) & \cdots & C(n-2) \\ \vdots & \vdots & \vdots & \vdots \\ C(n-1) & C(n-2) & \cdots & C(0) \end{pmatrix}. \tag{17}$$

The proof of the following theorem is similar to the one of Theorem 4 in Wang et al. (2016).

Theorem 5. *The auto-covariance function of any stationary process satisfies:*

① *$C(k) = C(-k)$ for all $k \in \mathbb{Z}$;*

② *$|C(k)| \leq C(0)$ for all $k \in \mathbb{Z}$;*

③ *the auto-covariance matrix $\mathbf{C}_n$ is positive semi-definite;*

④ *if $C(0) > 0$ and $(C(k))$ converges to to 0 then $\mathbf{C}_n$ is positives definite.*

Let $X_1, \ldots, X_T$ be a sample of a stationary extended interval time series (X_t) with expectation A. An unbiased estimator of A is given by

$$mX = \frac{X_1 + \cdots + X_T}{T} \tag{18}$$

and the sample-covariance is given by

$$\widehat{C}(k) = \frac{1}{T}\sum_{i=1}^{T-|k|} \int_0^1 (\nabla_{X_{i+|k|}}(t) - \nabla_{mX}(t))(\nabla_{X_{i+|k|}}(t) - \nabla_{mX}(t))\gamma(t)dt. \tag{19}$$

Theorem 6. *Let (X_t) be a stationary extended interval-valued time series with expectation A and auto-covariance function $C(k)$ such that $(C(k))$ converges to 0. Then mX is a consistent estimator of A; that is for any $\varepsilon > 0$, $\lim_{T\to\infty} P(d_\gamma(mX, A) \geq \varepsilon) = 0$.*

Proof. One has

$$\begin{aligned} Var(mX) = D_\gamma^2(mX, A) = E\langle S_{mX-A}, S_{mX-A}\rangle_\gamma = \frac{1}{T^2}\sum_{i,j=1}^{T} E\langle S_{X_i-A}, S_{X_j-A}\rangle_\gamma \\ = \frac{1}{T^2}\sum_{i,j=1}^{T} C(i-j) = \frac{1}{T^2}\sum_{i-j=-T}^{T} (n-|i-j|)C(i-j) = \frac{1}{T}\sum_{k=-T}^{T}\left(1-\frac{k}{n}\right)C(k). \end{aligned}$$

So, $Var(mX)$ goes to 0 as T goes to infinity since $(C(k))$ converges to 0. By Chebyshev inequality, $\forall \varepsilon > 0$, $P(d_\gamma(m, A) \geq \varepsilon) \leq Var(mX)/\varepsilon^2$ goes to 0 as T goes to infinity. □

As usually, $\widehat{C}(k)$ is not an unbiased estimator of $C(k)$ (unless $mX = A$) but,

Theorem 7. *If $(C(k))$ converges to 0 as k goes to infinity, then for any k, $\widehat{C}(k)$ is an asymptotically unbiased estimator of $C(k)$, that is $\lim_{T\to\infty} E[\widehat{C}(k)] = C(k)$.*

Proof.

$$\begin{aligned}
\widehat{C}(k) &= \frac{1}{T}\sum_{i=1}^{T-|k|}\int_0^1 (\nabla_{X_{i+|k|}}(t) - \nabla_{mX}(t))(\nabla_{X_i}(t) - \nabla_{mX}(t))\gamma(t)dt \\
&= \frac{1}{T}\sum_{i=1}^{T-|k|}\int_0^1 (\nabla_{X_{i+|k|}}(t) - \nabla_A(t))(\nabla_{X_i}(t) - \nabla_A(t))\gamma(t)dt + \frac{1}{T}\sum_{i=1}^{T-|k|}\int_0^1 (\nabla_{mX}(t) - \nabla_A(t))^2\gamma(t)dt \\
&- \frac{1}{T}\sum_{i=1}^{T-|k|}\int_0^1 (\nabla_{mX}(t) - \nabla_A(t))(\nabla_{X_{i+|k|}}(t) + \nabla_{X_i}(t) - 2\nabla_A(t))\gamma(t)dt
\end{aligned}$$

Hence,

$$\begin{aligned}
\lim_{T\to\infty} E[\widehat{C}(k)] &= \lim_{T\to\infty}\frac{1}{T}\sum_{i=1}^{T-|k|} E[C(k)] + \lim_{T\to\infty}\frac{1}{T}\sum_{i=1}^{T-|k|} Var(mX) \\
&- \lim_{T\to\infty}\frac{1}{T}\sum_{i=1}^{T-|k|}\big(Cov(mX, X_{i+|k|}) + Cov(mX, X_i)\big) \\
&= C(k) - \lim_{T\to\infty}\frac{1}{T^2}\sum_{i=1}^{T-|k|}\sum_{j=1}^{T}\big(Cov(X_j, X_{i+|k|}) + Cov(X_j, X_i)\big) \\
&= C(k) - \lim_{T\to\infty}\frac{1}{T^2}\sum_{j-i=-T}^{T}(T - |j-i|)\left(C(j-i-|k|) + C(j-i)\right) \\
&= C(k) - \lim_{T\to\infty}\frac{1}{T}\sum_{l=-T}^{T}\left(1 - \frac{|l|}{T}\right)\left(C(l-|k|) + C(l)\right) = C(k)
\end{aligned}$$

□

4.1 Extended Interval-valued AutoRegressive Moving-Average process

Let (X_t) be an extended interval-valued stationary time series with expectation A and autocovariance function $C(k)$. We say that (X_t) is an **interval autoregressive moving-average (I-ARMA)** process of order (p, q) when

$$X_t = K + \sum_{i=1}^{p}\theta_i X_{t-i} + \varepsilon_t + \sum_{i=1}^{q}\phi_i\varepsilon_{t-i}, \tag{20}$$

being K a constant extended interval, ϕ_i and θ_i are the parameters of the model, $(\varepsilon_t) \rightsquigarrow IID(\{0\}, \sigma^2)$ and for each t, ε_t is uncorrelated with the past of X_t. By taking expectation at the both sides of (20) one finds

$$\lambda A = K, \tag{21}$$

where $\lambda = 1 - \theta_1 - \cdots - \theta_p$. So, as in the case of real random variables, the expectation μ_t of X_t doesn't depend on t and the new series $X'_t = X_t - \frac{1}{\lambda}K$ is a zero-mean I-ARMA process, ie Equation (20) with $K = 0$. In what follows, till numerical study section, we assume that $K = 0$, that is (X_t) is a zero-mean stationary process. When $p = 0$, the process (X_t) is called an extended interval-valued moving-average time series process of order q, I-MA(q), and when

$q = 0$, one obtains an extended interval-valued autoregressive time series process of order p, I-AR(p). Let L be the delay operator, thus $LX_t = X_{t-1}$. Setting $\Theta(L) = 1 - \theta_1 L - \cdots - \theta_p L^p$ and $\Phi(L) = 1 + \phi_1 L + \cdots + \phi_q L^q$, equation (20) can be written as

$$\Theta(L)X_t = \Phi(L)\varepsilon_t. \tag{22}$$

The functions Θ and Φ are called autoregressive and moving-average polynomials respectively.

4.2 Extended interval-valued Moving-Average process of order q

If the autoregressive polynomial $\Theta = 1$ then (22) leads to

$$X_t = \varepsilon_t + \sum_{i=1}^{q} \phi_i \varepsilon_{t-i}, \tag{23}$$

which is a extended interval-valued moving-average process of order q, I-MA(q). It is clear that the latter has a unique solution (X_t) and moreover this solution is always a stationary process. In fact,

$$E[X_t] = \sum_{i=1}^{q} \phi_i E[\varepsilon_{t-i}] = 0,$$

and

$$C(k) = Cov(X_{t-k}, X_t) = \begin{cases} (1 + (\sum_{i=1}^{q} \phi_i^2)\sigma^2) & \text{if } k = 0 \\ (\phi_k + \sum_{i=k+1}^{q} \phi_i \phi_{i-k})\sigma^2 & \text{if } 1 \leq k \leq q-1 \\ \phi_k \sigma^2 & \text{if } k = q \\ 0 & \text{otherwise} \end{cases}.$$

In particular, if (X_t) is an I-MA(1) process: $X_t = \varepsilon_t + \phi\varepsilon_{t-1}$; then

$$C(1) = \phi\sigma^2. \tag{24}$$

In section 5 we show that any *non-deterministic* zero-mean stationary random extended interval process can be expressed as a $MA(\infty)$.

4.3 Extended interval-valued AutoRegressive process of order p

If the moving-average polynomial $\Phi = 1$ then (22) leads to

$$X_t = (1 - \Theta(L))X_t + \varepsilon_t. \tag{25}$$

which is an extended interval-valued autoregressive process of order p, I-AR(p). In this case, the existence and the uniqueness of a stationary solution is not guaranteed. However when a stationary solution exits, using Proposition 3.2 it is nothing to show that its auto-covariance function satisfies

$$C(k) - \sum_{i=1}^{p} \theta_i C(k-i) = 0, \text{ for any } 1 \leq k \leq p. \tag{26}$$

Hence the parameters of an I-AR(p) process satisfy the following Yule-Walker equation

$$\mathbf{C}_p \boldsymbol{\Theta} = \mathbf{c}_p, \tag{27}$$

where $\mathbf{c}_p = (C(1), \ldots, C(p))^T$, $\boldsymbol{\Theta} = (\theta_1, \ldots, \theta_p)^T$ and $\mathbf{C}_p$ is the auto-covariance matrix (17).

Theorem 8. *Any AR(1) process $X_t = \theta X_{t-1} + \varepsilon_t$, with $0 < \theta < 1$ and $\sup_t E\|\varepsilon_t\| < \infty$, possesses a unique stationary solution given by $X_t = \sum_{i=0}^{\infty} \theta^i \varepsilon_{t-i}$.*

Proof. One has

$$X_t = \theta X_{t-1} + \varepsilon_t = \theta^2 X_{t-2} + \theta\varepsilon_{t-1} + \varepsilon_t = \theta^{n+1} X_{t-n-1} + \sum_{i=0}^{n} \theta^i \varepsilon_{t-i}.$$

As $0 < \theta < 1$ one has that $\sum \theta^{2i} < \infty$. This together with $\sup_t E\|\varepsilon_t\| < \infty$ implies that $(S_n = \sum_{i=0}^{n} \theta^i \varepsilon_{t-i})$ converges in probability under the metric d_γ by Corollary 3.1. Since (X_t) is stationary $Var(X_t) = E\|X_t\|^2$ is constant and

$$E\left\| X_t - \sum_{i=0}^{n} \theta^i \varepsilon_{t-i} \right\|^2 = E\|\theta^{n+1} X_{t-n-1}\|^2 = \theta^{2(n+1)} E\|X_{t-n-1}\|^2$$

goes to 0 as n goes to infinity. Hence $E\left\|X_t - \sum_{i=0}^{\infty} \theta^i \varepsilon_{t-i}\right\|^2 = 0$. This implies that $X_t = \sum_{i=0}^{\infty} \theta^i \varepsilon_{t-i}$ a.e. From this solution we have

$$Cov(X_{t+k}, X_t) = \sigma^2 \sum_{i=k}^{\infty} \theta^k \theta^{i-k} = \sigma^2 \frac{\theta^k}{1-\theta^2}.$$

□

Now, if (X_t) is an I-ARMA$(1,1)$ process: $X_t = \theta X_{t-1} + \varepsilon_t + \phi\varepsilon_{t-1}$. Then

$$C(2) = \theta C(1) \qquad \text{and} \qquad C(1) = \theta C(0) + \phi\sigma^2. \tag{28}$$

5 Wold decomposition for extended interval-valued time series

Let $(X_t)_{t\in\mathbb{Z}}$ be a zero-mean extended interval-valued stationary process. The sets $S_t = \overline{Span(\{X_k\}_{k=-\infty}^{t})}$ and $S_{-\infty} = \bigcap_{t=-\infty}^{\infty} S_t$ are Hilbert spaces of $L^2[\Omega, \mathcal{K}(\mathbb{R})]_0$. For any $j \geq 0$, the projection $P_{S_{t-j}} X_t$ of X_t on S_{t-j} is called the prediction of X_t on S_{t-j}. We shall say that an extended interval-valued process $(X_t)_{t\in\mathbb{Z}}$ is **deterministic** if for any $t \in \mathbb{Z}$, $X_t \in S_{t-1}$. $X_t - P_{S_{t-1}} X_t$ is called the error in the projection of X_t on S_{t-1} and when $P_{S_{t-1}} X_t = X_t$ one says that $(X_t)_{t\in\mathbb{Z}}$ is (perfectly) predictable. As $(L^2[\Omega, \mathcal{K}(\mathbb{R})]_0, Cov)$ is a Hilbert space, we have the following Wold decomposition for extended interval time series.

Theorem 9. *Let $(X_t)_{t\in\mathbb{Z}}$ be a non-deterministic extended interval-valued stationary time series process with expectation $\{0\}$ and auto-covariance function $(C(k))$. Then X_t can be expressed as*

$$X_t = \sum_{k=0}^{\infty} \alpha_k \varepsilon_{t-k} + W_t \quad a.s \tag{29}$$

where:

(i) $\alpha_k = \frac{1}{\sigma^2} Cov(X_t, \varepsilon_{t-k})$, $\alpha_0 = 1$ *and* $\sum_{k=0}^{\infty} \alpha_k^2 < \infty$;

(ii) $\{\varepsilon_t\} \rightsquigarrow WN(\{0\}, \sigma^2)$, *with* $\sigma^2 = Var(X_t - P_{S_{t-1}} X_t)$;

(iii) $Cov(W_t, \varepsilon_s) = 0$ *for all* $t, s \in \mathbb{Z}$;

(iv) $(W_t)_{t\in\mathbb{Z}}$ *is zero-mean, stationary and deterministic.*

Proof. For any $t \in \mathbb{Z}$, application of the Theorem 4 in Bierens (2012) to the regular sequence $(X_{t-k})_{k=0}^{\infty}$ gives that X_t can be expressed as

$$X_t = \sum_{k=0}^{\infty} \theta_k e_{t-k} + W_t \quad \text{a.s} \tag{30}$$

where $\{e_{t-k}\}_{k=0}^{\infty}$ is an uncorrelated process with $Cov(e_i, e_j) = \delta_{ij}$, $\theta_k = Cov(X_t, e_{t-k})$, $\sum_{k=1}^{\infty} \theta_k^2 < \infty$, $W_t \in U_t^{\perp}$ with $U_t = \overline{Span(\{e_k\}_{k=-\infty}^{t})} \subset S_t$. Since the process $(X_t)_{t\in\mathbb{Z}}$ is non-deterministic, the residual $\varepsilon_t = X_t - P_{S_{t-1}} X_t$ is different from 0 and $\varepsilon_t = \|\varepsilon_t\| e_t$, hence (29) holds with $\alpha_k = \theta_k / \|\varepsilon_{t-k}\|$, and (ε_t) is also uncorrelated. As $W_t, \varepsilon_t \in L^2[\Omega, \mathcal{K}(\mathbb{R})]_0$, $E[W_t] = 0 = E[\varepsilon_t]$. $W_t \in U_t^{\perp}$ implies that $Cov(W_t, \varepsilon_s) = 0$ for any $s \leq t$. For $s > t$, taking scalar product of (30) with ε_s one has $Cov(W_t, \varepsilon_s) = Cov(X_t, \varepsilon_s) = 0$ since $\varepsilon_s \in S_{s-1}^{\perp}$ and $X_t \in S_t \subset S_{s-1}$ for $s > t$. This proves **(iii)**. Let $X_{t,n}$ be the projection of X_t on $S_{t,n} = span(\{X_{t-j}\}_{j=1}^{n})$ and $\varepsilon_{t,n}$ the residual. Then $X_{t,n}$ takes the form

$$X_{t,n} = \sum_{j=1}^{n} \beta_{j,n} X_{t-j},$$

where the scalars $\beta_{k,n}$ do not depend on t, since they are solutions of the system of equations

$$\sum_{j=1}^{n} \beta_{j,n} C(j-k) = C(k), \qquad k = 1, \ldots, n.$$

Hence $E[X_{t,n}] = 0$, $E[\varepsilon_{t,n}] = 0$. Moreover,

$$\begin{aligned} Var(\varepsilon_{t,n}) = \|X_t - X_{t,n}\|^2 &= \left\| X_t - \sum_{j=1}^{n} \beta_{j,n} X_{t-j} \right\|^2 \\ &= C(0) + \sum_{i,j=1}^{n} \beta_{i,n}\beta_{j,n} C(i-j) - 2\sum_{j=1}^{n} \beta_{j,n} C(j). \end{aligned}$$

Hence $Var(\varepsilon_{t,n}) = \sigma_n$ does not depend on t and so does for $\sigma = \|\varepsilon_t\| = \lim_{n\to\infty} \sigma_n$. Also,

$$Cov(X_{t+k}, \varepsilon_{t,n}) = C(k) - \sum_{j=1}^{n} \beta_{j,n} C(k+j),$$

which does not depend on t. Using Cauchy-Schwarz inequality,

$$\lim_{n\to\infty} |Cov(X_{t+k}, \varepsilon_{t,n} - \varepsilon_t)| \leq \sqrt{C(0)} \lim_{n\to\infty} \|\varepsilon_{t,n} - \varepsilon_t\| = 0,$$

which implies that $Cov(X_{t+k}, \varepsilon_t) = \lim_{n\to\infty} Cov(X_{t+k}, \varepsilon_{t,n})$ and does not depend on t. So,

$$\alpha_k = \frac{1}{\|\varepsilon_t\|} Cov(X_{t+k}, e_k) = \frac{1}{\|\varepsilon_t\|^2} Cov(X_{t+k}, \varepsilon_t)$$

does not depend on t. Moreover, $\alpha_0 = \frac{Cov(X_t, \varepsilon_t)}{\|\varepsilon_t\|^2} = 1$. All this completes the proof of **(i)** and

(ii). For $k \geq 0$,

$$\begin{aligned} Cov(W_t, W_{t-k}) &= Cov\left(X_{t-k} - \sum_{j=0}^{\infty} \alpha_j \varepsilon_{t-k-j}, X_t - \sum_{j=0}^{\infty} \alpha_j \varepsilon_{t-j}\right) \\ &= C(k) - \sum_{j=0}^{\infty} \alpha_j Cov(X_t, \varepsilon_{t-k-j}) - \sum_{j=k}^{\infty} \alpha_j Cov(X_{t-k}, \varepsilon_{t-j}) + \sigma^2 \sum_{j=0}^{\infty} \alpha_{j+k}\alpha_j \\ &= C(k) - \sigma^2 \sum_{j=0}^{\infty} \alpha_{j+k}\alpha_j, \end{aligned}$$

which does not depend on t. As $W_t \in S_t$, one can write $W_t = \sum_{k=0}^{\infty} a_k X_{t-k}$. Taking covariance with ε_t and using the fact that $\varepsilon_t \perp Span(X_{t-1}, X_{t-2}, \ldots)$ one gets $Cov(W_t, \varepsilon_t) = a_0 Cov(X_t, \varepsilon_t) = a_0 \|\varepsilon_t\|^2$. Since $Cov(W_t, \varepsilon_t) = 0$, one deduces that $a_0 = 0$ hence $W_t \in S_{t-1}$, thus (W_t) is deterministic from the past of (X_t). This completes the proof of **(iv)**. □

6 Numerical study

Let (X_t) is an AR(1) process:

$$X_t = K + \theta X_{t-1} + \varepsilon_t. \tag{31}$$

Then from Yule-Walker equation, the parameter θ can be estimated by $\widehat{\theta} = \frac{\widehat{C}(1)}{\widehat{C}(0)}$ with

$$\begin{aligned} \widehat{C}(0) &= \frac{1}{T} \sum_{i=1}^{T} \int_0^1 (\nabla_{X_i} - \nabla_{mX})^2 \gamma(t) dt = \frac{1}{T} \sum_{i=1}^{T} d_\gamma^2(X_i, mX), \\ \widehat{C}(1) &= \frac{1}{T} \sum_{i=1}^{T-1} \int_0^1 (\nabla_{X_{i+1}} - \nabla_{mX})(\nabla_{X_i} - \nabla_{mX}) \gamma(t) dt \\ &= \frac{1}{2T} \sum_{t=1}^{T-1} \left(d_\gamma^2(X_{i+1}, mX) + d_\gamma^2(X_i, mX) - d_\gamma^2(X_{i+1}, X_i) \right), \end{aligned}$$

where $\widehat{C}(1)$ and $\widehat{C}(0)$ are the sample-covariance.

More generally, if we assume that the I-AR(p) process (25) is stationary then from Theorem 5, when $C(0) > 0$ and $(C(k))$ converges to 0, Yule-Walker equation (27) is well-posed and from a large sample $X_1, \ldots, X_T$, the coefficients of the I-AR(p) process can be estimated by

$$\widehat{\mathbf{\Theta}} = \widehat{\mathbf{C}}_p \widehat{\mathbf{c}}_p.$$

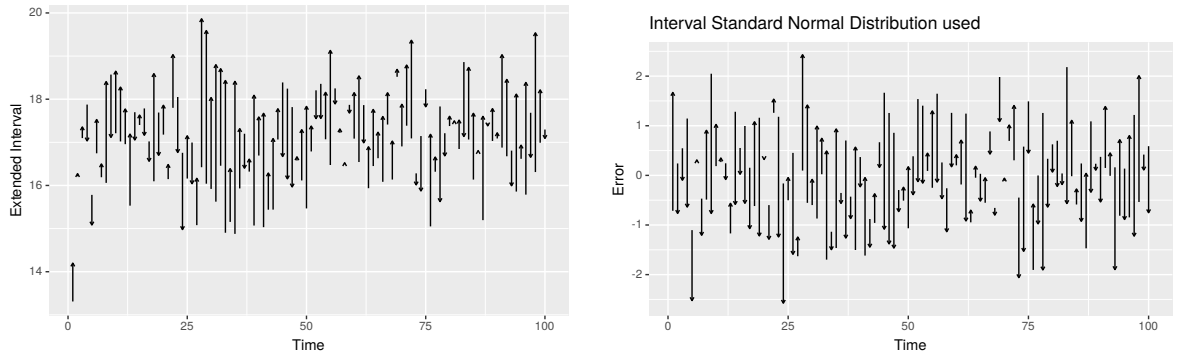
Using (10) and (19) the sample-covariance can be written as

$$\widehat{C}(k) = \frac{1}{2T} \sum_{i=1}^{T-|k|} \left(d_\gamma^2(X_{i+k}, mX) + d_\gamma^2(X_i, mX) - d_\gamma^2(X_{i+k}, X_i) \right). \tag{32}$$

It is natural to assume that $\gamma(t)dt$ is an adapted measure and in this case, the distance d_γ is given by Lemma 3.1 and it easy to numerically compute.

6.1 Simulations

Now, we plot the model (31) with $\theta = 0.2$, $K = [13.31, 14.2]$, $\overline{\varepsilon_t}$ and $\underline{\varepsilon_t}$ following independent standard normal distributions. Figure 2(a) shows a sample for this model for $T = 100$, when the interval standard normal distribution used is the one plotted on Figure 2(b). One sees that



(a) (b)

Figure 2: Simulation for the model (31) with $T = 100$

most of the outputs of this sample are standard intervals (indeed 71 standard intervals versus 29 decreasing ones) while for the error (interval standard normal distribution), they seem to be the same number (indeed 41 standard intervals versus 59 decreasing). Figure 3 displays the estimated auto-covariance function $C(k)$ and shows that it goes to 0 as k becomes large. Also, K is estimated using the formula $\widehat{K} = (1 - \widehat{\theta})mX$.

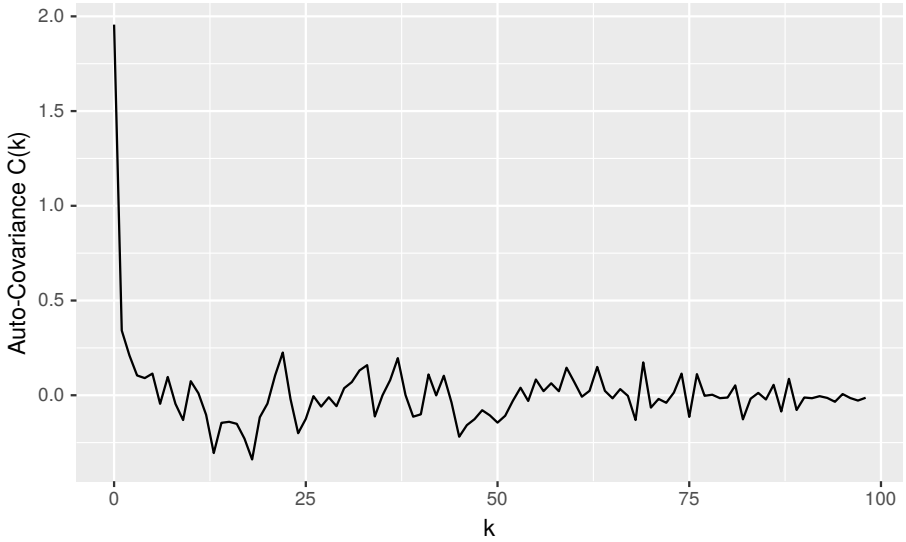


Figure 3: Auto-Covariance estimated for the model (31) for $T = 100$

T	$\widehat{K}$	$C(T-2)$	$\widehat{\theta}$	Error
100	$[13.31, 14.2]$	-0.02807759	0.1747072	0.02529285
500	$[13.51569, 14.41001]$	0.01240641	0.1892873	0.01071265

Table 1: Some estimations using simulation with R

6.2 Empirical studies

Figure 4 displays systolic (in blue) and diastolic (in red) blood pressure of a person recorded in the morning (left bounds) and in the afternoon (right bounds), over 4 days in 2004. One sees that on the 11.03.04, blood pressure recorded in the morning is higher than the one recorded in the afternoon, both for systolic and diastolic.

In Figure 5, we have plotted as standard min-max intervals (in blue) and extended intervals (in red), CAC 40 Stock Index from the 2nd January to the 31st May 2019 (105 trading days).

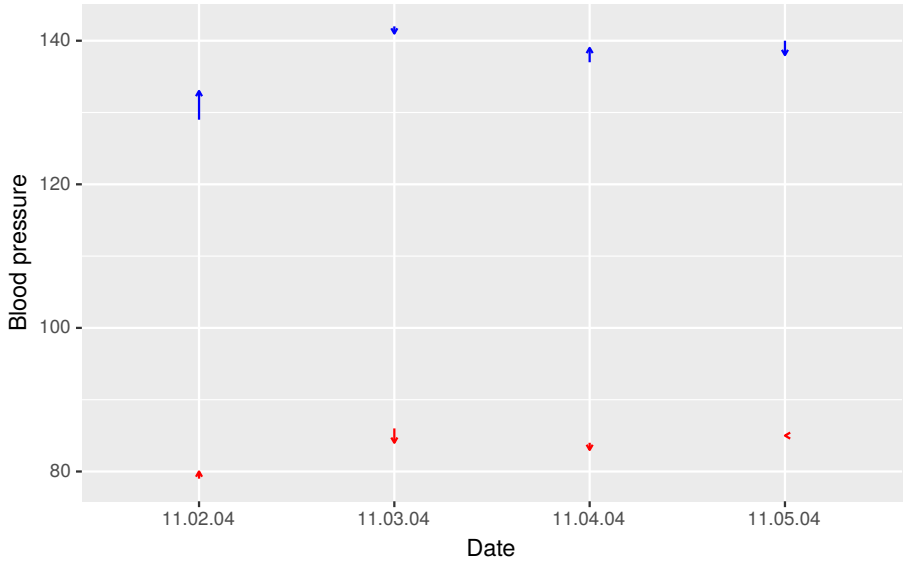


Figure 4: Systolic blood pressure in blue and diastolic blood pressure in red, of a person, recorded over 4 days in 2004. Left bounds are the morning records and right bounds are the afternoon records.

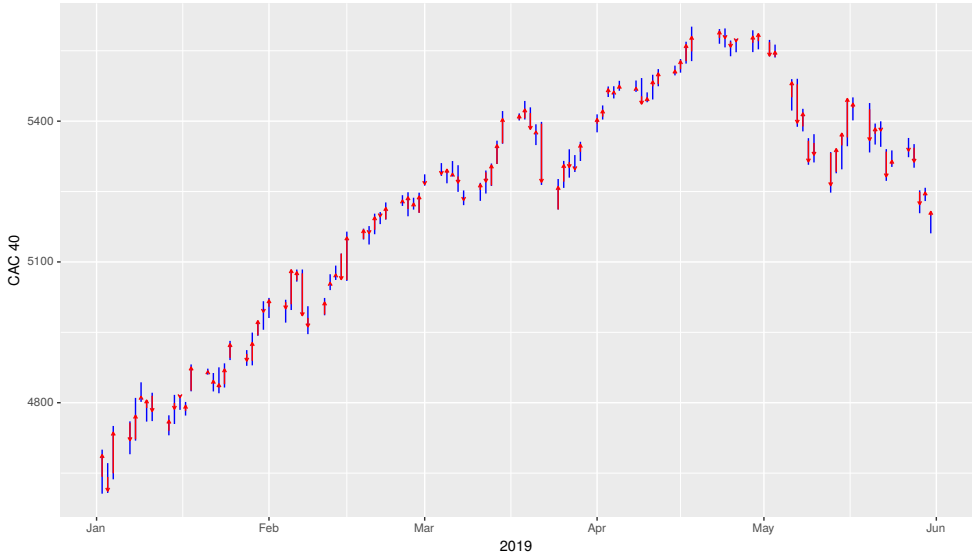


Figure 5: CAC 40 Stock Index from 2nd January to 31st May 2019. Red arrows represent the extended intervals with left bounds the opening values (in EUR) and right bounds the closing values. The blue line segments represent the interval-valued prices composed of the lowest and highest prices of each day.

Extended intervals are formed by the opening values (left bounds) and the closing values (right bounds). This figure shows that most often, neither opening nor closing values are the lowest or the highest value of the index for the day. Notice that in such an index, what is important most often is not just opening and closing values, but also to know how it has been fluctuating along the day. For instance, the plot shows many days where opening value and closing value are the same with a fluctuation along the day. Now, we wish to find the I-ARMA model which best fits)this data. The first step is to induce stationarity. Augmented Dickey–Fuller Test shows that neither the data nor its first-order differential are stationaries but its second-order differential is stationary. So, we differentiate data twice and use AIC to determine the optimal order (p, q). We define the AIC of the random interval to be the summation of the AIC of the bounds, and we assume that $p, q = 1, 2, 3, 4$. Figure 6 shows that the optimal order is $p = q = 1$. Finally, using equation (28) we estimated the coefficients of the I-ARMA model by $\widehat{\theta} = \frac{\widehat{C}(2)}{\widehat{C}(1)}, \widehat{\phi} = \widehat{C}(1) - \widehat{\theta}\widehat{C}(0)$ and we found

$$\widehat{\theta} = -0.2519991 \qquad \text{and} \qquad \widehat{\phi} = -0.5326387. \tag{33}$$

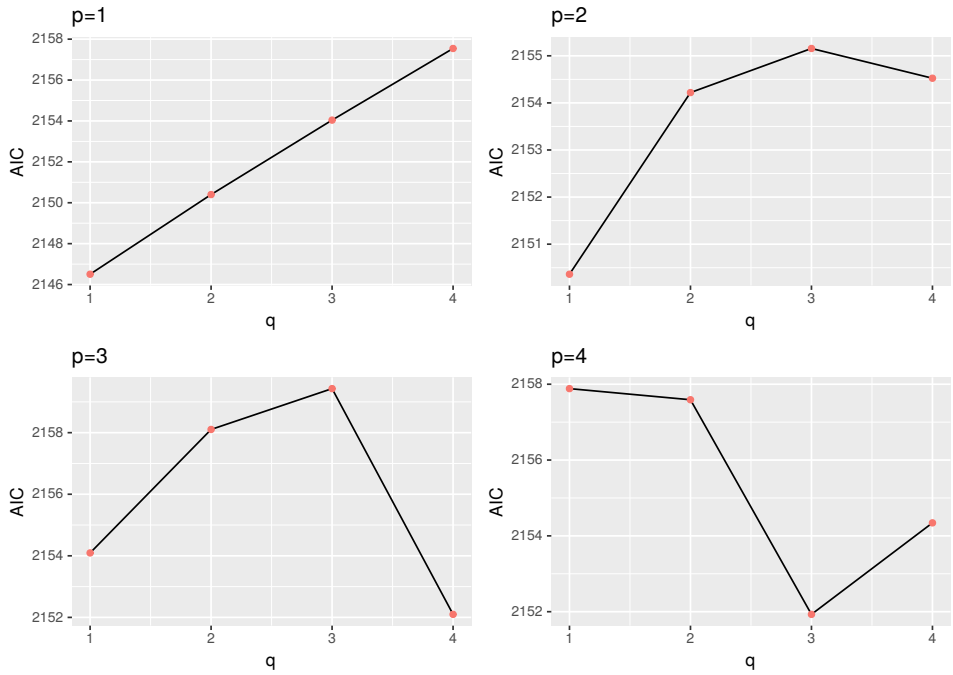


Figure 6: AIC as function of q for $p = 1, 2, 3, 4$.

Figure 7 shows the forecast of the differentiated CAC 40 for the next 40 trading days.

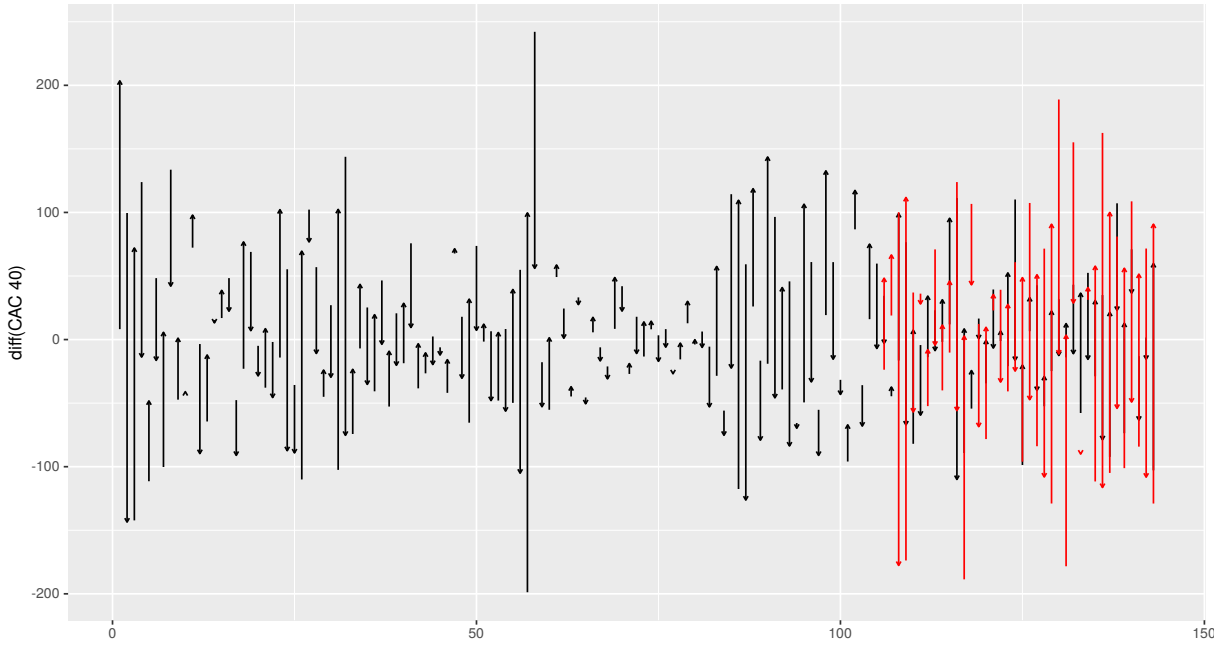


Figure 7: Forecast values from the 1st June to 26 July 2019 (red) and real values from the 2nd January to 26 July.

7 Conclusion

In this work, we have defined extended intervals and justified that they are relevant to study a variable that keep more or less one direction of variation each period T. To give a definition of extended random intervals which restricted on classical intervals coincide with the definition of random intervals, we had to introduce the set $\mathbb{R}_{\rightleftarrows} := (-\infty, \infty) \cup (\infty, -\infty)$ of real numbers for which intervals can be taken in the decreasing running direction. We further define intersection,

inclusion, and union on $\mathbb{R}_{\rightleftarrows}$ in such a way that they extend the definitions of these operators on $\mathbb{R}$. Those operators have been used to extend the topology of $\mathbb{R}$ on $\mathbb{R}_{\rightleftarrows}$. A suitable distance has been defined on extended intervals and used to define variance and covariance on random extended intervals, in a natural way. We have studied ARMA processes with extended interval-valued both theoretically and numerically. In the numerical part, we have made forecasting on CAC 40 stock index from the 2nd January to 26 July 2019.

It could be relevant to apply on extended interval time series nonlinear models such as GARCH and its extensions, as further research.

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