Fishery Management in a Regime Switching Environment: Utility Based Approach

Gaston Clément Nyassoke Titi, Jules Sadefo-Kamdem, Louis Aimé Fono

Results in Applied Mathematics 7 (2020) 100125

Contents lists available at ScienceDirect

Results in Applied Mathematics

journal homepage: www.elsevier.com/locate/results-in-applied-mathematics

Fishery management in a regime switching environment: Utility theory approach

Nyassoke Titi Gaston Clément [a,*], Sadefo Kamdem Jules [b,c], Fono Louis Aimé [a]

[a] Laboratoire de Mathématiques, Faculté des Sciences, Université de Douala, Cameroon
[b] MRE EA 7491, Université de Montpellier, France
[c] DFR SJE, Université de la Guyane, French Guiana

ARTICLE INFO

Article history:
Received 10 February 2020
Received in revised form 2 August 2020
Accepted 16 August 2020
Available online xxxx

Keywords:
Regime switching
Crra utility
Stochastic logistic growth
Mean-reverting prices
Viscosity solution
Howard's algorithm

ABSTRACT

In this paper, we study the problem of optimal fishing for regime switching in the growth dynamics of a given fish species which is described by the differential stochastic logistic model with two states: prior or during floods and after. The resulting dynamic programming principle leads to a system of two variational inequalities. By using viscosity solutions approach, we prove the existence and uniqueness of the value functions. Then numerical approximation of the obtained system is used to determine the optimal fishing effort for a sustainable fishery.

1. Introduction

The usual population model commonly used in fisheries is the logistic growth model extended to include catch (see [1]):

$$\frac{dB}{dt} = rB\left(1 - \frac{B}{K}\right) - qEB$$

where B is the biomass of the population, r is the intrinsic rate of growth, K (the carrying capacity) is the biomass the population would tend toward if unfished, q is a catchability parameter describing the efficiency of the fishing gear, and E is the fishing effort. Notice that $C = qEB$ is the catch proportional to fishing effort E and to population size B. However, it ignores the inherent environmental variability faced by fishers. It is more convenient to incorporate this random effect into a model by changing the differential equation to a stochastic differential equation. As Brites et al. [2], the state of Biomass $B(t)$ is described by the following stochastic differential equation:

$$dB(t) = rB(t)\left(1 - \frac{B(t)}{K}\right)dt - qE(t)B(t)dt + \sigma B(t)dW(t) \tag{1}$$

where σ is the positive constant volatility measuring the strength of environmental fluctuations, $W(t)$ is the one-dimensional standard brownian motion.

* Corresponding author.
E-mail address: titigaston@yahoo.fr (Nyassoke Titi G. C.).

https://doi.org/10.1016/j.rinam.2020.100125

Environmentally driven long-term changes in fish populations, which can play a major role in determining how such populations respond to fishing pressure, are rapidly being recognized as a critical problem in fisheries science [3]. The life cycle of African fish species of river is closely related to the seasons — reproduction almost always occurring just prior to, or during, floods [4–6]. Floods appear to be essential for the completion of their reproductive cycle for most species: the absence of floods due to the drought in the Sahel has caused a decline in fish reproduction in the central Niger Delta, the Senegal River and Lake Chad (Stauch, personal communication). There is some evidence that flood intensity acts in favor of reproduction, as it has been observed that the structured age class related to the high floods in the Kafu were more varied [7].

In this view, we consider only two seasons in this paper: (i) the flood period with reduced fishing and intensive reproduction, and (ii) the dry season with intensive fishing and reduced reproduction. This generates a drift and a volatility depending of the season. If we denote flood period by 1 and the dry season by 2, the above SDE Eq. (1) of the state process $B(t)$ is extended to the following SDE which captures the regime switching:

$$dB(t) = r_{\alpha(t)}B(t)\left(1 - \frac{B(t)}{K_{\alpha(t)}}\right)dt - q_{\alpha(t)}E_{\alpha(t)}(t)B(t)dt + \sigma_{\alpha(t)}B(t)dW(t) \tag{2}$$

where $\alpha(t) \in \{1, 2\}$ refers to regime.

In [2] and [8], the instantaneous profit from the harvest of the population biomass denoted by $\pi(B_t, h_t)$ is defined by:

$$\pi(B_t, h_t) = P_t h_t - c(B_t, h_t) \tag{3}$$

where $h_t(= qE_tB_t)$ is the volume of harvest, B_t is the stock of the resource, $c(B_t, h_t)$ is the cost function, and P_t is the price of the harvest at the time of decision making.

Many authors [8–10] considered the price of the harvest $P(t)$ as a stochastic process. In this paper, we assume as Kvamsdal et al. [9] that the price P_t is a mean-reverting or Ornstein–Uhlenbeck process defined by:

$$dP_t = \theta(\bar{p}_0 - \bar{p}_1 h_t - P_t)dt + \sigma_P dW_P(t) \tag{4}$$

where the positive constant parameters are, the reversion speed θ, the maximum price $\bar{p}_0$, the slope of the inverse demand curve $\bar{p}_1$, and the volatility of the spot price σ_P. Note that the mean (or long-term) price $\bar{p}_0 - \bar{p}_1 h_t$ may depend upon the harvest level and $W_P(t)$ is the standardized Brownian motion.

Authors [2,8,9,11–13] maximize the present value of the fisherman's profit. More formally, they solve the optimization problem, in the infinite horizon time, defined by:

$$max_{h_t} \int_0^{+\infty} e^{-\beta t}\pi(B_t, h_t)dt \tag{5}$$

where $\beta > 0$ is the continuous discount rate and $e^{-\beta t}$ is the opportunity cost of holding money in hands instead of investing it.

These frameworks did not take into account the utility of the fisherman. In others words, they did not consider the behavior of the fisherman with respect to risk. It is important to notice that the authors [14–16] find that all fishers are risk-averse. Consequently, we will examine this problem of fishery using expected utility approach of the Mathematical Economics. The main idea of this approach is to maximize the present value of the expected utility of the fisherman's profit. In this paper, we choose the constant relative risk-aversion (CRRA) utility function defined by $U(x) = \frac{x^{1-\gamma}}{1-\gamma}$ where x signifies the lottery prize and γ is the CRRA coefficient to be estimated: with $\gamma = 0$ denoting risk neutrality, $\gamma > 0$ indicating risk aversion, and $\gamma < 0$ denoting risk loving [17].

In addition, the approach considered in Eq. (5) use infinite time horizon. This requires the existence of linear growth conditions on drift part of the logistic process for our solution to hold, raising the question of whether another solution may exist or not. Since, the time horizon also plays a crucial role in optimal policies, we consider in this paper the finite time horizon T.

The main contribution of this paper is to study the fisherman problem in finite time horizon by using the expected utility approach in the regime switching environment.

The outline of this paper is as follows. In Section 2, we describe the model setup and we formulate a stochastic optimal control problem. In Section 3, we use the dynamic programming principle, under some additional assumptions, to prove the continuity of the value functions and we derive that these are unique viscosity solutions. In Section 4, we apply numerical method on optimal control problem to implement these value functions and we display some simulations for optimal effort given values of (known and new) parameters. In Section 5, we give some concluding remarks. In the Appendix, we prove some theorems, which allow us to derive the existence and uniqueness of the viscosity solution.

2. Mathematical model

2.1. Stochastic logistic growth model

Throughout this paper, $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P})$ is a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$ satisfying the usual conditions (i.e. it is increasing and right continuous while $\mathcal{F}_0$ contains all P-null sets). Let $W(t)$ and $W_P(t)$, $t \geq 0$, be scalar independent Brownian motions defined on this probability space.

As we specify in Introduction, the stochastic model for biological growth that we use in this paper is defined by the SDE (2). This equation can be rewritten as the following 2 equations:

$$dB(t) = f(t, B(t), i)dt + g(t, B(t), i)dW(t); \quad i = 1, 2 \tag{6}$$

where

$$f(t, B(t), i) = r_i B(t)\left(1 - \frac{B(t)}{K_i}\right) - q_i E_i(t)B(t) \;\; and \;\; g(t, B(t), i) = \sigma_i B(t), \tag{7}$$

with the initial condition $B(0) = b$ such that $0 < b < K$. In the following, we define more formally $\alpha(t) = i \in \{1, 2\}$ and its related parameters:

$\alpha(t)$ is a right-continuous-time Markov chain, $\mathcal{F}_t$-adapted with finite state space $\mathcal{S} = \{1, 2\}$ and generator $\mathcal{Q} = (q_{ij}) \in \mathbb{R}^2 \times \mathbb{R}^2$ such that $q_{ij} \geq 0$ for $i \neq j$ and $\sum_{j=1}^{2} q_{ij} = 0$. We assume that the Markov chain $\alpha(.)$ is independent of the Brownian motions $W_P(.)$ and $W(.)$,

$r_{\alpha(t)}$ intrinsic rate of growth in regime $\alpha(t)$,

$K_{\alpha(t)}$ carrying capacity in regime $\alpha(t)$,

$q_{\alpha(t)}$ catchability parameter in regime $\alpha(t)$,

$E_{\alpha(t)}(t)$ is the fishing effort which depends on the current regime $\alpha(t)$,

$\sigma_{\alpha(t)}$ is volatility in regime $\alpha(t)$.

Notice that we have

$$f : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathcal{S} \to \mathbb{R} \;\; and \;\; g : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathcal{S} \to \mathbb{R}.$$

The resulting stochastic differential equations do not satisfy the standard assumptions for existence and uniqueness of solutions, namely, linear growth and the Lipschitz condition. Nevertheless, for any positive initial condition, the solution exists and is unique under a hypothesis that both f and g satisfy the local Lipschitz condition. The solution of this equation is (for more details see supplementary material)

$$B_{t,i} = \frac{K_i \exp\left[\int_0^t \left(r_i - q_i E_i(s) - \frac{1}{2}\sigma_i^2\right) ds + \int_0^t \sigma_i dW_s\right]}{K_i/B_{0,i} + r_i \int_0^t \exp\left[\int_0^s \left(r_i - q_i E_i(\tau) - \frac{1}{2}\sigma_i^2\right) d\tau + \int_0^s \sigma_i dW_\tau\right] ds}.$$

2.2. Optimization problem

We consider, as the authors [2,18,19], the cost of harvest defined by the quadratic function in the effort:

$$c(B_t, E_t) = (c_1 + c_2 E(t))E(t) \tag{8}$$

where c_1, $c_2 > 0$ are constants. By substituting this cost function in Eq. (3), the profit function becomes:

$$\pi(B_t, P_t, E) = (qB_tP_t - c_1 - c_2E(t))E(t).$$

We introduce a performance criterion for each $i \in \mathcal{S}$ denoted by V_i and defined by: for a time t in the horizon $[0, T]$, for b_t and $p_t \in \mathbb{R}_+$,

$$V_i(t, b_t, p_t) = \mathrm{E}_{b_t,p_t,i}\left[\int_t^T e^{-\beta(s-t)} U(\pi(B_s^{t,b_t}, P_s^{t,p_t}, E_s))ds + e^{-\beta(T-t)} V_i(B_T^{t,b_t})\right]$$

where U is crra utility function and $\mathrm{E}_{b_t,p_t,i}$ is the conditional expectation given $B(t) = b_t$, $P(t) = p_t$ and $\alpha(t) = i$ under $\mathbb{P}$. In the details we have

$$V_i(t, b_t, p_t) = \mathrm{E}_{b_t,p_t,i}\left[\int_t^T e^{-\beta(s-t)} \frac{\pi(B_s^{t,b_t}, P_s^{t,p_t}, E_s, i)^{1-\gamma}}{1-\gamma} ds + e^{-\beta(T-t)} V_i(B_T^{b_t})\right].$$

It is important to notice that V_i is the objective function given the regime i, and the effort $E(t)$ is the control process. Throughout this paper, we set: $l(s, B_s^{t,b_t}, P_s^{t,p_t}, E_s, i) = \dfrac{\pi(B_s^{t,b_t}, P_s^{t,p_t}, E_s, i)^{1-\gamma}}{1-\gamma}$ and $m(T, B_T) = V_i(B_T^{b_t})$.

Considering the initial data $(t_0, b_0, p_0) = (0, b, p)$ the objective function becomes

$$V_i(0, b, p) = \mathrm{E}_{\mathrm{b,p,i}}\left[\int_0^T e^{-\beta s} l(s, B_s^b, P_s^p, E_s, i)ds + e^{-\beta T} m(T, B_T^b)\right]. \tag{9}$$

We say that the control process $E(t)$ is admissible if the following three conditions are satisfied:

1. the SDE Eq. (2) for the state process $B(t)$ has a unique strong solution;

2. the SDE Eq. (4) for the state process $P(t)$ has a unique strong solution;

3. $\mathrm{E}_{b,p,i}\left[\int_0^T \left|e^{-\beta t}\dfrac{\pi(B_t^b, P_t^p, E_t, i)^{1-\gamma}}{1-\gamma}\right| dt + \left|e^{-\beta T}V(B_T^b)\right|\right] < \infty.$

Throughout this paper, $\mathcal{A}$ is the set of admissible controls. Since the effort is bounded (for more details see [2]), $\mathcal{A}$ is the set of admissible bounded controls.

The stochastic control problem is to find an optimal control $E^* \in \mathcal{A}_i$ such that:

$$v_i(b,p) = \sup_{E\in\mathcal{A}_i} V_i(b,p).$$

3. Dynamic programming and viscosity solutions

The Hamilton–Jacobi–Bellman equations associated with this problem is a variational inequality involving, at least heuristically, a nonlinear second order parabolic differential equations:

$$\frac{\partial v_i}{\partial t}(t,b_t,p_t) + \sup_{E\in\mathcal{A}_i}\left\{-\beta v_i(t,b_t,p_t) + \frac{\pi(b_t,p_t,E_t)^{1-\gamma}}{1-\gamma} + \mathcal{L}v_i(t,b_t,p_t)\right\} = 0, \tag{10}$$

$$v_i(T,b_t,p_t) = \kappa^{\gamma}\frac{b_t^{1-\gamma}}{1-\gamma} \quad \textit{for} \quad i\in\{0,1\} \text{ and } \kappa>0 \tag{11}$$

where $\mathcal{L}$ is an operator defined by:

$$\begin{aligned}\mathcal{L}v_i(t,b_t,p_t) &= \theta(\bar{p}_0 - \bar{p}_1 qE_t b_t - p_t)\frac{\partial v_i}{\partial p}(t,b_t,p_t) + f(b_t,E_t)\frac{\partial v_i}{\partial b}(t,b_t,p_t)\\ &\quad + \frac{1}{2}\sigma_P^2\frac{\partial^2 v_i}{\partial p^2}(t,b_t,p_t) + \frac{1}{2}g^2(b_t,E_t)\frac{\partial^2 v_i}{\partial b^2}(t,b_t,p_t) + q_{ij}(v_j(t,b_t,p_t) - v_i(t,b_t,p_t)).\end{aligned} \tag{12}$$

As it is well-known, there is not, in general, a smooth solution of the Eq. (10) hence we find the solution in the viscosity sense, as introduced by [20], in Section 3.2. We denote by E^*_{re} the regular optimal solution in the absence of control constraints of Eq. (10). The optimal harvest $E^*(t)$ is a composite bang-regular-bang solution, consequently

$$E^*(t) = \begin{Bmatrix} 0 & \text{if } E^*_{\mathrm{re}}(t) < 0 \\ E^*_{\mathrm{re}}(t) & \text{if } 0 \le E^*_{\mathrm{re}}(t) \le E_{\max} \\ E_{\max} & \text{if } E_{\max} < E^*_{\mathrm{re}}(t) \end{Bmatrix}.$$

In addition to these, we know that the fishery is valueless if the population goes extinct and therefore add the condition $V_i(0,P_t) = 0$, which must hold for all P_t and i.

3.1. On the regularity of value functions

In this section, we study the growth and continuity properties of the value functions.

We shall make the following assumptions: there exists $\rho > 0$ such that for all $s,t\in[0,T]$, $b,b'\in\mathbb{R}_+$, $p,p'\in\mathbb{R}_+$ and $E\in\mathcal{A}$

$$|l(t,b,p,E) - l(s,b',p',E)| + |m(b,p) - m(b',p')| \le \rho\left[|t-s| + |b-b'| + |p-p'|\right], \tag{13}$$

and the global linear growth conditions:

$$|l(t,b,p,E)| + |m(b,p)| \le \rho\left[1+|b|+|p|\right]. \tag{14}$$

Lemma 3.1. *For any $k\in[0,2]$ there exists $C = C(k,K,T) > 0$ such that for all $h,t\in[0,T]$, $b,p,b_t,p_t\in\mathbb{R}_+$:*

$$\begin{aligned}&\mathrm{E}|B_h^{t,b_t}|^k \le C(1+|b_t|^k);\quad \mathrm{E}|P_h^{t,p_t}|^k \le C(1+|p_t|^k).\\ &\mathrm{E}|B_h^{t,b_t} - b_t|^k \le C(1+|b_t|^k)h^{\frac{k}{2}};\quad \mathrm{E}|P_h^{t,p_t} - p_t|^k \le C(1+|p_t|^k)h^{\frac{k}{2}}.\\ &\mathrm{E}|B_h^{t,b_t} - B_h^{t,b_t'}|^k \le C|b_t - b_t'|^2;\quad \mathrm{E}|P_h^{t,p_t} - P_h^{t,p_t'}|^k \le C|p_t - p_t'|^2.\\ &\mathrm{E}\Big[\sup_{0\le s\le h}|B_h^{t,b_t}|\Big]^k \le C(1+|b_t|^k)h^{\frac{k}{2}};\quad \mathrm{E}\Big[\sup_{0\le s\le h}|P_h^{t,p_t}|\Big]^k \le C(1+|p_t|^k)h^{\frac{k}{2}}.\end{aligned}$$

Proof of Lemma 3.1 is available as supplementary material.

Proposition 3.1. *For any $i \in \mathcal{S}$, the value function denoted by $v_i(s,b,p)$ satisfies a linear growth condition and is also Lipschitz in couple (b,p) uniformly in t. There exists a constant $C>0$, such that*

$$0 \le v_i(s,b_s,p_s) \le C(1+|b_s|+|p_s|), \quad \forall (s,b_s,p_s) \in [0,T]\times\mathbb{R}_+\times\mathbb{R}_+.$$

$$|v_i(s,b_s,p_s)-v_i(s,b_s',p_s')| \le C(|b_s-b_s'|+|p_s-p_s'|),\ \forall s\in[0,T],\ b_s,b_s'\in\mathbb{R}_+,\ p_s,p_s'\in\mathbb{R}_+.$$

Proposition 3.2. *Under assumptions (13) and (14) the value function $v \in \mathcal{C}^0([0,T]\times\mathbb{R}_+\times\mathbb{R}_+)$. More precisely, there exists a constant $C>0$ such that for all $t,s\in[0,T]$, $b_t,b_s\in\mathbb{R}_+$, $p_t,p_s\in\mathbb{R}_+$,*

$$|v_i(t,b_t,p_t)-v_i(s,b_s,p_s)| \le C\left[(1+|b_t|+|p_t|)|s-t|^{\frac{1}{2}}+|b_t-b_s|+|p_t-p_s|\right].$$

Proofs of Propositions 3.1 and 3.2 are available as supplementary material.

3.2. Existence of viscosity solution

In this section, we will first define what we mean by viscosity solutions. Then we will prove that the value function is a viscosity solution.

From the optimization problem given by Eqs. (10) and (11), we have the Hamilton–Jacobi–Bellman equations as follows:

$$\begin{aligned}&\frac{\partial v_i}{\partial t}(t,b_t,p_t)+\sup_{E\in\mathcal{A}_i}\Bigg\{-\beta v_i(t,b_t,p_t)+\frac{\pi^{1-\gamma}}{1-\gamma}+\theta(\bar{p}_0-\bar{p}_1 qE_ib_t-p_t)\frac{\partial v_i}{\partial p}(t,b_t,p_t)\\&+\left[r_ib_t\left(1-\frac{b_t}{K_i}\right)-q_iE_ib_t\right]\frac{\partial v_i}{\partial b}(t,b_t,p_t)+\frac{1}{2}\sigma_P^2\frac{\partial^2 v_i}{\partial p^2}(t,b_t,p_t)+\frac{1}{2}\sigma^2b_t^2\frac{\partial^2 v_i}{\partial b^2}(t,b_t,p_t)\\&+q_{ij}(v_j(t,b_t,p_t)-v_i(t,b_t,p_t))\Bigg\}=0.\end{aligned}\tag{15}$$

The corresponding pseudo-Hamiltonian has the following form:

$$\begin{aligned}&\mathcal{H}\left(i,s,b_s,p_s,u_i,\frac{\partial u_i}{\partial s},\frac{\partial u_i}{\partial b},\frac{\partial u_i}{\partial p},\frac{\partial^2 u_i}{\partial b^2},\frac{\partial^2 u_i}{\partial p^2}\right)\\&=\frac{\partial u_i}{\partial s}(s,b_s,p_s)+\sup_{E\in\mathcal{A}_i}\left\{-\beta u_i(s,b_s,p_s)+\frac{\pi(b_s,p_s,E_s)^{1-\gamma}}{1-\gamma}+\mathcal{L}u_i(s,b_s,p_s)\right\}=0.\end{aligned}$$

We have the following systems:

$$\begin{cases}\mathcal{H}\left(i,s,b_s,p_s,u_i,\frac{\partial u_i}{\partial s},\frac{\partial u_i}{\partial b},\frac{\partial u_i}{\partial p},\frac{\partial^2 u_i}{\partial b^2},\frac{\partial^2 u_i}{\partial p^2}\right)=0 \ \textit{for} \ \ (i,s,b_s,p_s)\in\mathcal{S}\times[0,T_i]\times\mathbb{R}_+\times\mathbb{R}_+\\ u_i(T,b_s,p_s)=\kappa^\gamma\frac{b_s^{1-\gamma}}{1-\gamma} \ \ \textit{for} \ \ i,j\in\{0,1\}\ \kappa>0.\end{cases}\tag{16}$$

We recall that

$$\pi(B_s,P_s,E_s)=(qB_sP_s-c_1-c_2E(s))E(s).$$

In order to study the possibility of existence and uniqueness of a solution of (16), we use a notion of viscosity solution introduced by [20].

Let denote the set of measurable functions on $[0,T]\times\mathbb{R}_+\times\mathbb{R}_+$ with polynomial growth of degree $q\ge 0$ as,

$$\begin{aligned}&\mathcal{C}_q([0,T]\times\mathbb{R}_+\times\mathbb{R}_+)\\&=\{\phi:[0,T]\times\mathbb{R}_+\times\mathbb{R}_+,\text{measurable}\mid \exists C>0,|\phi(t,b,p)|\le C(1+|b|^q+|p|^q)\}.\end{aligned}$$

Definition 3.1. We say that $u_i \in \mathcal{C}^0([0,T]\times\mathbb{R}_+\times\mathbb{R}_+)$ is called

i. a viscosity subsolution of (16) if for any $i\in\mathcal{S}$, $u_i(T,b,p)\le\kappa^\gamma\frac{b^{1-\gamma}}{1-\gamma}$, for all $b\in\mathbb{R}_+$, $p\in\mathbb{R}_+$ and for all functions $\phi\in\mathcal{C}^{1,2,2}([0,T]\times\mathbb{R}_+\times\mathbb{R}_+)\cap\mathcal{C}_2([0,T]\times\mathbb{R}_+\times\mathbb{R}_+)$ and $(\bar{t},\bar{b},\bar{p})$ such that $u_i-\phi$ attains its local maximum at $(\bar{t},\bar{b},\bar{p})$,

$$\mathcal{H}\left(i,\bar{t},\bar{b},\bar{p},\phi(\bar{t},\bar{b},\bar{p}),\frac{\partial\phi(\bar{t},\bar{b},\bar{p})}{\partial s},\frac{\partial\phi(\bar{t},\bar{b},\bar{p})}{\partial b},\frac{\partial\phi(\bar{t},\bar{b},\bar{p})}{\partial p},\frac{\partial^2\phi(\bar{t},\bar{b},\bar{p})}{\partial b^2},\frac{\partial^2\phi(\bar{t},\bar{b},\bar{p})}{\partial p^2}\right)\ge 0,\tag{17}$$

ii. a viscosity supersolution of (16) if for any $i \in \mathcal{S}$, $u_i(T,b,p) \geq \kappa^{\gamma}\frac{b^{1-\gamma}}{1-\gamma}$, for all $b \in \mathbb{R}_+$, $p \in \mathbb{R}_+$ and if for all functions $\phi \in \mathcal{C}^{1,2,2}([0,T]\times\mathbb{R}_+\times\mathbb{R}_+)\cap\mathcal{C}_2([0,T]\times\mathbb{R}_+\times\mathbb{R}_+)$ and $(\underline{t},\underline{b},\underline{p})$ such that $u_i-\phi$ attains its local minimum at $(\underline{t},\underline{b},\underline{p})$,

$$\mathcal{H}\left(i,\underline{t},\underline{b},\underline{p},\phi(\underline{t},\underline{b},\underline{p}),\frac{\partial\phi(\underline{t},\underline{b},\underline{p})}{\partial s},\frac{\partial\phi(\underline{t},\underline{b},\underline{p})}{\partial b},\frac{\partial\phi(\underline{t},\underline{b},\underline{p})}{\partial p},\frac{\partial^2\phi(\underline{t},\underline{b},\underline{p})}{\partial b^2},\frac{\partial^2\phi(\underline{t},\underline{b},\underline{p})}{\partial p^2}\right)\leq 0, \tag{18}$$

iii. a viscosity solution of (16) if it is both a viscosity sub- and a supersolution of Eq. (16).

Theorem 3.1. *Under assumption (13), the value function v is a viscosity solution of Eq. (15).*

Proof of Theorem 3.1. See Appendix A.

3.3. Comparison principle: uniqueness of the viscosity solution

In this section, we prove a comparison result from which we obtain the uniqueness of the solution of the coupled system of partial differential equations. In proving comparison results for viscosity solutions, the notion of parabolic superjet and subjet defined by Crandall, Ishii and Lions [20] is particularly useful. Thus, we begin by

Definition 3.2. Given $v \in \mathcal{C}^o([0,T]\times\mathbb{R}\times\mathbb{R}\times\mathcal{S})$ and $(t,b,p,i)\in[0,T)\times\mathbb{R}\times\mathbb{R}\times\mathcal{S}$, we define the parabolic superjet:

$$\begin{aligned}\mathcal{P}^{2,+}v(t,b,p,i) = \Big\{ &(c,q,M)\in\mathbb{R}\times\mathbb{R}^2\times\mathbb{S}^2 : v(s,b',p',i)\leq v(t,b,p,i)\\ &+c(s-t)+q.((b'-b),(p'-p))+\frac{1}{2}((b'-b),(p'-p)).M((b'-b),(p'-p))\\ &+o(|((b'-b),(p'-p))|^2)\ \text{ as }\ (s,b',p')\to(t,b,p)\Big\}\end{aligned}$$

and its closure:

$$\begin{aligned}\bar{\mathcal{P}}^{2,+}v(t,b,p,i) = \Big\{ &(c,q,M)=\lim_{n\to\infty}(c_n,q_n,M_n)\\ &\text{with }\ (c_n,q_n,M_n)\in\mathcal{P}^{2,+}v(t_n,b_n,p_n,i)\ \text{ and}\\ &\lim_{n\to\infty}(t_n,b_n,p_n,v(t_n,b_n,p_n,i))=(t,b,p,v(t,b,p,i))\Big\}.\end{aligned}$$

Similarly, we define the parabolic subjet $\bar{\mathcal{P}}^{2,-}v(t,b,p,i) = -\bar{\mathcal{P}}^{2,+}(-v)(t,b,p,i)$ and its closure $\bar{\mathcal{P}}^{2,-}v(t,b,p,i) = -\bar{\mathcal{P}}^{2,+}(-v)(t,b,p,i)$.

It is proved in [21] that

$$\begin{aligned}\mathcal{P}^{2,+(-)}v(t,b,p,i) = \Big\{\Big(&\frac{\phi}{\partial t}(t,b,p,i), D_{(b,p)}\phi(t,b,p,i), D^2_{(b,p)}\phi(t,b,p,i)\\ &\textit{and }\ v-\phi \text{ has a global maximum (minimum) at }\ (t,b,p,i)\Big)\Big\}.\end{aligned}$$

The previous notions lead to new definition of viscosity solutions.

Definition 3.3. $u_i \in \mathcal{C}^0([0,T]\times\mathbb{R}_+^*\times\mathbb{R}_+^*)$ satisfying the polynomial growth condition is a viscosity solution of (16) if

(1) for any test-function $\phi\in\mathcal{C}^{1,2,2}([0,T]\times\mathbb{R}_+^*\times\mathbb{R}_+^*)$, if (t,b,p) is a local maximum point of $u_i(.,.,.)-\phi(.,.,.)$ and if $(c,q,L_1)\in\bar{\mathcal{P}}^{2,+}u(t,b,p,i)$ with $c=\partial\phi(t,b,p)/\partial t$, $q=D_{(b,p)}\phi(t,b,p)$ and $L_1\leq D^2_{(b,p)}\phi(t,b,p)$, then

$$\mathcal{H}\left(i,s,b,p,u_i,\frac{\partial u_i}{\partial s},\frac{\partial u_i}{\partial b},\frac{\partial u_i}{\partial p},\frac{\partial^2 u_i}{\partial b^2},\frac{\partial^2 u_i}{\partial p^2}\right)\leq 0,$$

in this case u is a viscosity subsolution;

(2) for any test-function $\phi\in\mathcal{C}^{1,2,2}([0,T]\times\mathbb{R}_+^*\times\mathbb{R}_+^*)$, if (t,b,p) is a local minimum point of $u_i(.,.,.)-\phi(.,.,.)$ and if $(c,q,L_2)\in\bar{\mathcal{P}}^{2,-}u(t,b,p,i)$ with $c=\partial\phi(t,b,p)/\partial t$, $q=D_{(b,p)}\phi(t,b,p)$ and $L_2\geq D^2_{(b,p)}\phi(t,b,p)$, then

$$\mathcal{H}\left(i,s,b,p,u_i,\frac{\partial u_i}{\partial s},\frac{\partial u_i}{\partial b},\frac{\partial u_i}{\partial p},\frac{\partial^2 u_i}{\partial b^2},\frac{\partial^2 u_i}{\partial p^2}\right)\geq 0,$$

in this case u is a viscosity supersolution.

The authors [22] proved that Definitions 3.2 and 3.3 are equivalent. The second definition is particular suitable for the discussion of a maximum principle which is the backbone of the uniqueness problem for the viscosity solutions theory.

Before state next lemma, we first introduce the inf and sup-convolution operations we are going to use.

Definition 3.4. For any usc (upper semi-continuous) function $U : \mathbb{R}^m \to \mathbb{R}$ and any lsc (lower semi-continuous) function $V : \mathbb{R}^m \to \mathbb{R}$, we set

$$R^{\alpha}[U](z,r) = \sup_{|Z-z|\leq 1}\left\{U(Z) - r\cdot(Z-z) - \frac{|Z-z|}{2\alpha}\right\}$$

$$R_{\alpha}[V](z,r) = \inf_{|Z-z|\leq 1}\left\{V(Z) + r\cdot(Z-z) + \frac{|Z-z|}{2\alpha}\right\}$$

$R^{\alpha}[U](z,r)$ is called the modified sup-convolution and $R_{\alpha}[V](z,r)$ the modified inf-convolution. Notice that $R_{\alpha}[V](z,r) = -R^{\alpha}[-U](z,r)$.

Lemma 3.2 (*Nonlocal Jensen–Ishii's Lemma [22]*)**.** *For any $i \in \mathcal{S}$, let $u_i(.,.,.)$ and $v_i(.,.,.)$ be, respectively, a usc and lsc function defined on $[0,T]\times\mathbb{R}_+\times\mathbb{R}_+$ and $\phi \in \mathcal{C}^{1,2,2}([0,T]\times\mathbb{R}_+^2\times\mathbb{R}_+^2)\cap\mathcal{C}_2([0,T]\times\mathbb{R}_+^2\times\mathbb{R}_+^2)$ if $(\hat{t},(\hat{b}_1,\hat{p}_1),(\hat{b}_2,\hat{p}_2)) \in [0,T]\times\mathbb{R}_+^2\times\mathbb{R}_+^2$ is a zero global maximum point of $u_i(t,b,p) - v_i(t,b',p') - \phi(t,(b,p),(b',p'))$ and if $c - d := D_t\phi(\hat{t},(\hat{b}_1,\hat{p}_1),(\hat{b}_2,\hat{p}_2))$, $q := D_{(b,p)}\phi(\hat{t},(\hat{b}_1,\hat{p}_1),(\hat{b}_2,\hat{p}_2))$, $r := -D_{(b',p')}\phi(\hat{t},(\hat{b}_1,\hat{p}_1),(\hat{b}_2,\hat{p}_2))$, then for any $K>0$, there exists $\alpha(K)>0$ such that, for any $0<\alpha<\alpha(K)$, we have: there exist sequences $t_k \to \hat{t}$, $(b_k,p_k) \to (\hat{b}_1,\hat{p}_1)$, $(b'_k,p'_k) \to (\hat{b}_2,\hat{p}_2)$, $q_k \to q$, $r_k \to r$, matrices M_k, N_k and a sequence of functions ϕ_k, converging to the function $\phi_\alpha := R^{\alpha}[\phi](((b,p),(b',p')),(q,r))$ uniformly in $\mathbb{R}_+^2\times\mathbb{R}_+^2$ and in $C^2(B((\hat{t},(\hat{b}_1,\hat{p}_1),(\hat{b}_2,\hat{p}_2)),K))$, such that*

$$u_i(t_k,(b_k,p_k)) \to u_i(\hat{t},(\hat{b}_1,\hat{p}_1)), \qquad v_i(t_k,(b'_k,p'_k)) \to v_i(\hat{t},(\hat{b}_2,\hat{p}_2))$$

$(t_k,(b_k,p_k),(b'_k,p'_k))$ is a global maximum of $u_i(.,(.,.)) - v_i(.,(.,.)) - \phi(.,(.,.),(.,.))$

$$(c_k,q_k,M_k) \in \bar{\mathcal{P}}^{2,+}u_i(t_k,(b_k,p_k))$$

$$(-d_k,r_k,N_k) \in \bar{\mathcal{P}}^{2,-}v_i(t_k,(b'_k,p'_k))$$

$$-\frac{1}{\alpha}\begin{pmatrix} I & 0\\ 0 & I\end{pmatrix} \leq \begin{pmatrix} M_k & 0\\ 0 & -N_k\end{pmatrix} \leq D^2_{(b,p),(b',p')}\phi(t_k,(b_k,p_k),(b'_k,p'_k))$$

Here $c_k - d_k = \nabla_t\phi(t_k,(b_k,p_k),(b'_k,p'_k))$, $q_k = \nabla_{(b,p)}\phi(t_k,(b_k,p_k),(b'_k,p'_k))$, $r_k = \nabla_{(b',p')}\phi(t_k,(b_k,p_k),(b'_k,p'_k))$ and $\phi_\alpha(\hat{t},(\hat{b}_1,\hat{p}_1),(\hat{b}_2,\hat{p}_2)) = \phi(\hat{t},(\hat{b}_1,\hat{p}_1),(\hat{b}_2,\hat{p}_2))$, $\nabla\phi_\alpha(\hat{t},(\hat{b}_1,\hat{p}_1),(\hat{b}_2,\hat{p}_2)) = \nabla\phi(\hat{t},(\hat{b}_1,\hat{p}_1),(\hat{b}_2,\hat{p}_2))$.

We refer the reader to the mentioned paper for a proof. Now we can state our comparison result.

Theorem 3.2 (*Comparison Principle*)**.** *If $u_i(t,b,p)$ and $v_i(t,b,p)$ are continuous in (t,b,p) and are, respectively, viscosity subsolution and supersolution of the HJB system (15) with at most linear growth, then*

$$u_i(t,b,p) \leq v_i(t,b,p) \text{ for all } (t,b,p,i) \in [0,T]\times\mathbb{R}_+\times\mathbb{R}_+\times\mathcal{S}.$$

Proof of Theorem 3.2. See Appendix B.

The following corollary follows from Theorems 3.1 and 3.2.

Corollary 3.1. *The value function v is a unique viscosity solution of Eq. (15) that has at most a linear growth.*

4. Numerical approximation and simulation

To numerically approximate the optimal effort, we use the parameter values $(r_1, K_1, q, E_{max}, B_0, p_0, \beta, c_1, c_2)$ from [2] and $(\theta, \bar{p}_0, \bar{p}_1, \sigma, \sigma_p)$ from [9], to which we add assumed parameter values ($T, r_2, K_2, \kappa, B_{min}$ and γ). We get Table 1. Notice that we did not consider rigorous statistical methods to estimate stochastic parameters.

4.1. Monotone finite difference

In this section, we present a numerical solution. We consider the switching process $\alpha(t)$ where $\alpha(t) \in \mathcal{S} = \{1,2\}$ represents the season. In particular, $\alpha(t) = 1$ stands for the flood period with reduced fishing and intensive reproduction and $\alpha(t) = 2$ the dry season with intensive fishing and reduced reproduction. The generator of $\alpha(t)$ is given by

$$\begin{pmatrix} 0.5 & -0.5\\ -0.5 & 0.5\end{pmatrix}.$$

Note that the generator's values are chosen arbitrarily and one can choose other values without affecting the conclusion.

Table 1
Parameters and their values.

Parameters	Description	Values	Units
r_1, r_2	Intrinsic growth rate	0.71, 0.68	yr^{-1}
K_1, K_2	Carrying capacity	80.5×10^6, $0.75 \times K_1$	kg
$q = q_1 = q_2$	Catchability coefficient	3.30×10^{-6}	SFU^{-1} yr^{-1}
E_{max}	Maximum fishing effort	0.7r/q	SFU
B_o	Initial population size	$0.5K$	kg
B_{min}	Minimum population size	$0.4K$	kg
β	Discount factor	0.05	yr^{-1}
p_0	Price per unit yield	1.59	$\$kg^{-1}$
c_1	Linear cost parameter	0.96×10^{-6}	$\$SFU^{-1}$ yr^{-1}
c_2	Quadratic cost parameter	0.10×10^{-6}	$\$SFU^{-2}$ yr^{-1}
θ	Mean-reversion speed	0.59	
$\bar{p}_0$	Price of the harvest	12.65×10^{-1}	$\$kg^{-1}$
$\bar{p}_1$	Strength of demand	0.00839×10^{-1}	$\$kg^{-1}$
σ	Volatility of the fish stock	0.3	
σ_p	Volatility of the fish price	0.3	
T	Time horizon	5	yr
γ	Risk aversion coefficient	0.3	
κ	Utility function coefficient	0.7	

For our problems we need to ensure that our discretization methods converge to the viscosity solution and determine the optimal effort. Using the basic results of [23], this ensures that our numerical solutions convergence to the viscosity solution. Since it is well known that the fully implicit upwind scheme is unconditionally monotone, regardless of the size of the time step, we employ an implicit scheme and use upwind differences for the first order derivatives for stability reasons. Second order derivatives are approximated by central spaces differences. For details see [24].

Define a variable $\tau = T - t$ and a function $w(\tau, b, p) = v(t, b, p)$. The HJB equations (10) and (11) can be rewritten as

$$-\frac{\partial w_i}{\partial t}(\tau, b, p) + \sup_{E \in \mathcal{A}_i} \left\{ -\beta w_i(\tau, b, p) + \frac{\pi(b, p, E)^{1-\gamma}}{1-\gamma} + \mathcal{L} w_i(\tau, b, p) \right\} = 0, \tag{19}$$

$$w_i(0, b, p) = \kappa^{\gamma} \frac{b^{1-\gamma}}{1-\gamma} \quad \textit{for} \quad i \in \{0, 1\} \text{ and } \kappa > 0. \tag{20}$$

To approximate the solution to (19) we discretized variables τ, b and p with stepsizes $\Delta\tau$, Δb and Δp respectively. We consider $0 \le \tau \le T$, $B_{min} \le b \le 2K$ and also, $0 \le p \le p_{max}$.

Discretize time τ and spatial variables b and p:

$$\Delta\tau = \frac{T}{N}, \qquad \Delta b = \frac{2K - B_{min}}{N_b}, \qquad \Delta p = \frac{p_{max}}{N_p}$$

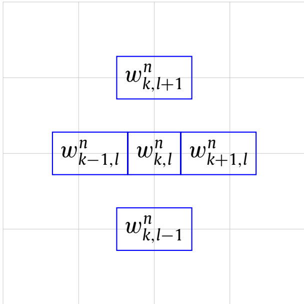
$$\tau_n = n\Delta\tau,\ 0 \le n \le N, \qquad b_k = B_{min} + k\Delta b,\ 0 \le k \le N_b, \qquad p_l = l\Delta p_{max},\ 0 \le l \le N_p.$$

The value of w_i at a grid point (τ_n, b_k, p_l) in the regime i is denoted by $w^n_{k,l}(i)$. The derivatives of w_i are approximated by
$$\frac{\partial w_i}{\partial \tau} \approx \frac{w^{n+1}_{k,l}(i) - w^n_{k,l}(i)}{\Delta\tau}, \ \frac{\partial^2 w_i}{\partial p^2} \approx \frac{w^{n+1}_{k,l+1}(i) + w^{n+1}_{k,l-1}(i) - 2w^{n+1}_{k,l}(i)}{(\Delta p)^2}, \ \frac{\partial^2 w_i}{\partial b^2} \approx \frac{w^{n+1}_{k+1,l}(i) + w^{n+1}_{k-1,l}(i) - 2w^{n+1}_{k,l}(i)}{(\Delta b)^2},$$

$$\Theta_i \frac{\partial w_i}{\partial b} \approx \begin{cases} \Theta_i \dfrac{w^{n+1}_{k+1,l}(i) - w^{n+1}_{k,l}(i)}{\Delta b} & \text{if} \quad \Theta_i > 0 \\ \Theta_i \dfrac{w^{n+1}_{k,l}(i) - w^{n+1}_{k-1,l}(i)}{\Delta b} & \text{if} \quad \Theta_i < 0 \end{cases} \quad \text{and}$$

$$\Phi_i \frac{\partial w_i}{\partial p} \approx \begin{cases} \Phi_i \dfrac{w^{n+1}_{k,l+1}(i) - w^{n+1}_{k,l}(i)}{\Delta p} & \text{if} \quad \Phi_i > 0 \\ \Phi_i \dfrac{w^{n+1}_{k,l}(i) - w^{n+1}_{k,l-1}(i)}{\Delta p} & \text{if} \quad \Phi_i < 0. \end{cases}$$

Discretizing Eq. (19) with an initial condition $w_i(0, b, p) = \kappa^{\gamma} \dfrac{b^{1-\gamma}}{1-\gamma}$. We get the following five-point stencil.



If we define the vector and constants

$$\mathbf{w}^{\mathbf{n}'} = \left[w^n_{1,1}, w^n_{2,1}, \ldots, w^n_{N_b,1}, w^n_{1,2}, w^n_{2,2}, \ldots, w^n_{N_b,2}, \ldots, w^n_{N_b,N_p}\right].$$

$$a_i = 1 + \beta\Delta\tau + \frac{\sigma_p^2\Delta\tau}{(\Delta p)^2} + \frac{\sigma_B^2 b_k^2\Delta\tau}{(\Delta b)^2} + q_{ij}\Delta\tau + \frac{\Delta\tau}{\Delta p}\theta\,|\bar{p}_0 - \bar{p}_1 qE_i b_k - p_l| + \frac{\Delta\tau}{\Delta b}\left|r_i b_k\left(1 - \frac{b_k}{K_i}\right) - qE_i b_k\right|.$$

$$b_i = -\frac{\sigma_B^2 b_k^2\Delta\tau}{2(\Delta b)^2} - \frac{\max\left(r_i b_k\left(1 - \frac{b_k}{K_i}\right) - qE_i b_k; 0\right)}{\Delta b}\Delta\tau.$$

$$c_i = -\frac{\sigma_B^2 b_k^2\Delta\tau}{2(\Delta b)^2} + \frac{\min\left(r_i b_k\left(1 - \frac{b_k}{K_i}\right) - qE_i b_k; 0\right)}{\Delta b}\Delta\tau.$$

$$d_i = -\frac{\sigma_p^2\Delta\tau}{2(\Delta p)^2} - \frac{\max\left(\theta(\bar{p}_0 - \bar{p}_1 qE_i b_k - p_l); 0\right)}{\Delta p}\Delta\tau.$$

$$e_i = -\frac{\sigma_p^2\Delta\tau}{2(\Delta p)^2} + \frac{\min\left(\theta(\bar{p}_0 - \bar{p}_1 qE_i b_k - p_l); 0\right)}{\Delta p}\Delta\tau.$$

$$f_i = -\frac{(qE_i b_k p_l - c_1 E_i - c_2 E_i^2)^{1-\gamma}}{1-\gamma}\Delta\tau.$$

We obtain a more manageable form of the difference equation:

$$\inf_{E\in\mathcal{A}_i}\left\{ a_i w^{n+1}_{k,l} + b_i w^{n+1}_{k+1,l} + c_i w^{n+1}_{k-1,l} + d_i w^{n+1}_{k,l+1} + e_i w^{n+1}_{k,l-1} - q_{ij}\Delta t\, w^{n+1}_{k,l}(j) + f_i \right\} = w^n_{k,l}. \tag{21}$$

Writing (21) in an appropriate matrix form,

$$\inf_{E\in\mathcal{A}_i}\left\{\mathbf{A}_i^E\mathbf{w}_i^{n+1} - \mathbf{\Lambda}_{ji}\mathbf{w}_j^{n+1} + \mathbf{F}_i^{n+1}\right\} = \mathbf{w}_i^n.$$

4.2. Howard's algorithm and the optimal effort

The study of the convergence of Howard's algorithm (also known as the "policy iteration algorithm") was already done by previous authors. We refer the reader to [25].

We denote by $\mathbf{w}_i^n$ and $\mathbf{w}_i^{n+1}$ the approximations at time n and $n+1$.

Step 0: start with an initial value for the control E^0 and with an initial value for the value function w_i^0. Compute the solution w_i^1 of $\mathbf{A}_i^{E^0} x - \mathbf{\Lambda}_{ji}\mathbf{w}_j^{n+1} + \mathbf{F}_i^{n+1} - \mathbf{w}_i^n = 0$.

Step $j \to j+1$: given w_h^j, find $E^{j+1} \in \mathcal{A}_i$ minimizing $\mathbf{A}_i^E w - \mathbf{\Lambda}_{ji}\mathbf{w}_j^{n+1} + \mathbf{F}_i^{n+1}$. Then compute the solution w_h^{j+1} of $\mathbf{A}_i^{E^{j+1}} x - \mathbf{\Lambda}_{ji}\mathbf{w}_j^{n+1} + \mathbf{F}_i^{n+1} - \mathbf{w}_i^n = 0$.

Final step: if $|\mathbf{w}_i^{j+1} - \mathbf{w}_i^j| < \epsilon$, then set $\mathbf{w}_i^{n+1} = \mathbf{w}_i^{j+1}$.

In the practical implementation of the Howard's algorithm, our main focus is estimating effort value i.e find the argument of the minimum at each node. For doing so, we need to replace the controls set $\mathcal{A}$ by a finite subset of controls $\mathcal{A}_N$ (see [26], Appendix A). This idea is compatible with the fishing activity.

Table 2
Sustainable effort and the sensitivity analysis.

Regime	Initial population size	Constant harvesting effort
1	$0.5 \times K_1$	1.5061×10^3
2	$0.5 \times K_2$	1.4424×10^3
1	$0.6 \times K_1$ (sensitivity)	1.5061×10^3
2	$0.6 \times K_2$ (sensitivity)	1.4424×10^3
1	$0.7 \times K_1$ (sensitivity)	1.5061×10^3
2	$0.7 \times K_2$ (sensitivity)	1.4424×10^3

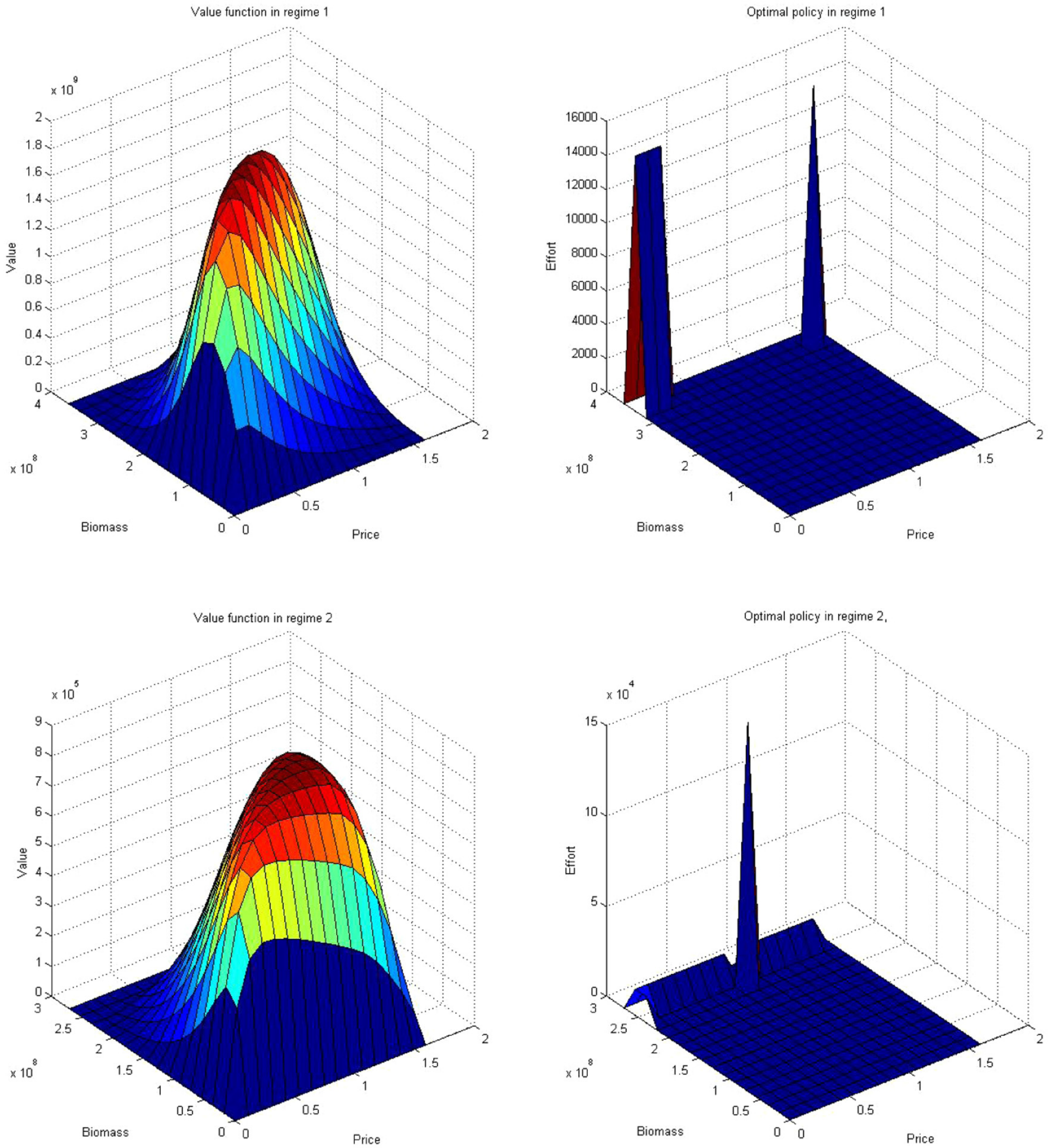


Fig. 1. The value functions and the corresponding optimal policy. On top $K_1 = 180.5 \times 10^6$, $r_1 = 0.71$ for regime 1, on bottom $K_2 = 0.75 \times K_1$, $r_2 = 0.68$ for regime 2, and $\mathcal{A}_2$.

Finite subsets for numerical tests are specified as following: $\mathcal{A}_1 = \{0, E_{max}\}$: bang–bang type optimal controls set, $\mathcal{A}_2 = \{E_0, E_1, E_2, \dots, E_{10}\}$, $\mathcal{A}_3 = \{E'_0, E'_1, E'_2, \dots, E'_{100}\}$, where $E_n = 0.1n \times E_{max}$, and $E'_n = 0.01n \times E_{max}$ respectively. The reader might choose his own discretization of controls set.

As shown in Fig. 1, the Howard's algorithm based on finite subsets produces the estimate 0 except in few points of the grid where we have exactly the same value, E_1 or E'_1. Furthermore, the value function is not typically monotone with respect to the biomass population and the fish price when the subset contains estimate 0. That motivates us to establish the optimal sustainable policy i.e estimate a constant harvesting effort.

We repeated exactly the same simulations as previously except that we have $\mathcal{A}_4 = \{0.01 \times E_{max}, 0.1 \times E_{max}, E_{max}\}$. We found that the monotone convergence of the scheme based on the last subset generally occurs in $100 - 160$ iterations and in each point of the grid, we have the value $0.01 \times E_{max}$. So, the maximal fishing effort E_{max} is not optimal for averse risk fishers.

Finally, we varied initial population size in all simulations and displayed the estimate of the constant harvesting effort.

From Table 2, we found that the level of optimal effort was higher when the state of nature $i = 1$ i.e during the floods with the higher intrinsic growth rate, and it was lower when the state of nature $i = 2$ i.e in the dry season with the lower intrinsic growth rate. Our results suggest that, the fishers should consider the climate changes in order to establish their risk-averse decision rules.

5. Conclusion

This paper studied an finite-horizon optimal fishery problem in switching diffusion models. Using the dynamic programming principle and stating some estimates, we proved that the value function is the solution of the associated system of HJB equations in the viscosity sense. As an application, the optimal effort is deduced by using Howard's algorithm. Based on our initial choice of parameter values, the major result was that dry season and flood period have a very strong effect on the fish reproduction and on the optimal effort. The dry season corresponds to the reduced reproduction and the lower optimal effort than the flood period with the intensive reproduction.

These methodologies can be applied to similar comparison studies and other fishery models. Regarding further research, one could investigate the effects of risk aversion in fishing effort, and the sensitivity of fishing policies to the parameter values.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

The parameter values used to support the findings of this study are included within the article.

Acknowledgments

We would like to thank the editor, Professor Jichun Li, and the anonymous reviewers for their valuable contributions. We also thank Dr MOMEYA Hervé Romuald and M. EBENE Roland for their useful suggestions in the new version.

Appendix A. Proof of Theorem 3.1

We establish the viscosity super- and sub-solution properties, respectively in the following two steps.

Step 1. $v_i(t, b_t, p_t)$, $i = 1, 2$ is a viscosity super-solution of Eq. (15).

We already know that $v \in \mathcal{C}^0([0, T] \times \mathbb{R}_+ \times \mathbb{R}_+)$. We first note that $v_i(T, b, p) = \kappa^{\gamma} \frac{b^{1-\gamma}}{1-\gamma}$ so, the boundary condition at time $t = T$ is clearly satisfied. Let $(s, b_s, p_s) \in [0, T] \times \mathbb{R}_+ \times \mathbb{R}_+$, $i \in \mathcal{S}$ and $\phi \in \mathcal{C}^{1,2,2}([0, T] \times \mathbb{R}_+ \times \mathbb{R}_+) \cap \mathcal{C}_2([0, T] \times \mathbb{R}_+ \times \mathbb{R}_+)$ such that $v_i(., ., .) - \phi(., ., .)$ has a local minimum at (s, b_s, p_s). Let $\mathtt{N}(b_s, p_s)$ to be a neighborhood of (s, b_s, p_s) where $v_i(., ., .) - \phi(., ., .)$ take its minimum, we introduce a new test-function ψ as follows:

$$\psi(., ., ., j) = \begin{cases} \phi(., ., .) + [v_i(s, b_s, p_s) - \phi(s, b_s, p_s)], & \text{if } j = i, \\ v_i(., ., .), & \text{if } j \neq i. \end{cases} \tag{A.1}$$

This helps us to suppose without any loss of generality that this minimum is equal to 0.

Let τ_α be the first jump time of $\alpha(t)\big(= \alpha(t)^{b_s, p_s, i}\big)$, i.e. $\tau_\alpha = \min\{t \geq s : \alpha(t) \neq i\}$. Then $\tau_\alpha > s$, a.s. Let $\theta_s \in (s, \tau_\alpha)$ be such that the state $(B_t^{b_s,i}, P_t^{p_s,i})$ starts at (b_s, p_s) and stays in $\mathtt{N}(b_s, p_s)$ for $s \leq t \leq \theta_s$. Applying the generalized

Itô's formula to the switching process $e^{-\beta t}\psi(t, B_t, P_t, \alpha(t))$, taking integral from $t = s$ to $t = \theta_s \wedge h$, where $h > 0$ is a positive constant, and then taking expectation we have

$$
\begin{aligned}
&\mathrm{E}_{b_s,p_s,i}\left[e^{-\beta\theta_s\wedge h}\psi(\theta_s \wedge h, B_{\theta_s\wedge h}, P_{\theta_s\wedge h}, \alpha(\theta_s \wedge h)) \right] \\
&= e^{-\beta s}\psi(s, B_s, P_s, i) + \mathrm{E}_{b_s,p_s,i}\left[\int_s^{\theta_s\wedge h} e^{-\beta t} \left\{ -\beta\psi(t, B_t, P_t, \alpha(t)) + \frac{\partial\psi(t, B_t, P_t, \alpha(t))}{\partial t} \right.\right. \\
&\quad + \left[r_i B_t\left(1 - \frac{B_t}{K}\right) - qEB_t\right]\frac{\partial\psi(t, B_t, P_t, \alpha(t))}{\partial b} + \theta(\bar{p}_0 - \bar{p}_1 qEB_t - P_t)\frac{\partial\psi(t, B_t, P_t, \alpha(t))}{\partial p} \\
&\quad + \frac{1}{2}\sigma^2 B_t^2\frac{\partial^2\psi(t, B_t, P_t, \alpha(t))}{\partial b^2} + \frac{1}{2}\sigma_P^2\frac{\partial^2\psi(t, B_t, P_t, \alpha(t))}{\partial p^2} \\
&\quad \left.\left. + q_{\alpha(t)j}(\psi(t, B_t, P_t, j) - \psi(t, B_t, P_t, \alpha(t))) \right\} dt \right], \quad \alpha(t) \neq j.
\end{aligned} \tag{A.2}
$$

From hypothesis, for any $t \in [s, \theta_s \wedge h]$

$$
v_i(t, B_t^{b_s}, P_t^{p_s}) \geq \phi(t, B_t^{b_s}, P_t^{p_s}) + v_i(s, b_s, p_s) - \phi(s, b_s, p_s) \geq \psi(t, B_t^{b_s}, P_t^{p_s}, i). \tag{A.3}
$$

Recalling that $(B_s^{b_s}, P_s^{p_s}) = (b_s, p_s)$ and using Eqs. (A.1) and (A.3), we have

$$
\begin{aligned}
&\mathrm{E}_{b_s,p_s,i}\left[e^{-\beta\theta_s\wedge h}\psi(\theta_s \wedge h, B_{\theta_s\wedge h}, P_{\theta_s\wedge h}, \alpha(\theta_s \wedge h)) \right] \geq \\
&\quad + e^{-\beta s}v_i(s, b_s, p_s) + \mathrm{E}_{b_s,p_s,i}\left[\int_s^{\theta_s\wedge h} e^{-\beta t} \left\{ -\beta v_i(t, B_t, P_t) + \frac{\partial\psi(t, B_t, P_t, \alpha(t))}{\partial t} \right.\right. \\
&\quad + \left[r_i B_t\left(1 - \frac{B_t}{K}\right) - qEB_t\right]\frac{\partial\psi(t, B_t, P_t, \alpha(t))}{\partial b} + \theta(\bar{p}_0 - \bar{p}_1 qEB_t - P_t)\frac{\partial\psi(t, B_t, P_t, \alpha(t))}{\partial p} \\
&\quad + \frac{1}{2}\sigma^2 B_t^2\frac{\partial^2\psi(t, B_t, P_t, \alpha(t))}{\partial b^2} + \frac{1}{2}\sigma_P^2\frac{\partial^2\psi(t, B_t, P_t, \alpha(t))}{\partial p^2} \\
&\quad \left.\left. + q_{ij}(v_j(t, B_t, P_t) - v_i(t, B_t, P_t)) \right\} dt \right].
\end{aligned} \tag{A.4}
$$

By Bellman's principle

$$
\begin{aligned}
e^{-\beta s}\psi(s, b_s, p_s, i) = e^{-\beta s}v_i(s, b_s, p_s) &= \sup_{E\in\mathcal{A}_i} \mathrm{E}_{b_s,p_s,i}\left[\int_s^{\theta_s\wedge h} e^{-\beta t} l(i, t, B_t^{s,b_s}, P_t^{s,p_s}, E_t)dt \right. \\
&\quad \left. + e^{-\beta(\theta_s\wedge h)}v_i(\theta_s \wedge h, B_{\theta_s\wedge h}^{s,b_s}, P_{\theta_s\wedge h}^{s,p_s}) \right] \\
&\geq \sup_{E\in\mathcal{A}_i} \mathrm{E}_{b_s,p_s,i}\left[\int_s^{\theta_s\wedge h} e^{-\beta t} l(i, t, B_t^{s,b_s}, P_t^{s,p_s}, E_t)dt \right. \\
&\quad \left. + e^{-\beta(\theta_s\wedge h)}\psi(\theta_s \wedge h, B_{\theta_s\wedge h}^{s,b_s}, P_{\theta_s\wedge h}^{s,p_s}, i) \right].
\end{aligned} \tag{A.5}
$$

Setting $\tau = \mathrm{E}(\theta_s \wedge h)$ combining (A.4) and (A.5) and multiplying both sides by $1/(\tau - s) > 0$, we obtain

$$
\begin{aligned}
&\sup_{E\in\mathcal{A}_i} \mathrm{E}_{b_s,p_s,i}\left[\frac{1}{\tau - s}\int_s^{\theta_s\wedge h} e^{-\beta t} \left\{ \beta v_i(t, B_t, P_t) - \frac{\partial\psi(t, B_t, P_t, \alpha(t))}{\partial t} \right.\right. \\
&\quad - \left[r_i B_t\left(1 - \frac{B_t}{K}\right) - qEB_t\right]\frac{\partial\psi(t, B_t, P_t, \alpha(t))}{\partial b} - \theta(\bar{p}_0 - \bar{p}_1 qEB_t - P_t)\frac{\partial\psi(t, B_t, P_t, \alpha(t))}{\partial p} \\
&\quad - \frac{1}{2}\sigma^2 B_t^2\frac{\partial^2\psi(t, B_t, P_t, \alpha(t))}{\partial b^2} - \frac{1}{2}\sigma_P^2\frac{\partial^2\psi(t, B_t, P_t, \alpha(t))}{\partial p^2} \\
&\quad \left.\left. - q_{ij}[v_j(t, B_t, P_t) - v_i(t, B_t, P_t)] - l(i, t, B_t^{s,b_s}, P_t^{s,p_s}, E_t) \right\} dt \right] \geq 0.
\end{aligned} \tag{A.6}
$$

Letting $\tau \downarrow s$ and using the dominated convergence theorem, it turns out that

$$e^{-\beta s}\Bigg[-\frac{\partial\psi(s,b_s,p_s,i)}{\partial t}+\inf_{E\in\mathcal{A}_i}\Bigg\{\beta v_i(s,b_s,p_s)- \\ \Big[r_i b_s\Big(1-\frac{b_s}{K}\Big)-qEb_s\Big]\frac{\partial\psi(s,b_s,p_s,i)}{\partial b}-\theta(\bar{p}_0-\bar{p}_1 qEb_s-p_s)\frac{\partial\psi(s,b_s,p_s,i)}{\partial p} \\ -\frac{1}{2}\sigma^2 b_s^2\frac{\partial^2\psi(s,b_s,p_s,i)}{\partial b^2}-\frac{1}{2}\sigma_P^2\frac{\partial^2\psi(s,b_s,p_s,i)}{\partial p^2} \\ -q_{ij}[v_j(s,b_s,p_s)-v_i(s,b_s,p_s)]-l(i,s,b_s,p_s,E_s)\Bigg\}\Bigg]\geq 0. \tag{A.7}$$

This shows that the value function $v_i(t,b_t,p_t)$, $i=1,2$, satisfies the viscosity super-solution property (18).

Step 2. $v_i(t,b_t,p_t)$, $i=1,2$, is a viscosity sub-solution of (15).

We argue by contradiction. Assume that there exist an $i_0\in\mathcal{S}$, a point $(s,b_s,p_s)\in[0,T]\times\mathbb{R}_+^*\times\mathbb{R}_+^*$ and a testing function $\phi_{i_0}\in\mathcal{C}^{1,2,2}([0,T]\times\mathbb{R}_+^*\times\mathbb{R}_+^*)\cap\mathcal{C}_2([0,T]\times\mathbb{R}_+^*\times\mathbb{R}_+^*)$ such that $v_{i_0}(.,.,.)-\phi_{i_0}(.,.,.)$ has a local maximum at (s,b_s,p_s) in a bounded neighborhood $\mathtt{N}(b_s,p_s)$, $v_{i_0}(s,b_s,p_s)=\phi_{i_0}(s,b_s,p_s)$, and

$$\min\Bigg[-\frac{\partial\phi_{i_0}(s,b_s,p_s)}{\partial t}+\inf_{E\in\mathcal{A}_{i_0}}\Bigg\{\beta v_{i_0}(s,b_s,p_s)- \\ \Big[r_{i_0}b_s\Big(1-\frac{b_s}{K}\Big)-qEb_s\Big]\frac{\partial\phi_{i_0}(s,b_s,p_s)}{\partial b}-\theta(\bar{p}_0-\bar{p}_1 qEb_s-p_s)\frac{\partial\phi_{i_0}(s,b_s,p_s)}{\partial p} \\ -\frac{1}{2}\sigma^2 b_s^2\frac{\partial^2\phi_{i_0}(s,b_s,p_s)}{\partial b^2}-\frac{1}{2}\sigma_P^2\frac{\partial^2\phi_{i_0}(s,b_s,p_s)}{\partial p^2}-q_{i_0 j}[v_j(s,b_s,p_s)-v_{i_0}(s,b_s,p_s)] \\ -l(_{i_0},s,b_s,p_s,E_s)\Bigg\},\quad v_{i_0}(T,b_s,p_s)-\kappa^\gamma\frac{b_s^{1-\gamma}}{1-\gamma}\Bigg]>0,\quad i_0\neq j. \tag{A.8}$$

By the continuity of all functions involved in (A.8)($v_{i_0},\phi'_{i_0},\phi''_{i_0},q_{ij},l,\ldots$), there exists a $\delta>0$ and an open ball $\mathtt{B}_\delta(b_s,p_s)\subset\mathtt{N}(b_s,p_s)$ such that,

$$-\frac{\partial\phi_{i_0}(t,b_t,p_t)}{\partial t}+\inf_{E\in\mathcal{A}_{i_0}}\Bigg\{\beta v_{i_0}(t,b_t,p_t)- \\ \Big[r_{i_0}b_t\Big(1-\frac{b_t}{K}\Big)-qEb_t\Big]\frac{\partial\phi_{i_0}(t,b_t,p_t)}{\partial b}-\theta(\bar{p}_0-\bar{p}_1 qEb_t-p_t)\frac{\partial\phi_{i_0}(t,b_t,p_t)}{\partial p} \\ -\frac{1}{2}\sigma^2 b_s^2\frac{\partial^2\phi_{i_0}(t,b_t,p_t)}{\partial b^2}-\frac{1}{2}\sigma_P^2\frac{\partial^2\phi_{i_0}(t,b_t,p_t)}{\partial p^2}-q_{i_0 j}[v_j(t,b_t,p_t)-v_{i_0}(t,b_t,p_t)] \\ -l(_{i_0},t,b_t,p_t,E_t)\Bigg\}>\delta,\quad i_0\neq j,\quad (t,b_t,p_t)\in\mathtt{B}_\delta(b_s,p_s) \tag{A.9}$$

and

$$v_{i_0}(T,b_t,p_t)-\kappa^\gamma\frac{b_t^{1-\gamma}}{1-\gamma}>\delta\quad(t,b_t,p_t)\in\mathtt{B}_\delta(b_s,p_s).$$

Let $\theta_\delta=\min\{t\geq s:(t,B_t,P_t)\notin\mathtt{B}_\delta(b_s,p_s)\}$ be the first exit time of (t,B_t,P_t) $(=(t,B_t^{s,b_s},P_t^{s,p_s}))$ from $\mathtt{B}_\delta(b_s,p_s)$. Let $\theta_s=\theta_\delta\wedge\tau_\alpha$ where τ_α is the first stopping time of $\alpha(t)^{b_s,p_s,i_0}$. Then $\theta_s>0$, a.s. For $0\leq t\leq\theta_s$, we have

$$\beta v_{i_0}(t,B_t,P_t)-\frac{\partial\phi_{i_0}(t,B_t,P_t)}{\partial t} \\ -\Big[r_{i_0}b_t\Big(1-\frac{B_t}{K}\Big)-qE_{i_0}B_t\Big]\frac{\partial\phi_{i_0}(t,B_t,P_t)}{\partial b}-\theta(\bar{p}_0-\bar{p}_1 qEB_t-P_t)\frac{\partial\phi_{i_0}(t,B_t,P_t)}{\partial p} \\ -\frac{1}{2}\sigma^2 b_s^2\frac{\partial^2\phi_{i_0}(t,B_t,P_t)}{\partial b^2}-\frac{1}{2}\sigma_P^2\frac{\partial^2\phi_{i_0}(t,B_t,P_t)}{\partial p^2}-q_{i_0 j}[v_j(t,B_t,P_t)-v_{i_0}(t,B_t,P_t)] \\ -l(_{i_0},t,B_t,P_t,E_t)>\delta,\quad i_0\neq j,\quad (t,B_t,P_t)\in\mathtt{B}_\delta(b_s,p_s) \tag{A.10}$$

and

$$v_{i_0}(T,b_t,p_t)-\kappa^\gamma\frac{b_t^{1-\gamma}}{1-\gamma}>\delta\quad(t,b_t,p_t)\in\mathtt{B}_\delta(b_s,p_s). \tag{A.11}$$

As previously, we can replace ϕ_{i_0} by a new test-function ψ defined as follows:

$$\psi(.,.,.,j) = \begin{cases} \phi_{i_0}(.,.,.), & \text{if } j = i_0, \\ v_{i_0}(.,.,.), & \text{if } j \neq i_0. \end{cases} \tag{A.12}$$

For any first exit time $\tau \in [s,T]$. Applying Itô's formula to the switching process $e^{-\beta t}\psi(t,B_t,P_t,\alpha(t))$, taking integral from $t=s$ to $t=(\theta_s\wedge\tau)-$ and then taking expectation yield

$$\begin{aligned}
&\mathrm{E}_{b_s,p_s,i}\Big[e^{-\beta\theta\wedge\tau}\psi(\theta_s\wedge\tau, B_{\theta_s\wedge\tau}, P_{\theta_s\wedge\tau}, \alpha(\theta_s\wedge\tau))\Big] \\
&\quad = e^{-\beta s}v_{i_0}(s,b_s,p_s) + \mathrm{E}_{b_s,p_s,i}\Bigg[\int_s^{(\theta_s\wedge\tau)-} e^{-\beta t}\Bigg\{-\beta\psi(t,B_t,P_t,\alpha(t)) + \frac{\partial\psi(t,B_t,P_t,\alpha(t))}{\partial t} \\
&\qquad + \Big[r_iB_t\Big(1-\frac{B_t}{K}\Big) - qE_iB_t\Big]\frac{\partial\psi(t,B_t,P_t,\alpha(t))}{\partial b} + \theta(\bar p_0 - \bar p_1 qEB_t - P_t)\frac{\partial\psi(t,B_t,P_t,\alpha(t))}{\partial p} \\
&\qquad + \frac{1}{2}\sigma^2B_t^2\frac{\partial^2\psi(t,B_t,P_t,\alpha(t))}{\partial b^2} + \frac{1}{2}\sigma_P^2\frac{\partial^2\psi(t,B_t,P_t,\alpha(t))}{\partial p^2} \\
&\qquad + q_{\alpha(t)j}[v_j(t,B_t,P_t) - \psi(t,B_t,P_t,\alpha(t))]\Bigg\}\,dt\Bigg], \quad \alpha(t)\neq j
\end{aligned} \tag{A.13}$$

in which we used $\mathrm{E}_{b_s,p_s,i}\Big[e^{-\beta\theta_s\wedge\tau}\psi(\theta_s\wedge\tau, B_{\theta_s\wedge\tau}, P_{\theta_s\wedge\tau}, \alpha(\theta_s\wedge\tau))\Big] = \mathrm{E}_{b_s,p_s,i}\Big[e^{-\beta\theta_s\wedge\tau}\psi(\theta_s\wedge\tau, B_{\theta_s\wedge\tau}, P_{\theta_s\wedge\tau}, \alpha(\theta_s\wedge\tau)-)\Big]$ due to continuity. Noting that the integrand in the RHS of (A.13) is continuous in t. Using (A.10), (A.11) and that $v_{i_0}(t,B_t,P_t)\le\phi_{i_0}(t,B_t,P_t)$ in (A.13). Also noting that $\alpha(t)=i_0$ for $0\le t\le\theta_s$, it follows

$$\begin{aligned}
&e^{-\beta s}v_{i_0}(s,b_s,p_s) \\
&\quad \ge \mathrm{E}_{b_s,p_s,i_0}\Bigg[e^{-\beta\theta_s\wedge\tau}\phi_{i_0}(\theta_s\wedge\tau, B_{\theta_s\wedge\tau}, P_{\theta_s\wedge\tau}, \alpha(\theta_s\wedge\tau)) \\
&\qquad + \int_s^{(\theta_s\wedge\tau)} e^{-\beta t}\Bigg\{\beta v_{i_0}(t,B_t,P_t) - \frac{\partial\phi_{i_0}(t,B_t,P_t)}{\partial t} \\
&\qquad - \Big[r_iB_t\Big(1-\frac{B_t}{K}\Big) - qE_iB_t\Big]\frac{\partial\phi_{i_0}(t,B_t,P_t)}{\partial b} - \theta(\bar p_0 - \bar p_1 qEB_t - P_t)\frac{\partial\phi_{i_0}(t,B_t,P_t)}{\partial p} \\
&\qquad - \frac{1}{2}\sigma^2B_t^2\frac{\partial^2\phi_{i_0}(t,B_t,P_t)}{\partial b^2} - \frac{1}{2}\sigma_P^2\frac{\partial^2\phi_{i_0}(t,B_t,P_t)}{\partial p^2} \\
&\qquad - q_{i_0j}[v_j(t,B_t,P_t) - v_{i_0}(t,B_t,P_t)]\Bigg\}\,dt\Bigg], \quad i_0\neq j
\end{aligned} \tag{A.14}$$

i.e

$$\begin{aligned}
&e^{-\beta s}v_{i_0}(s,b_s,p_s) \\
&\quad \ge \mathrm{E}_{b_s,p_s,i_0}\Bigg[e^{-\beta\tau}v_{i_0}(\tau,B_\tau,P_\tau,\alpha(\tau))\mathbf{1}_{\{\tau<\theta\}} + e^{-\beta\theta_s}v_{i_0}(\theta,B_{\theta_s},P_{\theta_s},\alpha(\theta_s))\mathbf{1}_{\{\tau\ge\theta_s\}} \\
&\qquad + \int_s^{(\theta_s\wedge\tau)} e^{-\beta t}\Big\{l(_{i_0},t,B_t,P_t,E_t)+\delta\Big\}dt\Bigg] \\
&\quad \ge \mathrm{E}_{b_s,p_s,i_0}\Bigg[e^{-\beta\tau}[\kappa^\gamma\frac{B_T^{1-\gamma}}{1-\gamma}+\delta]\mathbf{1}_{\{\tau<\theta_s\}} + e^{-\beta\theta_s}v_{i_0}(\theta_s,B_{\theta_s},P_{\theta_s},\alpha(\theta_s))\mathbf{1}_{\{\tau\ge\theta_s\}} \\
&\qquad + \int_s^{(\theta_s\wedge\tau)} e^{-\beta t}\Big\{l(_{i_0},t,B_t,P_t,E_t)+\delta\Big\}dt\Bigg] \\
&\quad \ge \mathrm{E}_{b_s,p_s,i_0}\Bigg[+\int_s^{(\theta_s\wedge\tau)} e^{-\beta t}\Big\{l(_{i_0},t,B_t,P_t,E_t)\Big\}dt + e^{-\beta\theta}v_{i_0}(\theta_s,B_{\theta_s},P_{\theta_s},\alpha(\theta_s))\mathbf{1}_{\{\tau\ge\theta_s\}} \\
&\qquad + e^{-\beta\tau}[\kappa^\gamma\frac{b_s^{1-\gamma}}{1-\gamma}]\mathbf{1}_{\{\tau<\theta_s\}}\Bigg] + \delta\mathrm{E}_{b_s,p_s,i_0}\Bigg[\int_s^{(\theta_s\wedge\tau)} e^{-\beta t}dt + e^{-\beta\tau}\mathbf{1}_{\{\tau<\theta_s\}}\Bigg].
\end{aligned} \tag{A.15}$$

Now considering the estimate of the term $E_{b_s,p_s,i_0}\left[\int_s^{(\theta_s\wedge\tau)} e^{-\beta t}dt + e^{-\beta\tau}\mathbf{1}_{\{\tau<\theta_s\}}\right]$, there exists a positive constant C_0 such that,

$$E_{b_s,p_s,i_0}\left[\int_s^{(\theta_s\wedge\tau)} e^{-\beta t}dt + e^{-\beta\tau}\mathbf{1}_{\{\tau<\theta_s\}}\right] \geq C_0\big(1 - E_{b_s,p_s,i_0}\big[e^{-\beta\tau_\alpha}\big]\big).$$

For details see [27]. It follows that

$$\begin{aligned}
&v_{i_0}(s, b_s, p_s)\\
&\geq \sup_{\tau\in[s,T],E\in\mathcal{A}} E_{b_s,p_s,i_0}\Bigg[+ \int_s^{(\theta\wedge\tau)} e^{-\beta t}\big\{l(_{i_0}, t, B_t, P_t, E_t)\big\}dt\\
&\quad +e^{-\beta\theta}v_{i_0}(\theta, B_\theta, P_\theta, \alpha(\theta))\mathbf{1}_{\{\tau\geq\theta\}} + e^{-\beta\tau}[\kappa^\gamma\frac{B_T^{1-\gamma}}{1-\gamma}]\mathbf{1}_{\{\tau<\theta\}}\Bigg]\\
&\quad +C_0\delta\big(1 - E_{b_s,p_s,i_0}\big[e^{-\beta\tau_\alpha}\big]\big)
\end{aligned}\tag{A.16}$$

which is a contradiction to the DP principle since $E_{b_s,p_s,i_0}\big[e^{-\beta\tau_\alpha}\big] < 1$. Therefore the value function $v_i(t, b_t, p_t)$, $i = 1, 2$, is a viscosity sub-solution of the system (2.8).

This completes the proof of Theorem 3.1.

Appendix B. Proof of Theorem 3.2

For $\varrho, \epsilon, \delta, \lambda > 0$, we define the auxiliary functions $\phi : (0, T] \times \mathbb{R}_+^2 \times \mathbb{R}_+^2 \to \mathbb{R}$ and Ξ: $[0; T] \times \mathbb{R}_+^2 \times \mathbb{R}_+^2 \times \mathcal{S}$ by

$$\phi(t, (b, p), (b', p')) = \frac{\varrho}{t} + \frac{1}{2\epsilon}|(b, p) - (b', p')|^2 + \delta e^{\lambda(T-t)}(|(b, p)|^2 + |(b', p')|^2)$$

and

$$\Xi(t, (b, p), (b', p'), i) = v_i(t, b, p) - u_i(t, b', p') - \phi(t, (b, p), (b', p')).$$

By using the linear growth of v_i and u_i, we have for each $i \in \mathcal{S}$

$$\lim_{|(b,p)|+|(b',p')|\to\infty} \Xi(t, (b, p), (b', p'), i) = -\infty.$$

Then, since v_i and u_i are uniformly continuous with respect to (t, b, p) on each compact subset of $[0, T]\times\mathbb{R}_+\times\mathbb{R}_+\times\mathcal{S}$ and that $\mathcal{S}$ is a finite set, Ξ attains its global maximum at some finite point belonging to a compact $K \subset [0, T]\times\mathbb{R}_+^2\times\mathbb{R}_+^2\times\mathcal{S}$ say, $(t_{\delta\epsilon}, (b_{1\delta\epsilon}, p_{1\delta\epsilon}), (b_{2\delta\epsilon}, p_{2\delta\epsilon}), \alpha_{\delta\epsilon})$. Observing that $2\Xi\big(t_{\delta\epsilon}, (b_{1\delta\epsilon}, p_{1\delta\epsilon}), (b_{2\delta\epsilon}, p_{2\delta\epsilon}), \alpha_{\delta\epsilon}\big) \geq \Xi\big(t_{\delta\epsilon}, (b_{1\delta\epsilon}, p_{1\delta\epsilon}), (b_{2\delta\epsilon}, p_{2\delta\epsilon}), \alpha_{\delta\epsilon}\big)$ $+\ \Xi\big(t_{\delta\epsilon}, (b_{1\delta\epsilon}, p_{1\delta\epsilon}), (b_{2\delta\epsilon}, p_{2\delta\epsilon}), \alpha_{\delta\epsilon}\big)$ and using the uniform continuity of v_i and u_i on K we have

$$\begin{aligned}
&\frac{1}{\epsilon}|(b_{1\delta\epsilon}, p_{1\delta\epsilon}) - (b_{2\delta\epsilon}, p_{2\delta\epsilon})|^2\\
&\leq v_i\,(t_{\delta\epsilon}, (b_{1\delta\epsilon}, p_{1\delta\epsilon})) - v_i\big(\hat{t}_{\delta\epsilon}, (b_{2\delta\epsilon}, p_{2\delta\epsilon})\big) + u_i\,(t_{\delta\epsilon}, (b_{1\delta\epsilon}, p_{1\delta\epsilon})) - u_i\,(t_{\delta\epsilon}, (b_{2\delta\epsilon}, p_{2\delta\epsilon}))\\
&\leq 2C|(b_{1\delta\epsilon}, p_{1\delta\epsilon}) - (b_{2\delta\epsilon}, p_{2\delta\epsilon})|.
\end{aligned}$$

Thus,

$$|(b_{1\delta\epsilon}, p_{1\delta\epsilon}) - (b_{2\delta\epsilon}, p_{2\delta\epsilon})| \leq 2C\epsilon \tag{B.1}$$

where C is a positive constant independent of $\varrho, \epsilon, \delta, \lambda$. From the inequality,

$$2\Xi\,(T, (0, 0), (0, 0), \alpha_{\delta\epsilon}) \leq 2\Xi\,(t_{\delta\epsilon}, (b_{1\delta\epsilon}, p_{1\delta\epsilon}), (b_{2\delta\epsilon}, p_{2\delta\epsilon}), \alpha_{\delta\epsilon})$$

and the linear growth for v_i and u_i, we have:

$$\begin{aligned}
\delta\,(|(b_{1\delta\epsilon}, p_{1\delta\epsilon}))\,|^2+|\big(b_{2\delta\epsilon}, p_{2\delta\epsilon}|^2\big) &\leq e^{-\lambda(T-t_{\delta\epsilon})}\Big[\ v_i\,(t_{\delta\epsilon}, b_{1\delta\epsilon}, p_{1\delta\epsilon}) - v_i\,(T, 0, 0)\\
&\qquad +u_i\big(T, 0, 0\big) - u_i\,(t_{\delta\epsilon}, b_{2\delta\epsilon}, p_{2\delta\epsilon})\ \Big]\\
&\leq e^{-\lambda(T-t_{\delta\epsilon})}C_2\,(1 + |(b_{1\delta\epsilon}, p_{1\delta\epsilon})| + |(b_{2\delta\epsilon}, p_{2\delta\epsilon})|)\,.
\end{aligned}\tag{B.2}$$

It follows that

$$\frac{\delta\big(|(b_{1\delta\epsilon}, p_{1\delta\epsilon})\big)|^2+|(b_{2\delta\epsilon}, p_{2\delta\epsilon}|^2)}{(1 + |(b_{1\delta\epsilon}, p_{1\delta\epsilon})|+|(b_{2\delta\epsilon}, p_{2\delta\epsilon})|)} \leq C_2.$$

Consequently, there exists $C_\delta > 0$ such that,

$$|(b_{1\delta\epsilon}, p_{1\delta\epsilon})| + |(b_{2\delta\epsilon}, p_{2\delta\epsilon})| \leq C_\delta. \tag{B.3}$$

This inequality implies that for any fixed $\delta \in (0, 1)$, the sets $\{(b_{1\delta\epsilon}, p_{1\delta\epsilon}), \epsilon > 0\}$ and $\{(b_{2\delta\epsilon}, p_{2\delta\epsilon}), \epsilon > 0\}$ are bounded by C_δ independent of ϵ. It follows from inequalities (B.2) and (B.3) that, possibly if necessary along a subsequence, named again $\big(t_{\delta\epsilon}, (b_{1\delta\epsilon}, p_{1\delta\epsilon}), (b_{2\delta\epsilon}, p_{2\delta\epsilon}), \alpha_{\delta\epsilon}\big)$ that there exists $(b_{1\delta 0}, p_{1\delta 0}) \in \mathbb{R}_+^2$, $t_{\delta\epsilon 0} \in (0, T]$ and $\alpha_{\delta\epsilon 0} \in \mathcal{S}$ such that: $\lim_{\epsilon\downarrow 0}(b_{1\delta\epsilon}, p_{1\delta\epsilon}) = (b_{1\delta 0}, p_{1\delta 0}) = \lim_{\epsilon\downarrow 0}(b_{1\delta\epsilon}, p_{1\delta\epsilon})$, $\lim_{\epsilon\downarrow 0} t_{\delta\epsilon} = t_{\delta 0}$, $\lim_{\epsilon\downarrow 0} \alpha_{\delta\epsilon} = \alpha_{\delta 0}$.

If $t_{\delta\epsilon} = T$ then writing that $\Xi\big(t, (b, p), (b, p), \alpha_{\delta\epsilon}\big) \leq \Xi\big(T, (b_{1\delta\epsilon}, p_{1\delta\epsilon}), (b_{2\delta\epsilon}, p_{2\delta\epsilon}), \alpha_{\delta\epsilon}\big)$, we have

$$\begin{aligned}
&u_i(t, b, p) - v_i(t, b, p) - \frac{\varrho}{t} - 2\delta e^{\lambda(T-t)}(|(b, p)|^2) \\
&\leq u_i(T, (b_{1\delta\epsilon}, p_{1\delta\epsilon})) - v_i(T, (b_{2\delta\epsilon}, p_{2\delta\epsilon})) - \frac{\varrho}{T} \\
&\quad - \frac{1}{2\epsilon}|(b_{1\delta\epsilon}, p_{1\delta\epsilon}) - (b_{2\delta\epsilon}, p_{2\delta\epsilon})|^2 - \delta(|(b_{1\delta\epsilon}, p_{1\delta\epsilon})|^2 + |(b_{2\delta\epsilon}, p_{2\delta\epsilon})|^2) \\
&\leq u_i(T, (b_{1\delta\epsilon}, p_{1\delta\epsilon})) - v_i(T, (b_{2\delta\epsilon}, p_{2\delta\epsilon})) \\
&= [u_i(T, (b_{1\delta\epsilon}, p_{1\delta\epsilon})) - v_i(T, (b_{1\delta\epsilon}, p_{1\delta\epsilon}))] \\
&\quad + [v_i(T, (b_{1\delta\epsilon}, p_{1\delta\epsilon})) - v_i(T, (b_{2\delta\epsilon}, p_{2\delta\epsilon}))] \\
&\quad \leq C_1|(b_{1\delta\epsilon}, p_{1\delta\epsilon}) - (b_{2\delta\epsilon}, p_{2\delta\epsilon})|
\end{aligned}$$

where the last inequality follows from the uniform continuity of v_i and by assumption that $u_i(T, (b_{1\delta\epsilon}, p_{1\delta\epsilon})) = \kappa^\gamma \frac{b_{1\delta\epsilon}^{1-\gamma}}{1-\gamma} = v_i(T, (b_{1\delta\epsilon}, p_{1\delta\epsilon}))$. Sending $\varrho, \epsilon, \delta \downarrow 0$ and using estimate (B.1), we have: $u_i(t, b, p) \leq v_i(t, b, p)$. Assume now that $t_{\delta\epsilon} < T$.

Applying Lemma 3.2 with u_i, v_i and $\phi(t, (b, p), (b', p'))$ at the point $(t_{\delta\epsilon}, (b_{1\delta\epsilon}, p_{1\delta\epsilon}), (b_{2\delta\epsilon}, p_{2\delta\epsilon}), \alpha_{\delta\epsilon}) \in (0, T) \times \mathbb{R}_+^2 \times \mathbb{R}_+^2 \times \mathbb{S}$, for any $\zeta \in (0, 1)$ there are $d \in \mathbb{R}$, $M_{\delta\epsilon}, N_{\delta\epsilon} \in \mathbb{S}^2$ such that:

$$\begin{aligned}
&\Bigg(d - \frac{\varrho}{t_{\delta\epsilon}^2} - \lambda\delta e^{\lambda(T-t_{\delta\epsilon})}(|(b_{\delta\epsilon}, p_{\delta\epsilon})|^2 + |(b'_{\delta\epsilon}, p'_{\delta\epsilon})|^2), \frac{1}{\epsilon}((b_{\delta\epsilon}, p_{\delta\epsilon}) - (b'_{\delta\epsilon}, p'_{\delta\epsilon})) \\
&+ 2\delta e^{\lambda(T-t_{\delta\epsilon})}(b_{\delta\epsilon}, p_{\delta\epsilon}), M_{\delta\epsilon} + 2\delta e^{\lambda(T-t_{\delta\epsilon})} I\Bigg) \in \bar{\mathcal{P}}^{2,+} u(t_{\delta\epsilon}, b_{\delta\epsilon}, p_{\delta\epsilon}, i) \\
&\left(d, \frac{1}{\epsilon}((b_{\delta\epsilon}, p_{\delta\epsilon}) - (b'_{\delta\epsilon}, p'_{\delta\epsilon})) - 2\delta e^{\lambda(T-t_{\delta\epsilon})}(b'_{\delta\epsilon}, p'_{\delta\epsilon}), N_{\delta\epsilon} - 2\delta e^{\lambda(T-t_{\delta\epsilon})} I\right) \in \bar{\mathcal{P}}^{2,-} v(t_{\delta\epsilon}, b'_{\delta\epsilon}, p'_{\delta\epsilon}, i)
\end{aligned}$$

and

$$\begin{aligned}
-\frac{1}{\zeta}\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} M_{\delta\epsilon} & 0 \\ 0 & -N_{\delta\epsilon} \end{pmatrix} &\leq D^2_{(b,p),(b',p')}\phi(t_{\delta\epsilon}, (b_{\delta\epsilon}, p_{\delta\epsilon}), (b'_{\delta\epsilon}, p'_{\delta\epsilon})) \\
&\quad + \zeta\left(D^2_{(b,p),(b',p')}\phi(t_{\delta\epsilon}, (b_{\delta\epsilon}, p_{\delta\epsilon}), (b'_{\delta\epsilon}, p'_{\delta\epsilon}))\right)^2 \\
&\leq \frac{\epsilon + \zeta(2 + 4\delta\epsilon e^{\lambda(T-t)})}{\epsilon^2}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + (2\delta + 4\zeta\delta^2\epsilon e^{\lambda(T-t)})e^{\lambda(T-t)}\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.
\end{aligned}$$

Letting $\delta \downarrow 0$ and taking $\zeta = \dfrac{\epsilon}{2}$, we obtain

$$-\frac{1}{\epsilon}\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} M_{\delta\epsilon} & 0 \\ 0 & -N_{\delta\epsilon} \end{pmatrix} \leq \frac{2}{\epsilon}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}.$$

It follows that

$$\begin{aligned}
&(b_{\delta\epsilon}, p_{\delta\epsilon})M_{\delta\epsilon}\begin{pmatrix} b_{\delta\epsilon} \\ p_{\delta\epsilon} \end{pmatrix} - (b'_{\delta\epsilon}, p'_{\delta\epsilon})N_{\delta\epsilon}\begin{pmatrix} b'_{\delta\epsilon} \\ p'_{\delta\epsilon} \end{pmatrix} \\
&= ((b_{\delta\epsilon}, p_{\delta\epsilon}), (b'_{\delta\epsilon}, p'_{\delta\epsilon}))\begin{pmatrix} M_{\delta\epsilon} & 0 \\ 0 & -N_{\delta\epsilon} \end{pmatrix}\begin{pmatrix} \begin{pmatrix} b_{\delta\epsilon} \\ p_{\delta\epsilon} \end{pmatrix} \\ \begin{pmatrix} b'_{\delta\epsilon} \\ p'_{\delta\epsilon} \end{pmatrix} \end{pmatrix}
\end{aligned}$$

$$\leq ((b_{\delta\epsilon}, p_{\delta\epsilon}), (b'_{\delta\epsilon}, p'_{\delta\epsilon})) \left[\frac{2}{\epsilon}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}\right] \begin{pmatrix} \begin{pmatrix} b_{\delta\epsilon} \\ p_{\delta\epsilon} \end{pmatrix} \\ \begin{pmatrix} b'_{\delta\epsilon} \\ p'_{\delta\epsilon} \end{pmatrix} \end{pmatrix}$$

$$\leq \frac{2}{\epsilon}|(b_{\delta\epsilon}, p_{\delta\epsilon}) - (b'_{\delta\epsilon}, p'_{\delta\epsilon})|^2.$$

Furthermore, the definition of the viscosity subsolution u_i and supersolution v_i implies that

$$\min\Bigg[\ \beta u_{i_0}(t_{\delta\epsilon}, b_{\delta\epsilon}, p_{\delta\epsilon}) - \Big(d - \frac{\varrho}{t^2_{\delta\epsilon}} - \lambda\delta e^{\lambda(T-t_{\delta\epsilon})}(|(b_{\delta\epsilon}, p_{\delta\epsilon})|^2 + |(b'_{\delta\epsilon}, p'_{\delta\epsilon})|^2)\Big)$$

$$+ \inf_{E\in\mathcal{A}_{i_0}} \Bigg\{ -\Big[r_{i_0} b_{\delta\epsilon}\Big(1 - \frac{b_{\delta\epsilon}}{K}\Big) - qEb_{\delta\epsilon}\Big]\Big(\frac{1}{\epsilon}(b_{\delta\epsilon} - b'_{\delta\epsilon}) + 2\delta e^{\lambda(T-t)} b_{\delta\epsilon}\Big)$$

$$-\theta(\bar{p}_0 - \bar{p}_1 qEb_{\delta\epsilon} - p_s)\Big(\frac{1}{\epsilon}(p_{\delta\epsilon} - p'_{\delta\epsilon}) + 2\delta e^{\lambda(T-t)} p_{\delta\epsilon}\Big) - \frac{1}{2}(\sigma b_{\delta\epsilon}; \sigma_P)(M_{\delta\epsilon} + 2\delta e^{\lambda(T-t)} I)\begin{pmatrix} \sigma b_{\delta\epsilon} \\ \sigma_p \end{pmatrix}$$

$$-q_{i_0 j}[u_j(t_{\delta\epsilon}, b_{\delta\epsilon}, p_{\delta\epsilon}) - u_{i_0}(t_{\delta\epsilon}, b_{\delta\epsilon}, p_{\delta\epsilon})]$$

$$-l(_{i_0}, t_{\delta\epsilon}, b_{\delta\epsilon}, p_{\delta\epsilon}, E_{t_{\delta\epsilon}})\Bigg\},\quad u_{i_0}(T, b_{\delta\epsilon}, p_{\delta\epsilon}) - \kappa^{\gamma}\frac{b^{1-\gamma}_{\delta\epsilon}}{1-\gamma}\ \Bigg] \leq 0,\quad i_0 \neq j$$

and

$$\min\Bigg[\ \beta v_{i_0}(t_{\delta\epsilon}, b_{\delta\epsilon}, p_{\delta\epsilon}) - d + \inf_{E\in\mathcal{A}_{i_0}} \Bigg\{ -\Big[r_{i_0} b_{\delta\epsilon}\Big(1 - \frac{b_{\delta\epsilon}}{K}\Big) - qEb_{\delta\epsilon}\Big]\Big(\frac{1}{\epsilon}(b_{\delta\epsilon} - b'_{\delta\epsilon}) + 2\delta e^{\lambda(T-t)} b'_{\delta\epsilon}\Big)$$

$$-\theta(\bar{p}_0 - \bar{p}_1 qEb_{\delta\epsilon} - p_s)\Big(\frac{1}{\epsilon}(p_{\delta\epsilon} - p'_{\delta\epsilon}) + 2\delta e^{\lambda(T-t)} p'_{\delta\epsilon}\Big) - \frac{1}{2}(\sigma b'_{\delta\epsilon}; \sigma_P)(N_{\delta\epsilon} - 2\delta e^{\lambda(T-t)} I)\begin{pmatrix} \sigma b'_{\delta\epsilon} \\ \sigma_p \end{pmatrix}$$

$$-q_{i_0 j}[v_j(t_{\delta\epsilon}, b'_{\delta\epsilon}, p'_{\delta\epsilon}) - v_{i_0}(t_{\delta\epsilon}, b'_{\delta\epsilon}, p'_{\delta\epsilon})]$$

$$-l(_{i_0}, t_{\delta\epsilon}, b'_{\delta\epsilon}, p'_{\delta\epsilon}, E_{t_{\delta\epsilon}})\Bigg\},\quad v_{i_0}(T, b_{\delta\epsilon}, p_{\delta\epsilon}) - \kappa^{\gamma}\frac{b^{1-\gamma}_{\delta\epsilon}}{1-\gamma}\ \Bigg] \geq 0,\quad i_0 \neq j.$$

Let us define operators $A^E(x, v, \phi, X, Z)$ and $B^E(x, v)$.

$$A^E(t, b, p, w, X, YZ) = \Big[r_{i_0} b\Big(1 - \frac{b}{K}\Big) - qEb\Big]X + \theta(\bar{p}_0 - \bar{p}_1 qEb - p_s)Y + \frac{1}{2}wZw'$$

$$B^E(t, b, p, v) = q_{i_0 j}[v_j(t, b, p) - v_{i_0}(t, b, p)].$$

By subtracting these last two inequalities and remarking that $\min(x; y) - \min(z; t) \leq 0$ implies either $x - z \leq 0$ or $y - t \leq 0$, we divide our consideration into two cases:

Case 1

$$\beta\big[u_{i_0}(t_{\delta\epsilon}, b_{\delta\epsilon}, p_{\delta\epsilon}) - v_{i_0}(t_{\delta\epsilon}, b_{\delta\epsilon}, p_{\delta\epsilon})\big] + \frac{\varrho}{t^2_{\delta\epsilon}} + \lambda\delta e^{\lambda(T-t_{\delta\epsilon})}(|(b_{\delta\epsilon}, p_{\delta\epsilon})|^2 + |(b'_{\delta\epsilon}, p'_{\delta\epsilon})|^2)$$

$$\leq \sup_{E\in\mathcal{A}_{i_0}}\big\{l(_{i_0}, t_{\delta\epsilon}, b_{\delta\epsilon}, p_{\delta\epsilon}, E_{t_{\delta\epsilon}}) - l(_{i_0}, t_{\delta\epsilon}, b'_{\delta\epsilon}, p'_{\delta\epsilon}, E_{t_{\delta\epsilon}})\big\}$$

$$+ \sup_{E\in\mathcal{A}_{i_0}} \Bigg\{ A^E\ \Big(\ t_{\delta\epsilon}, b_{\delta\epsilon}, p_{\delta\epsilon}, (\sigma b_{\delta\epsilon}; \sigma_P), \frac{1}{\epsilon}(b_{\delta\epsilon} - b'_{\delta\epsilon}) + 2\delta e^{\lambda(T-t_{\delta\epsilon})} b_{\delta\epsilon},$$

$$\frac{1}{\epsilon}(p_{\delta\epsilon} - p'_{\delta\epsilon}) + 2\delta e^{\lambda(T-t_{\delta\epsilon})} p_{\delta\epsilon}, M_{\delta\epsilon} + 2\delta e^{\lambda(T-t_{\delta\epsilon})} I\ \Big)$$

$$-A^E\ \Big(\ t_{\delta\epsilon}, b'_{\delta\epsilon}, p'_{\delta\epsilon}, (\sigma b'_{\delta\epsilon}; \sigma_P), \frac{1}{\epsilon}(b_{\delta\epsilon} - b'_{\delta\epsilon}) + 2\delta e^{\lambda(T-t)} b'_{\delta\epsilon}, \frac{1}{\epsilon}(p_{\delta\epsilon} - p'_{\delta\epsilon}) + 2\delta e^{\lambda(T-t_{\delta\epsilon})} p'_{\delta\epsilon},$$

$$N_{\delta\epsilon} - 2\delta e^{\lambda(T-t_{\delta\epsilon})} I\ \Big)\Bigg\}$$

$$+ \sup_{E\in\mathcal{A}_{i_0}}\ \Bigg\{\ B^E(t_{\delta\epsilon}, b_{\delta\epsilon}, p_{\delta\epsilon} u) - B^E(t_{\delta\epsilon}, b'_{\delta\epsilon}, p'_{\delta\epsilon} v)\ \Bigg\} \equiv \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3. \qquad \text{(B.4)}$$

In view of condition (13) on l and from estimate 3.1, we have the classical estimates of $\mathcal{I}_1$ and $\mathcal{I}_2$:

$$\mathcal{I}_1 \leq C|(b_{\delta\epsilon}, p_{\delta\epsilon}) - (b'_{\delta\epsilon}, p'_{\delta\epsilon})|$$

$$\mathcal{I}_2 \leq C(\frac{1}{\epsilon}|(b_{\delta\epsilon}, p_{\delta\epsilon}) - (b'_{\delta\epsilon}, p'_{\delta\epsilon})|^2) + 2\delta e^{\lambda(T-t_{\delta\epsilon})}(1 + |(b_{\delta\epsilon}, p_{\delta\epsilon})|^2 + |(b'_{\delta\epsilon}, p'_{\delta\epsilon})|^2).$$

Using the Lipschitz condition for u and v, we have

$$\mathcal{I}_3 \leq 2C|(b_{\delta\epsilon}, p_{\delta\epsilon}) - (b'_{\delta\epsilon}, p'_{\delta\epsilon})|.$$

Writing that $\Xi(t, (b, p), (b, p), i) \leq \Xi(t_{\delta\epsilon}, (b_{\delta\epsilon}, p_{\delta\epsilon}), (b_{\delta\epsilon}, p_{\delta\epsilon}), i)$ for $i \in \mathcal{S}$ and using the inequality (B.4),

$$u_i(t, b, p) - v_i(t, b, p) - \frac{\varrho}{t} - 2\delta e^{\lambda(T-t)}|(b, p)|^2 \leq$$
$$v_i(t_{\delta\epsilon}, b_{\delta\epsilon}, p_{\delta\epsilon}) - u_i(t_{\delta\epsilon}, b_{\delta\epsilon}, p_{\delta\epsilon}) - \frac{\varrho}{t_{\delta\epsilon}} - 2\delta e^{\lambda(T-t)}|(b_{\delta\epsilon}, p_{\delta\epsilon})|^2 \leq$$
$$\frac{1}{\beta}\Big[\mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3\Big] - \frac{\varrho}{\beta t_{\delta\epsilon}^2} - \frac{\lambda}{\beta}\delta e^{\lambda(T-t_{\delta\epsilon})}(|(b_{\delta\epsilon}, p_{\delta\epsilon})|^2 + |(b'_{\delta\epsilon}, p'_{\delta\epsilon})|^2)$$

this implies

$$u_i(t, b, p) - v_i(t, b, p) - \frac{\varrho}{t} - 2\delta e^{\lambda(T-t)}|(b, p)|^2 \leq$$
$$\frac{1}{\beta}\Big[\mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3\Big] - \frac{\lambda}{\beta}\delta e^{\lambda(T-t_{\delta\epsilon})}(|(b_{\delta\epsilon}, p_{\delta\epsilon})|^2 + |(b'_{\delta\epsilon}, p'_{\delta\epsilon})|^2).$$

Sending $\epsilon \downarrow 0$, with the above estimates of $(\mathcal{I}_1) - (\mathcal{I}_2) - (\mathcal{I}_3)$, we obtain:

$$u_i(t, b, p) - v_i(t, b, p) - \frac{\varrho}{t} - 2\delta e^{\lambda(T-t)}|(b, p)|^2 \leq \frac{2\delta}{\beta}e^{\lambda(T-t_0)}\Big[C(1 + 2|(b_0, p_0)|^2) - \lambda|(b_0, p_0)|^2\Big].$$

Choose λ sufficiently large positive ($\lambda \geq 2C$) and send $\varrho, \delta \to 0^+$ to conclude that $u_i(t, b, p) \leq v_i(t, b, p)$.

Case 2 The second case occurs if

$$u_{i_0}(T, b_{\delta\epsilon}, p_{\delta\epsilon}) - v_{i_0}(T, b_{\delta\epsilon}, p_{\delta\epsilon}) \leq 0$$

and finally that $u_i(t, b, p) \leq v_i(t, b, p)$.

This completes the proof.

Appendix C. Supplementary data

Supplementary material related to this article can be found online at https://doi.org/10.1016/j.rinam.2020.100125.

References

[1] Schaefer MB. Some aspects of the dynamics of populations important to the management of the commercial marine fisheries. Inter-Amer Trop Tuna Comm Bull 1954;1(2):23–56.
[2] Brites NM, Braumann CA. Fisheries management in random environments: Comparison of harvesting policies for the logistic model. Fish Res 2017;195:238–46.
[3] MacCall AD. Fishery-management and stock-rebuilding prospects under conditions of low-frequency environmental variability and species interactions. Bull Mar Sci 2002;70(2):613–28.
[4] Daget J. Les poission du niger superior. Mem IFAN 1954;36:39pp.
[5] Svensson GS. Fresh water fishes from the Gambia river (British West Africa): results of the Swedish expedition 1931. Almqvist & Wiksells; 1933.
[6] Carey T, Bell-Cross G. Breeding seasons and quantitative data on gonads and ova for certain fish species. Fish Res Bull Zambia 1967;3:12–22.
[7] Dudley RG. Growth of tilapia of the kafue floodplain, zambia: predicted effects of the kafue gorge dam. Trans Amer Fish Soc 1974;103(2):281–91.
[8] Murillas A, Chamorro JM. Valuation and management of fishing resources under price uncertainty. Environ Resour Econ 2006;33(1):39–71.
[9] Kvamsdal SF, Poudel D, Sandal LK. Harvesting in a fishery with stochastic growth and a mean-reverting price. Environ Resour Econ 2016;63(3):643–63.
[10] Insley M. A real options approach to the valuation of a forestry investment. J Environ Econ Manag 2002;44(3):471–92.
[11] Braumann CA. Stochastic differential equation models of fisheries in an uncertain world: extinction probabilities, optimal fishing effort, and parameter estimation. In: Mathematics in biology and medicine. Springer; 1985, p. 201–6.
[12] Nøstbakken L. Regime switching in a fishery with stochastic stock and price. J Environ Econ Manag 2006;51(2):231–41.
[13] Sarkar S. Optimal fishery harvesting rules under uncertainty. Resour Energy Econ 2009;31(4):272–86.
[14] Bockstael NE, Opaluch JJ. Discrete modelling of supply response under uncertainty: the case of the fishery. J Environ Econ Manag 1983;10(2):125–37.
[15] Dupont DP. Price uncertainty, expectations formation and fishers' location choices. Mar Resour Econ 1993;8(3):219–47.
[16] Mistiaen JA, Strand IE. Location choice of commercial fishermen with heterogeneous risk preferences. Amer J Agric Econ 2000;82(5):1184–90.
[17] Andersen S, Harrison GW, Lau MI, Elisabet Rutström E. Risk aversion in game shows. In: Risk aversion in experiments. Emerald Group Publishing Limited; 2008, p. 359–404.

[18] Clark W. Mathematical bioeconomics: The optimal management resources. John Wiley & Sons; 1976.
[19] Clark C. Mathematical bioeconomics. John Wiley & Sons; 1990.
[20] Crandall MG, Ishii H, Lions P-L. User's guide to viscosity solutions of second order partial differential equations. Bull Amer Math Soc 1992;27(1):1–67.
[21] Linos PL. Optimal control of diffusion processes and hamilton-jacobi-bellman equations part i: the dynamic programming principle and application. Comm Partial Differential Equations 1983;8(10):1101–74.
[22] Barles G, Imbert C. Second-order elliptic integro-differential equations: viscosity solutions' theory revisited. In: Annales de l'IHP analyse non linéaire, Vol. 25. 2008, p. 567–85.
[23] Barles G, Souganidis PE. Convergence of approximation schemes for fully nonlinear second order equations. Asymptot Anal 1991;4(3):271–83.
[24] Kushner H, Dupuis PG. Numerical methods for stochastic control problems in continuous time, Vol. 24. Springer Science & Business Media; 2013.
[25] Bokanowski O, Maroso S, Zidani H. Some convergence results for howard's algorithm. SIAM J Numer Anal 2009;47(4):3001–26.
[26] Bardi M, Capuzzo-Dolcetta I. Optimal control and viscosity solutions of Hamilton-Jacobi-bellman equations. Modern birkhäuser classics, Birkhäuser Boston; 2009, URL https://books.google.cm/books?id=dyjrBwAAQBAJ.
[27] Liu R. Optimal stopping of switching diffusions with state dependent switching rates. Stochastics 2016;88(4):586–605.