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Abstract

The main purpose of this work is to contribute to the study of set-valued random variables by providing a kind of Wold decomposition theorem for interval-valued processes. As the set of set-valued random variables is not a vector space, the Wold decomposition theorem as established in 1938 by Herman Wold is not applicable for them. So, a notion of pseudovector space is introduced and used to establish a generalization of the Wold decomposition theorem that works for interval-valued covariance stationary time series processes. Before this, Set-valued autoregressive and moving average (S-ARMA) time series process is defined by taking into account an arithmetical difference between random sets and random real variables.

Keywords: Wold décomposition, stationary time series, interval-valued time series processes, ARMA model

1 Introduction

No one can contest the great role played by point-valued time series in forecasting, merely because they intervene in several disciplines ranging from econometrics, astronomy to meteorology. But, the forecasts are not always efficient especially in cases where the values of the analyzed vary a lot during the period of one recording. In those cases, it should be assumed that the values belong to a range or even is equal to a set or an interval. For instance, economic increasing rates are most often assumed to belong to a range. In weather forecasting, the weather of the next day is always given as an interval bounded by the highest and lowest temperatures of the next day. An example where point-valued are sometimes wrongly used is in forecasting stock prices. For studying them, the closing prices are usually considered as the value of the studied index for the day. It should be better to consider the smallest interval containing all the prices of the day or to consider the set of the prices of the day as the value of the index. An interval-valued observation in a time period contains more information than a point-valued observation in the same time period as Ai Han and al. stated in [HHW12]. For some other references regarding the applications of intervals time series in finance, see [SKJ12, MMA14b, MMA14a] and some references therein.

A set-valued random variable is a random variable whose values are the subsets of the $d-$dimensional Euclidean space $\mathbb{R}^d$. When $d = 1$ and values are compact and convex, one obtains an interval-valued random variable. Random sets started to be intensively studied after the works of Robert Aumann [Aum65], where the author defined the expectation of a random set on a probability space. Since the set of random variables is not a vector space, this expectation does not allow the definition of variance as in the case of random real variables. It was necessary to wait for Xuhua Yang and Shoumei Li [YL05] for the definition of the variance of a random set, using the D_p-metric introduced in the same paper.

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The development of the modeling of time series begins with the autoregressive (AR) process of yule (1927). For interval-valued data, the Interval autoregression (IAR) model has been introduced in [WL11]. In the latter paper, the prediction of interval-valued time series is addressed using mean square error. In the present work, we give a new definition for I-AR and extend it to interval-valued autoregressive moving average time series (I-ARMA) processes. This definition takes into account the arithmetical differences between random sets and random real variables. More precisely for point-valued time series, the equations $X_t - \sum_{i=1}^{p} \phi_i X_{t-i} = K + \varepsilon_t + \sum_{i=1}^{q} \theta_i \varepsilon_{t-i}$ and $X_t = \sum_{i=1}^{p} \phi_i X_{t-i} + K + \varepsilon_t + \sum_{i=1}^{q} \theta_i \varepsilon_{t-i}$ are equivalent, which is not the case with set-valued time series. We will work mainly with the ARMA process because it is a simple model that is both linear in variable and in some of the parameters. The first linearity provides an easy to use prediction formulas and the second allows estimation of the parameters. The main theoretical justification for the use of ARMA processes instead of AR processes is the Wold theorem (1954) which states that any stochastic, weakly stationary process can be rewritten as an infinite moving average process.

Wold's decomposition theorem for covariance stationary processes [Wol38] in theoretical econometrics generalizes the idea that autoregressive processes can have a moving average representation. Wold's theorem consists of showing how to decompose any covariance stationary time series process into a non-deterministic component motivated by uncorrelated linear innovations and a deterministic component. It should be noted here that an abstract part of the Wold decomposition theorem is stated in functional analysis and more precisely in algebra. This abstract theorem makes it possible to decompose a Hilbert space as a sum of orthogonal subspaces. However, Wold's Theorem cannot be applied to interval-valued time series, since the set of random intervals (or more generally random sets) is not a vector space.

To deal with Wold decomposition for interval-valued time series processes, this paper introduces a notion of pseudovector space on the set of set random variables. The main difference of this with vector spaces is that opposites (inverses for addition) of vectors do not exist in general. To recover a kind of opposite, generalized Hukuhara difference [Ste10] is used.

This paper is structured as follows. In the present section, the work is introduced and motivated. The next section is devoted to the theoretical framework, while Sections 3 and 4 describe set-valued time series processes. In Section 5, pseudo-vector space with some resulting definitions and operations are introduced. The notion of difference operator is also defined as well as the notion of Hilbert pseudo-vector space. The last section establishes properly and in detail the Wold decomposition theorem for covariance stationary interval-valued processes. The idea behind this proof comes from the classical one in [Bie].

2 Set-valued random variables

Let $(\Omega, \mathscr{A}, P)$ be a probability space with values in $\mathbb{R}^d$, $K(\mathbb{R}^d)$ the set of nonempty closed subset of $\mathbb{R}^d$ and $K_{kc}(\mathbb{R}^d)$ the set of nonempty compact and convex subsets of $\mathbb{R}^d$. For $A, B \in K(\mathbb{R}^d)$ and $\lambda \in \mathbb{R}$, we recall the operations

$$A + B = \{a + b \ ; \ a \in A, b \in B\} \tag{1}$$

$$\lambda A = \{\lambda a \ ; \ a \in A\}. \tag{2}$$

It is noteworthy that $K_{kc}(\mathbb{R}^d)$ is closed under those operations, but is not a vector space, since $A + (-1)A$ is not necessarily $\{0\}$, unless $A = \{0\}$. For $A \in K_{kc}(\mathbb{R}^d)$, the support function of A is the function $s(\cdot, A) : \mathbb{R}^d \to \mathbb{R}$ defined by

$$s(x, A) = \sup\{ax \ ; \ a \in A\}, \tag{3}$$

for every $x \in \mathbb{R}^d$. From the definition, the following Proposition is easy to verify.

Proposition 2.1. *For every $x \in \mathbb{R}^d$, the map $s(x, \cdot) : K_{kc}(\mathbb{R}^d) \to \mathbb{R}$ satisfies*

$$s(x, A + B) = s(x, A) + s(x, B) \tag{4}$$
$$s(x, \lambda A) = \lambda s(x, A), \tag{5}$$

for all $A, B \in K_{kc}(\mathbb{R}^d)$ and $\lambda \geq 0$.

For $1 \leq p < \infty$, the d_p metric is defined for $A, B \in K_{kc}(\mathbb{R}^d)$ by

$$d_p(A, B) = \left(\int_{S^{d-1}} |s(x, A) - s(x, B)|^p dx \right)^{1/p}, \tag{6}$$

where $S^{d-1} = \{x \in \mathbb{R}^d \; ; \; \|x\| = 1\}$ is the unit sphere. A proof for the following Theorem can be found in [DK94].

Theorem 1 ([DK94]). *For each $1 \leq p < \infty$, $(K_{kc}(\mathbb{R}^d), d_p)$ is a complete and separable metric space. Moreover the metrics d_p induce the same topology on $K_{kc}(\mathbb{R}^d)$.*

Example 1. *In $\mathbb{R}^2$, $S^1 = \{(x_1, x_2) \in \mathbb{R}^2 \; ; \; x_1^2 + x_2^2 = 1\} = \{\theta^\star = (\cos\theta, \sin\theta) \; ; \; \theta \in [0; 2\pi]\}$. Let $A_1 = [a_1, b_1] \times [c_1, d_1]$ and $A_2 = [a_2, b_2] \times [c_2, d_2]$ be two quadrilaterals in $\mathbb{R}^2$, with $a_1 \leq a_2$, $b_2 \leq b_1$, $c_1 \leq c_2$ and $d_2 \leq d_1$. It is obvious to see that*

$$s(\theta^\star, A_1) = \begin{cases} b_1 \cos\theta + d_1 \sin\theta & \text{if } \theta \in [0, \pi/2] \\ a_1 \cos\theta + d_1 \sin\theta & \text{if } \theta \in [\pi/2, \pi] \\ a_1 \cos\theta + c_1 \sin\theta & \text{if } \theta \in [\pi, 3\pi/2] \\ b_1 \cos\theta + c_1 \sin\theta & \text{if } \theta \in [3\pi/2, 2\pi] \end{cases}. \tag{7}$$

It follows that

$$d_1(A_1, A_2) = \int_0^{\pi/2} |(b_1 - b_2)\cos\theta + (d_1 - d_2)\sin\theta|d\theta + \int_{\pi/2}^{\pi} |(a_1 - a_2)\cos\theta + (d_1 - d_2)\sin\theta|d\theta$$

$$+ \int_{\pi}^{3\pi/2} |(a_1 - a_2)\cos\theta + (c_1 - c_2)\sin\theta|d\theta + \int_{3\pi/2}^{2\pi} |(b_1 - b_2)\cos\theta + (c_1 - c_2)\sin\theta|d\theta$$

$$d_1(A_1, A_2) = 2(a_1 - a_2 + b_1 - b_2 + c_1 - c_2 + d_1 - d_2).$$

We prove the following result which will be used to show that the covariance operator on random sets shares some useful properties with the covariance operator on random real variables.

Lemma 2.1. *For all $A, B \in K_{kc}(\mathbb{R}^d)$ and $\lambda \in \mathbb{R}$, $d_p(\lambda A, \lambda B) = |\lambda| d_p(A, B)$.*

Proof. For $\lambda \geq 0$, the result follows from the fact that $s(x, \lambda A) = \lambda s(x, A)$ and the same for B. Now, since the maps $x \mapsto -x$ and $a \mapsto -a$ are bijective on S^{d-1} and A respectively, $d_p(-A, -B) = d_p(A, B)$. Finally for $\lambda < 0$, $d(\lambda A, \lambda B) = d(|\lambda|(-A), |\lambda|(-B)) = |\lambda| d(-A, -B) = |\lambda| d(A, B)$. $\square$

A **set-valued random variable** (or a **random set**) is a map $F : \Omega \to K(\mathbb{R}^d)$ such that for every open subset $\mathcal{O}$ of $\mathbb{R}^d$, one has $F^{-1}(\mathcal{O}) \in \mathscr{A}$, where $F^{-1}(\mathcal{O}) = \{\omega \in \Omega \; ; \; F(\omega) \cap \mathcal{O} \neq \emptyset\}$. We denote by $\mathcal{U}[\Omega, K_{kc}(\mathbb{R}^d)]$ the set of set-valued random variables with values in $K_{kc}(\mathbb{R}^d)$. The D_p metric is defined for $F_1, F_2 \in \mathcal{U}[\Omega, K_{kc}(\mathbb{R}^d)]$ by

$$D_p(F_1, F_2) = \left(E\left[d_p^p(F_1, F_2) \right] \right)^{1/p}. \tag{8}$$

Let $\mathcal{L}^p[\Omega, K_{kc}(\mathbb{R}^d)] = \{F \in \mathcal{U}[\Omega, K_{kc}(\mathbb{R}^d)] \; ; \; E\|F\|_{d_p}^p < +\infty\}$, where $\|F\|_{d_p} = d_p(F, \{0\})$. From [YL05] it is known that for any $p \geq 1$, $(\mathcal{L}^p[\Omega, K_{kc}(\mathbb{R}^d)], D_p)$ is a complete metric space. In 1965, the expectation of a random set F is defined by Robert Aumann [Aum65] as

$$E[F] = \left\{ \int_\Omega f dP \; ; \; f \in S_F \right\}, \tag{9}$$

where $S_F = \{f : \Omega \to \mathbb{R}^d \; ; \; f(\omega) \in F(\omega) \text{ a.s and } f \text{ is integrable}\}$.

Example 2. *For $F(\omega) = [a(\omega), b(\omega)]$, $E[F] = [E(a), E(b)]$.*

Since the set of random sets is not a vector space (with the addition (1) and the scalar multiplication (2)), it was not easy to define variance and covariance for set-valued random variables. It is about 40 years ago after the definition of the expectation that Yang and Li [YL05] used the D_p metric on $\mathcal{U}[\Omega, K_{kc}(\mathbb{R}^d)]$ to define covariance of set-valued random variables $F_1, F_2 \in \mathcal{U}[\Omega, K_{kc}(\mathbb{R}^d)]$ as

$$Cov(F_1, F_2) = E\left[\int_{S^{d-1}} (s(x, F_1) - s(x, E(F_1)))(s(x, F_2) - s(x, E(F_2)))\, \mu(dx)\right], \quad (10)$$

and the variance of $F \in \mathcal{U}[\Omega, K_{kc}(\mathbb{R}^d)]$ as

$$Var(F) = Cov(F, F) = E\left[\int_{S^{d-1}} (s(x, F) - s(x, E(F)))^2\, \mu(dx)\right] = D_2^2(F, E(F)). \quad (11)$$

Proposition 2.2. *For all $F, F_1, F_2, F_3 \in \mathcal{U}[\Omega, K_{kc}(\mathbb{R}^d)]$ the following hold:*

① *$Var(C) = 0$, for every constant $C \in \mathcal{U}[\Omega, K_{kc}(\mathbb{R}^d)]$;*

② *$Var(F_1 + F_2) = Var(F_1) + 2Cov(F_1, F_2) + Var(F_2)$;*

③ *$Cov(F_1, F_2) = Cov(F_2, F_1)$;*

④ *$Cov(F_1 + F_2, F_3) = Cov(F_1, F_3) + Cov(F_2, F_3)$.*

⑤ *$Var(\lambda F) = \lambda^2 Var(F)$, for every $\lambda \in \mathbb{R}$;*

Proof. Proof of items ①-② can be found in [YL05]. Items ③-④ are obvious. The last item can be proven using Lemma 2.1, $Var(\lambda F) = E\left[d_2^2(\lambda F, E(\lambda F))\right] = \lambda^2 E\left[d_2^2(F, E(F))\right]$ □

Example 3. *For $d = 1$ and $p = 2$, $d_2^2([a_1, b_1], [a_2, b_2]) = |a_1 - a_2|^2 + |b_1 - b_2|^2$ which implies that for $F(\omega) = [a(\omega), b(\omega)]$, $Var(F) = E\left[|a(\omega) - E(a)|^2 + |b(\omega) - E(b)|^2\right]$. Hence $Var(F) = 0$ if and only if $F = E[F]$ a.s, ie $F(\omega) = E[F]$ for almost everywhere $\omega \in \Omega$.*

3 Set-valued time series

Let $(X_t)_{t \in \mathbb{Z}}$ be a set-valued time series with $X_t \in \mathcal{U}(\Omega, K_{kc}(\mathbb{R}^d))$.

Definition 3.1. *We say that $(X_t)_{t \in \mathbb{Z}}$ is (covariance) stationary when:*

① *$\forall t \in \mathbb{Z}$, $E[X_t] = A$, being A a constant compact convex set;*

② *$\forall t, s \in \mathbb{Z}$, $Cov(X_t, X_s) = \gamma(t - s)$.*

The set K and the sequence $(\gamma(k))_{k \in \mathbb{Z}}$ are called expectation and auto-covariance function of the time series $(X_t)_{t \in \mathbb{Z}}$ respectively. The autocorrelation function $(\rho(k))_{k \in \mathbb{Z}}$ of $(X_t)_{t \in \mathbb{Z}}$ is defined by

$$\rho(k) = \frac{\gamma(k)}{\gamma(0)}. \quad (12)$$

Definition 3.2. *Two set-valued time series $(X_t)_{t \in \mathbb{Z}}$ and $(Y_t)_{t \in \mathbb{Z}}$ are said uncorrelated when*

$$\forall t, s \in \mathbb{Z}, \qquad Cov(X_t, Y_s) = 0. \quad (13)$$

Definition 3.3. *A set-valued time series* $(\varepsilon_t)_{t \in \mathbb{Z}}$ *is called white nose if for any* $t, s \in \mathbb{Z}$,

$$\begin{cases} E[\varepsilon_t] = \{0\} \\ Cov(\varepsilon_t, \varepsilon_s) = \sigma^2 \delta_{ts} \end{cases}, \tag{14}$$

and we write

$$(\varepsilon_t) \rightsquigarrow WN(\{0\}, \sigma^2). \tag{15}$$

If moreover random intervals ε_t *are independently and identically distributed then we write*

$$(\varepsilon_t) \rightsquigarrow IID(\{0\}, \sigma^2). \tag{16}$$

4 Set-valued autoregressive moving-average process

Interval-valued autoregressive time series were introduced in [WL11] and studied in [WZL16]. This section introduces the set-valued autoregressive moving-average time series process in a more suitable way.

Let $(X_t)_{t \in \mathbb{Z}}$ be a stationary set-valued time series with expectation A and auto-covariance function $(\gamma(k))$. We will say that X_t is a **set-valued autoregressive moving-average (S-ARMA)** time series process of order (p, q) when the series satisfies

$$X_t - \sum_{i=1}^{p} \phi_i X_{t-i} = K + \varepsilon_t + \sum_{i=1}^{q} \theta_i \varepsilon_{t-i} \quad \text{or} \quad X_t = \sum_{i=1}^{p} \phi_i X_{t-i} + K + \varepsilon_t + \sum_{i=1}^{q} \theta_i \varepsilon_{t-i}, \tag{17}$$

being K a constant compact convex set, $\phi_i \geq 0$ and $\theta_i \geq 0$ are the parameters of the model, $(\varepsilon_t) \rightsquigarrow IID(\{0\}, \sigma^2)$ and each ε_t is uncorrelated with $X_1, \ldots, X_{t-1}$. By taking expectation at the both sides of (17) one finds

$$\lambda A = K, \tag{18}$$

where $\lambda = 1 - \phi_1 - \cdots - \phi_p$. So if $\lambda = 0$ or the time series is centered (ie $A = \{0\}$) then $K = \{0\}$. But contrary to real random variables, the new series $X_t' = X_t - \frac{1}{\lambda}K$ does not satisfy a centered ARMA equation, ie Equation (17) with $K = \{0\}$. In fact replacing $X_t = X_t' + \frac{1}{\lambda}K$ in (17) one obtains

$$X_t' - \sum_{i=1}^{p} \phi_i X_{t-i}' + \frac{1}{\lambda}\left(K - \sum_{i=1}^{p} \phi_i K\right) = K + \varepsilon_t + \sum_{i=1}^{q} \theta_i \varepsilon_{t-i}, \tag{19}$$

and $K - \phi_1 K - \cdots - \phi_p K \neq (1 - \phi_1 - \cdot - \phi_p)K$ unless $K = \{0\}$ or $\phi_1 = \cdots = \phi_p = 0$. When $p = 0$, the time series (X_t) is called a set-valued moving-average time series of order q, S-MA(q), and when $q = 0$, one obtains a set-valued autoregressive time series of order p, S-AR(p). Let L be the delay operator, thus $LX_t = X_{t-1}$. Setting $\Phi(L) = \phi_1 L + \cdots + \phi_p L^p$ and $\Theta(L) = 1 + \theta_1 L + \cdots + \theta_q L^q$, equation (17) can be written as

$$X_t - \Phi(L)X_t = K + \Theta(L)\varepsilon_t \quad \text{or} \quad X_t = \Phi(L)X_t + K + \Theta(L)\varepsilon_t. \tag{20}$$

The functions Φ and Θ are called autoregressive and moving-average polynomials respectively.

Example 4 (S-MA(q)). *If the autoregressive polynomial* $\Phi = 0$ *then* (20) *leads to*

$$X_t = K + \varepsilon_t + \sum_{i=1}^{q} \theta_i \varepsilon_{t-i}, \tag{21}$$

which is a set-valued moving-average process of order q, S-MA(q). It is clear that the latter has a unique solution (X_t) *and moreover this solution is always a stationary process. In fact,*

$$E[X_t] = E[K] + \sum_{i=1}^{q} \theta_i E[\varepsilon_{t-i}] = K,$$

and

$$\gamma(k) = Cov(X_{t-k}, X_t) = \begin{cases} \sum\limits_{i=k}^{q} \theta_i \theta_{i-k} \sigma^2 & \text{if } 0 \le k \le q \\ 0 & \text{otherwise} \end{cases},$$

where we have set $\theta_0 = 1$. The autocorrelation function is given by

$$\rho(k) = \begin{cases} \dfrac{\sum\limits_{i=k}^{q} \theta_i \theta_{i-k}}{\sum\limits_{i=0}^{q} \theta_i^2} & \text{if } 1 \le k \le q \\ 0 & \text{otherwise} \end{cases}. \tag{22}$$

Hence the autocorrelation function is vanishing for $k > q$ and $\rho(q) = \theta_q/(1 + \theta_1^2 + \cdots + \theta_q^2)$ is not vanishing when $\theta_q \ne 0$. So in practice for determining the parameter q, one can compute empirical autocorrelations and look when they become significantly zero.

Example 5 (S-AR(p)). *If the moving-average polynomial $\Theta = 1$ then (20) leads to*

$$X_t - \Phi(L)X_t = K + \varepsilon_t \quad or \quad X_t = \Phi(L)X_t + K + \varepsilon_t. \tag{23}$$

which is a set-valued autoregressive process of order p, S-AR(p). In this case, the existence and the uniqueness of a stationary solution is not guaranteed. However when a stationary solution exits, its expectation A is given by (18) and using Proposition 2.2 it is nothing to show that its auto-covariance function satisfies

$$\gamma(k) - \sum_{i=1}^{p} \phi_i \gamma(k-i) = 0, \text{ for any } k \ge 1. \tag{24}$$

The above formula is for the particular case of S-AR processes. For the general case of an S-ARMA process, one can prove the following. Taking covariance with X_{t-k} on both sides of (17) we see that

$$\gamma(k) - \sum_{i=1}^{p} \phi_i \gamma(k-i) = 0, \text{ for } k \ge q+1 \tag{25}$$

$$\gamma(q) - \sum_{i=1}^{p} \phi_i \gamma(q-i) = \theta_q \sigma^2. \tag{26}$$

In the case of classical random variables, Wold decomposition is used to write any stationary process as a sum of an $MA(\infty)$ plus a deterministic part. As the set of set-valued random variables is not a vector space, it is not possible to apply directly Word decomposition theorem in this case. To deal with Wold decomposition for set-valued random variables, we are going to introduce a concept of *pseudovector space*, this is a space where opposites of vectors might not exist.

5 Pseudovector space

Definition 5.1. *A pseudovector space is a triplet $(\mathscr{E}, +, \cdot)$ where $\mathscr{E}$ is a nonempty set, $+ : \mathscr{E} \times \mathscr{E} \to \mathscr{E}$ is an inner composition law, $\cdot : \mathbb{R} \times \mathscr{E} \times \to \mathscr{E}$ is an external composition law such that:*

① $u + (v + w) = (u + v) + w$, $\forall u, v, w \in \mathscr{E}$ *(associativity)*;

② $u + v = v + u$, $\forall u, v \in \mathscr{E}$ *(commutativity)*;

③ $\exists \vec{0} \in \mathscr{E}$;

    (a) $\vec{0} + u = u$, $\forall u \in \mathscr{E}$;

    (b) $\lambda \cdot \vec{0} = \vec{0}$, $\forall \lambda \in \mathbb{R}_+$;

    (c) $0 \cdot u = \vec{0}$, $\forall u \in \mathscr{E}$;

④ $\lambda \cdot (u + v) = \lambda u + \lambda v$, $\forall \lambda \in \mathbb{R}_+$, $\forall u, v \in \mathscr{E}$;

⑤ $(\lambda \beta) \cdot u = \lambda \cdot (\beta \cdot u)$, $\forall \lambda, \beta \in \mathbb{R}$, $\forall u \in \mathscr{E}$;

⑥ $1 \cdot u = u$, $\forall u \in \mathscr{E}$.

Let $(\mathscr{E}, +, \cdot)$ be a pseudovector space. Elements of $\mathscr{E}$ are called pseudovectors. In what follows, $\lambda \cdot u$ will be denoted just λu and $u + (-1)v$ is denoted $u - v$. Observe that in a pseudovector space, there is no notion of inverse for the addition and $u - u \neq \vec{0}$ in general.

Proposition 5.1. *the pseudovector $\vec{0}$ defined at the item ③ of the definition is unique and called the **null pseudovector**.*

Proof. Let $\vec{0}'$ be another pseudovector satisfying the points ③ of Definition 5.1. Then for $u = \vec{0}$ we have $\vec{0} = \vec{0}' + \vec{0}$. As $\vec{0}$ also satisfies item ③ of the definition, for $u = \vec{0}'$ one has $\vec{0}' = \vec{0} + \vec{0}'$. The commutativity property gives us $\vec{0}' = \vec{0} + \vec{0}' = \vec{0}' + \vec{0} = \vec{0}$. $\qquad\square$

Example 6.   • *Any vector space is a pseudovector space.*

- *Let $\mathscr{E}$ and $\mathscr{E}'$ be two pseudovector spaces. Then the Cartesian product $\mathscr{E} \times \mathscr{E}'$ is a pseudovector space, laws being given by*

$$(u, u') + (v, v') = (u + v, u' + v') \qquad and \qquad \lambda(u, u') = (\lambda u, \lambda u').$$

- *It is nothing to check that $K_{kc}(\mathbb{R}^d)$ endowed with the addition (1) and the scalar multiplication (2) is a pseudovector space, $\{0\}$ being the null pseudovector.*

- *$\mathcal{U}[\Omega, K_{kc}(\mathbb{R}^d)]$ inherits from the pseudovector space structure of $K_{kc}(\mathbb{R}^d)$.*

Definition 5.2. *A subset $\mathscr{F}$ of $\mathscr{E}$ is called a **pseudovector subspace** if:*

① $\vec{0} \in \mathscr{F}$;

② $u + v \in \mathscr{F}$, $\forall u, v \in \mathscr{F}$;

③ $\lambda \cdot u \in \mathscr{F}$, $\forall \lambda \in \mathbb{R}$, $\forall u \in \mathscr{F}$.

Example 7. $\mathscr{E} = \{[a, b] \ ; \ a \leq 0 \ and \ b \geq 0\}$ *is a pseudovector subspace of $K_{kc}(\mathbb{R})$.*

Proposition 5.2. *Any pseudovector subspace is a pseudovector space, endowed with the restrictions of the composition laws.*

Proof. With items ①−③ of Definition 5.2 the restriction of the laws of $\mathscr{E}$ on $\mathscr{F}$ are with values in $\mathscr{F}$ and since $\mathscr{F}$ is a subset of $\mathscr{E}$, the axioms ①−⑥ of Definition 5.1 are satisfied. $\qquad\square$

Proposition 5.3. *Let $(\mathscr{F}_i)_{i \in I}$ be a family of pseudovector subspaces of $\mathscr{E}$. Then their intersection $\mathscr{F} = \cap_{i \in I} \mathscr{F}_i$ is a pseudovector subspace of $\mathscr{E}$.*

Proof. As $\mathscr{F}_i$ are pseudovector subspaces of $\mathscr{E}$, $\vec{0} \in \mathscr{F}_i$ for all $i \in I$. Hence $\vec{0} \in \mathscr{F}$. Let $u, v \in \mathscr{F}$ and $\lambda \in \mathbb{R}$ then for every $i \in I$, $u, v \in \mathscr{F}_i$. As F_i are pseudovector subspaces, it follows that $u + v \in \mathscr{F}_i$ and $\lambda u \in \mathscr{F}_i$, $\forall i \in I$. Hence $u + v \in \mathscr{F}$ and $\lambda u \in \mathscr{F}$. Thus $\mathscr{F}$ is a pseudovector subspace. $\qquad\square$

Definition 5.3. *Let $\mathcal{F} = (u_i)_{i \in I}$ be a family of pseudovectors of $\mathcal{E}$. A **Linear Combination** (LC) of $\mathcal{F}$ is a pseudovector $u \in \mathcal{E}$ such that there exists a family $(\lambda_i)_{i \in I}$ of scalars almost all zero such that*

$$u = \sum_{i \in I} \lambda_i u_i.$$

Proposition 5.4. *Let $\mathcal{F} = (u_i)_{i \in I}$ be a nonempty family of elements of $\mathcal{E}$. Then the set $\mathscr{F}$ of all LC of $\mathcal{F}$ is a pseudovector subspace.*

Proof. Let $u \in \mathscr{F}$, then $0 \cdot u = \vec{0} \in \mathscr{F}$. Let $u, v \in \mathscr{F}$ and $\lambda \in \mathbb{R}$. From the definition of $\mathscr{F}$, the pseudovectors u and v are LC of $\mathscr{F}$ and so do for $u + v$ and λu. Hence $u + v \in \mathscr{F}$ and $\lambda u \in \mathscr{F}$. $\square$

Proposition 5.5. *Let $\mathscr{A}$ be a subset of $\mathcal{E}$. The smallest pseudovector subspace of $\mathcal{E}$ containing $\mathscr{A}$ exists. We call it the pseudovector subspace spanned by $\mathscr{A}$, denoted by $Span(\mathscr{A})$.*

Proof. Let $(\mathscr{F}_i)_{i \in I}$ be the family of all pseudovector subspaces of $\mathcal{E}$ containing $\mathscr{A}$. (This family is not empty since $\mathcal{E}$ is a pseudovector subspace of $\mathcal{E}$ containing $\mathscr{A}$.) Then from Proposition 5.3, their intersection $\mathscr{F} = \cap_{i \in I} \mathscr{F}_i$ is a pseudovector subspace of $\mathcal{E}$ containing $\mathscr{A}$, and it is the smallest in the sense of inclusion. $\square$

Proposition 5.6. *Let $\mathcal{F} = (u_i)_{i \in I}$ be a nonempty family of elements of $\mathcal{E}$. The pseudovector subspace $Span(\mathcal{F})$ is the set of all LC of $\mathcal{F}$.*

Proof. Let $\mathscr{F}$ be the set of all LC of $\mathcal{F}$. As $Span(\mathcal{F})$ is a pseudovector subspace containing $\mathcal{F}$, it contains all LC of $\mathcal{F}$, thus $\mathscr{F} \subset Span(\mathcal{F})$. From Proposition 5.4, $\mathscr{F}$ is a pseudovector subspace containing $\mathcal{F}$. Hence $Span(\mathcal{F}) \subset \mathscr{F}$. $\square$

Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be two pseudovector subspaces of $\mathcal{E}$.

Definition 5.4. *We call sum of $\mathcal{E}_1$ and $\mathcal{E}_2$ and we denote $\mathcal{E}_1 + \mathcal{E}_2$, the pseudovector subspace $Span(\mathcal{E}_1 \cup \mathcal{E}_2)$.*

Definition 5.5. *We say that two pseudovector subspaces $\mathcal{E}_1$ and $\mathcal{E}_2$ are complementaries, or that $\mathcal{E}$ is the **direct sum** of $\mathcal{E}_1$ and $\mathcal{E}_2$ and we denote $\mathcal{E} = \mathcal{E}_1 \oplus \mathcal{E}_2$ if the following are satisfied:*

① $\mathcal{E} = \mathcal{E}_1 + \mathcal{E}_2$;

② $\mathcal{E}_1 \cap \mathcal{E}_2 = \{\vec{0}\}$.

Example 8. *For any $d \geq 1$, $K_c(\mathbb{R}^d)$ is a pseudovector space which is not a vector space, and $\{0\}$ is the null pseudovector. For $d = 1$, the subsets $\mathcal{E}_1 = \{[a,b]; \ a, b \in \mathbb{R}\} = K_{kc}(\mathbb{R}^d)$ and $\mathcal{E}_2 = \{]-\infty, b], [a, +\infty[; \ a, b \in \mathbb{R}\} \cup \{\{0\}\}$ are two complementary pseudovector subspaces of $K_c(\mathbb{R})$.*

Definition 5.6. *Let $\mathcal{F} = (u_i)_{i \in I}$ be a family of elements of $\mathcal{E}$. We say that $\mathcal{F}$ is **linearly independent** when for any family $(\lambda_i)_{i \in I}$ of scalars almost all zero, if $\sum_{i \in I} \lambda_i u_i = 0$ then for all $i \in I$, $\lambda_i = 0$. Otherwise we say that $\mathcal{F}$ is **linearly dependent**.*

A linearly independent family cannot contain the null pseudovector.

Definition 5.7. *Saying that a family $\mathcal{F} = (u_i)_{i \in I}$ spans $\mathcal{E}$ means that $\mathcal{E} = Span(\mathcal{F})$.*

Definition 5.8. *A basis for $\mathcal{E}$ is a linearly independent family which spans $\mathcal{E}$.*

Example 9. *$\mathcal{B} = \{[0, 1]\}$ is a basis of $\mathcal{E} = \{[a, b] \ ; \ a \leq 0 \text{ and } b \geq 0\}$.*

It is noteworthy that with operations of a pseudovector space, it is not possible to solve the equations

$$u = v + x \qquad \text{and} \qquad v = u - x \qquad (27)$$

of unknown x. The following definition introduces an operator which for any given couple $(u, v) \in \mathscr{E}^2$, allows to solve at less one of the two equations in (27).

Definition 5.9. *A **difference operator** in a pseudovector space $\mathscr{E}$ is a binary operator $\ominus :$ $\mathscr{E} \times \mathscr{E} \to \mathscr{E}$ such that for all $u, v, w \in \mathscr{E}$:*

- *$w = u \ominus v \iff u = v + w$ or $v = u - w$;*

- *$u \ominus v = \vec{0} \iff u = v$;*

- *$v \ominus u = -(u \ominus v)$.*

Example 10.   • *In a vector space, $u \ominus v = u - v$ is a difference operator.*

- *The generalized Hukuhara difference [Ste10] is giving on $K_{kc}(\mathbb{R})$ by*

$$[u_1, u_2] \ominus [v_1, v_2] = [min(u_i - v_i), max(u_i - v_i)]. \qquad (28)$$

In absence of additional precision, $K_{kc}(\mathbb{R})$ will always be endowed with this difference operator.

We say that a given couple (u, v) has an even signature when $u = v + (u \ominus v)$; otherwise we say that (u, v) has an odd signature (this means that $v = u - (u \ominus v)$). In the case where the both equalities $u = v + (u \ominus v)$ and $v = u - (u \ominus v)$ hold (in a vector space for example) and without other precision, we will say that (u, v) has an even signature and use rather $u = v + (u \ominus v)$.

Definition 5.10. *We define the signature map $\varepsilon : \mathscr{E} \times \mathscr{E} \to \{-1, +1\}$, by:*

$$\varepsilon(u, v) = \begin{cases} +1 & \text{if } (u, v) \text{ has an even signature} \\ -1 & \text{otherwise} \end{cases}. \qquad (29)$$

It is clear that $\varepsilon(v, u) = -\varepsilon(u, v)$.

Notation 1. *We set $\varepsilon_u^v := \varepsilon(u, v)$ and $u \ominus_\varepsilon v := \varepsilon(u, v)(u \ominus v)$.*

It is easy to verify that for all $u, v \in \mathscr{E}$ and $\lambda \in \mathbb{R}$, $\lambda(u \ominus v) = (\lambda u) \ominus (\lambda v)$.

Lemma 5.1. *Let $\mathscr{E}$ be a pseudovector space endowed with a difference operator. Then*

$$(u \ominus_\varepsilon v) + (u' \ominus_\varepsilon v') = \begin{cases} (u + u') \ominus_\varepsilon (v + v') & \text{if } \varepsilon_u^{u'} = \varepsilon_v^{v'} \\ (u + v) \ominus_\varepsilon (u' + v') & \text{otherwise} \end{cases}. \qquad (30)$$

Proof. If $\varepsilon(u, v) = \varepsilon(u', v') = +1$ then $u = v + (u \ominus v)$ and $u' = v' + (u' \ominus v')$. Hence $u + u' = (v + v') + ((u \ominus v) + (u' \ominus v'))$, which implies that $(u \ominus v) + (u' \ominus v') = (u + u') \ominus (v + v')$. Similarly if $\varepsilon(u, v) = \varepsilon(u', v') = -1$ then $v = u - (u \ominus v)$ and $v' = u' - (u' \ominus v')$. Hence $v + v' = (u + u') - ((u \ominus v) + (u' \ominus v'))$, which implies that $(u \ominus v) + (u' \ominus v') = (v + v') \ominus (u + u')$. Finally if $\varepsilon(u, v) = +1$ and $\varepsilon(u', v') = -1$ then $u = v + (u \ominus v)$ and $v' = u' - (u' \ominus v')$. Hence $u + v' = (v + u') + ((u \ominus v) - (u' \ominus v'))$, which implies that $(u \ominus v) - (u' \ominus v') = (u + v') \ominus (v + u')$. $\square$

Proposition 5.7. *Let $\mathscr{E}$ be a pseudovector space endowed with a difference operator. Then*

$$(u \ominus_\varepsilon v) + (u \ominus_\varepsilon v') = \begin{cases} (u + u) \ominus_\varepsilon (v + v') & \text{if } \varepsilon_u^v = \varepsilon_u^{v'} \\ v \ominus_\varepsilon v' & \text{otherwise} \end{cases}. \qquad (31)$$

Proof. The first line is deduced from the first line of (30) by replacing u' by u. For the second line, it is nothing to check that $v\ominus_\varepsilon v' = (u\ominus_\varepsilon v) + (u\ominus_\varepsilon v')$ when $\varepsilon_u^v \neq \varepsilon_u^{v'}$. $\square$

We need the following Lemma to prove the next Proposition.

Lemma 5.2. *Let $\mathscr{E}$ be a pseudovector space endowed with a difference operator. Then*

$$(u \ominus v) + w = \begin{cases} (u + w) \ominus v & \text{if } \varepsilon(u, v) = +1 \\ u \ominus (v - w) & \text{otherwise} \end{cases}. \tag{32}$$

Proof. If $\varepsilon(u, v) = +1$ then $u = v + (u \ominus v)$. Hence $u + w = v + ((u \ominus v) + w)$, which implies that $(u+w)\ominus v = (u\ominus v)+w$. Finally if $\varepsilon(u,v) = -1$ then $v = u-(u\ominus v)$. Hence $v-w = u-((u\ominus v)+w)$, which implies that $u \ominus (v - w) = (u \ominus v) + w$. $\square$

Proposition 5.8. *Let $\mathscr{E}$ be a pseudovector space endowed with a difference operator. Then*

$$(u\ominus_\varepsilon v)\ominus_\varepsilon(u\ominus_\varepsilon v') = \begin{cases} v\ominus_\varepsilon v' & \text{if } \varepsilon_u^v = \varepsilon_u^{v'} \\ (u + u)\ominus_\varepsilon(v + v') & \text{otherwise} \end{cases}. \tag{33}$$

Proof. If $\varepsilon(u, v) = \varepsilon(u, v') = -1$ then $v = u - (u \ominus v)$ and $v' = u - (u \ominus v')$. Hence $v \ominus v' = (u + (v \ominus u)) \ominus (u + (v' \ominus u))$. Using (30) one finds $v \ominus v' = (u \ominus u) + ((v \ominus u) \ominus (v' \ominus u)) = (v \ominus u) \ominus (v' \ominus u)$. If $\varepsilon(u, v) = \varepsilon(u, v') = +1$ then $u = v + (u \ominus v)$ and $u = v' + (u \ominus v')$. If $\varepsilon_v^{v'} = -1$ then we write $u \ominus v = (v' + (u \ominus v')) \ominus v$ and use (32) to turn the latter into $u \ominus v = (v' \ominus v) + (u \ominus v')$, which implies that $(v \ominus_\varepsilon v') = (u \ominus_\varepsilon v) \ominus_\varepsilon (u \ominus_\varepsilon v')$. If rather $\varepsilon_v^{v'} = +1$ then we write $v'\ominus u = v'\ominus(v-(v\ominus u))$ and use (32) to turn the latter into $v'\ominus u = (v'\ominus v)+(v\ominus u)$, which implies that $(v \ominus_\varepsilon v') = (u \ominus_\varepsilon v) \ominus_\varepsilon (u \ominus_\varepsilon v')$. Finally if $\varepsilon(u, v) = +1$ and $\varepsilon(u, v') = -1$ then $u = v + (u \ominus v)$ and $v' = u - (u \ominus v')$. Hence

$$v + v' + (u \ominus v) = u + u - (u \ominus v'). \tag{34}$$

Now, if $\varepsilon(u + u, v + v') = -1$ then we turn (34) into $u \ominus v' = (u + u) \ominus (v + v' + (u \ominus v))$. The latter together with (32) gives us $u \ominus v' = ((u + u) \ominus (v + v')) - (u \ominus v)$, which implies $(u \ominus v) \ominus (v' \ominus u) = (u + u) \ominus (v + v')$. On the contrary if $\varepsilon(u + u, v + v') = +1$ then we turn (34) into $u \ominus v = ((u + u) - (u \ominus v')) \ominus (v + v')$. The latter together with (32) gives us $u\ominus v = -((u+u)\ominus(v+v'))-(u\ominus v')$, which implies $(u\ominus v)\ominus(v'\ominus u) = -((u+u)\ominus(v+v'))$. $\square$

From equations (31) and (33) one deduces that

$$u\ominus_\varepsilon v = \begin{cases} (w\ominus_\varepsilon u)\ominus_\varepsilon(w\ominus_\varepsilon v) & \text{if } \varepsilon_w^u = \varepsilon_w^v \\ (w\ominus_\varepsilon u) + (w\ominus_\varepsilon v) & \text{otherwise} \end{cases}. \tag{35}$$

Definition 5.11. *We say that a pseudovector space $\mathscr{E}$ has compatibles laws when for all $u \in \mathscr{E}$ and α, β two scalar of the same sign, one has $\alpha u + \beta u = (\alpha + \beta)u$.*

Proposition 5.9. *The pseudovector subspace $K_{kc}(\mathbb{R})$ has compatible laws.*

Proof. Let $u = [u_1, u_2]$ and $\alpha, \beta \in \mathbb{R}$. If $\alpha, \beta \geq 0$ then,

$$\alpha u + \beta u = [\alpha u_1, \alpha u_2] + [\beta u_1, \beta u_2] = [(\alpha + \beta)u_1, (\alpha + \beta)u_2] = (\alpha + \beta)u.$$

On the contrary if $\alpha, \beta \leq 0$ then,

$$\alpha u + \beta u = [\alpha u_2, \alpha u_1] + [\beta u_2, \beta u_1] = [(\alpha + \beta)u_2, (\alpha + \beta)u_1] = (\alpha + \beta)u.$$

$\square$

5.1 Hilbert pseudovector space

Definition 5.12. *A **pseudometric** on a nonempty set E is a map $d : E \times E \to \mathbb{R}_+$ such that for every $x, y, z \in E$,*

① *$d(x, x) = 0$;*

② *$d(x, y) = d(y, x)$;*

③ *$d(x, z) \leq d(x, y) + d(y, z)$.*

The couple (E, d) is then called a pseudometric space. It is well-known that a pseudometric space is not distinguishable. This implies that a sequence $(x_n)_n$ may converge to many different points.

Let $\mathscr{E}$ and $\mathscr{E}'$ be two pseudovector spaces.

Definition 5.13. *A **linear map** from $\mathscr{E}$ to $\mathscr{E}'$ is a map $f : \mathscr{E} \to \mathscr{E}'$ such that*

① *$f(u + v) = f(u) + f(v)$, $\forall u, v \in \mathscr{E}$;*

② *$f(\lambda u) = \lambda f(u)$, $\forall \lambda \in \mathbb{R}$, $\forall u \in \mathscr{E}$.*

As a consequence of the definition, one has $f(\vec{0}) = \vec{0}'$.

Lemma 5.3. *Let $f : \mathscr{E} \to \mathscr{E}'$ be a linear map between pseudovector spaces admitting difference operators. Then*

$$f(u \ominus v) = f(u) \ominus f(v), \qquad \forall u, v \in \mathscr{E}. \tag{36}$$

Proof. If $\varepsilon_u^v = +1$ then $u = v + (u \ominus v)$ which implies that $f(u) = f(v + (u \ominus v))$. As f is linear, the latter equality leads to $f(u) = f(v) + f(u \ominus v)$ and hence $f(u \ominus v) = f(u) \ominus f(v)$. Similarly if $\varepsilon_u^v = -1$ then $v = u - (u \ominus v)$ and one shows that in this case too $f(u \ominus v) = f(u) \ominus f(v)$. $\square$

Definition 5.14. *The kernel and the image of f are defined respectively by*

$$Ker\, f = \{u \in \mathscr{E};\ f(u) = \vec{0}'\} \qquad and \qquad Im\, f = \{f(u);\ u \in \mathscr{E}\}. \tag{37}$$

Proposition 5.10. *The subset $Ker\, f$ and $Im\, f$ are pseudovector subspaces of $\mathscr{E}$ and $\mathscr{E}'$ respectively.*

A linear map is above all a map between sets and notions such that injectivity, surjectivity and bijectivity are applicable to them. However, linear maps between pseudovector spaces don't have the same properties as linear maps between vector spaces. It is obvious that if a linear map $f : \mathscr{E} \to \mathscr{E}$ is injective then $Ker\, f = \{\vec{0}\}$, but the converse is not true in general. However when $\mathscr{E}$ admits a difference operator, we may not expect a linear map to be injective since, $f(u - u) = f(u \ominus u) = f(\vec{0}) = \vec{0}$ but $u - u$ is not necessarily $\vec{0}$. Hence, the kernel of f contains $\{u - u;\ u \in \mathscr{E}\}$.

Definition 5.15. *A **linear form** on $\mathscr{E}$ is a linear map $\omega : \mathscr{E} \to \mathbb{R}$. We denote by $\mathscr{E}^\star$ the set of all linear form on $\mathscr{E}$.*

*A **bilinear form** on $\mathscr{E}$ is a map $\langle \cdot, \cdot \rangle : \mathscr{E} \times \mathscr{E} \to \mathbb{R}$, which is linear with respect to each of the two entries. In the same way, we define a multi-linear form on $\mathscr{E}$.*

The following comes directly from the definition of the difference operator.

Lemma 5.4. *Let $\mathscr{E}$ be a pseudovector space admitting a difference operator. For any linear form $f : \mathscr{E} \to \mathbb{R}$ one has $f(u \ominus v) = f(u) - f(v) = f(u - v)$ for all $u, v \in \mathscr{E}$.*

Example 11. *The kernel of the linear form $f : K_{kc}(\mathbb{R}) \to \mathbb{R}$, $[a, b] \mapsto a + b$ is $Ker\, f = \{[-a, a], a \in \mathbb{R}_+\}$.*

Definition 5.16. *A pseudo-norm on $\mathscr{E}$ is a map $\|\cdot\| : \mathscr{E} \to \mathbb{R}_+$ such that $\forall u, v \in \mathscr{E}$, $\forall \lambda \in \mathbb{R}$,*

    ① $\|\vec{0}\| = 0$;

    ② $\|\lambda u\| = |\lambda|\|u\|$ *(homogeneity)*;

    ③ $\|u + v\| \leq \|u\| + \|v\|$ *(triangular inequality)*.

*We call $(\mathscr{E}, \|\cdot\|)$ a **pseudovector norm space**.*

From now on, $\mathscr{E}$ is a pseudovector space admitting a difference operator. Let $\|\cdot\| : \mathscr{E} \to \mathbb{R}_+$ be a pseudo-norm on $\mathscr{E}$ such that

$$\|u \ominus v\| = \|u - v\|, \qquad \forall u, v \in \mathscr{E}. \tag{38}$$

We define on $\mathscr{E}$ the following relation

$$u \Re v \iff \|u \ominus v\| = 0. \tag{39}$$

Using (35) one shows that $\Re$ defines an equivalence relation on $\mathscr{E}$. We denote $cl(u)$ the equivalence class of an element $u \in \mathscr{E}$ and $cls(\mathscr{E})$ the set of all equivalence classes. Observe that $\|u\| = 0 \iff u \in cl(\vec{0})$ and $\|u - v\| = 0 \iff cl(u) = cl(v)$.

Definition 5.17. *A **scalar product** on $\mathscr{E}$ is a positive definite symmetric bilinear form $\langle , \rangle : \mathscr{E} \times \mathscr{E} \to \mathbb{R}$, that is a bilinear form satisfying:*

    ① $\langle u, v \rangle = \langle v, u \rangle$ *(symmetric)*;

    ② $\langle u, u \rangle \geq 0$ *(positive)*;

    ③ *if $\langle u, u \rangle = 0$ then $\langle u, v \rangle = 0$ for every $v \in \mathscr{E}$ (definite)*.

Let $\langle \cdot, \cdot \rangle$ be a scalar product on $\mathscr{E}$.

Proposition 5.11. *Let $\|\cdot\| : \mathscr{E} \to \mathbb{R}_+$ defined by*

$$\|u\| = \sqrt{\langle u, u \rangle}. \tag{40}$$

Then $\|\cdot\|$ is a pseudo-norm, called pseudo-norm associated to $\langle \cdot, \cdot \rangle$.

Proof. One has $\|\vec{0}\| = \sqrt{\langle \vec{0}, \vec{0} \rangle} = 0$ as $\langle \cdot, \cdot \rangle$ is bilinear and from Lemma 5.4, one deduces that.

$$\langle u \ominus v, w \rangle = \langle u, w \rangle - \langle v, w \rangle \quad \text{and} \quad \|u \ominus v\| = \|u - v\|. \tag{41}$$

Let $u, v \in \mathscr{E}$ and $\lambda \in \mathbb{R}$. $\|\lambda u\| = \sqrt{\langle \lambda u, \lambda u \rangle} = |\lambda|\|u\|$ and

$$\|u + v\|^2 = \langle u + v, u + v \rangle = \|u\|^2 + \|v\|^2 + 2\langle u, v \rangle.$$

It remains to show the following Cauchy-Schwarz inequality:

$$|\langle u, v \rangle| \leq \|u\|\|v\|. \tag{42}$$

If $\langle u, u \rangle = 0$ then (42) is satisfied. If $\langle u, u \rangle > 0$,

$$0 \leq \langle \frac{\langle u, v \rangle}{\langle u, u \rangle} u - v, \frac{\langle u, v \rangle}{\langle u, u \rangle} u - v \rangle = \langle v, v \rangle - \frac{\langle u, v \rangle^2}{\langle u, u \rangle}.$$

$\square$

Remark 5.1. *Since $\langle u-u, v\rangle = \langle u \ominus u, v\rangle = 0$, we may expect that for $u \in \mathscr{E}$, ($\langle u, v\rangle = 0 \; \forall v \in \mathscr{E}$) $\implies u = \vec{0}$. This is why the classical definite property is replaced by point ③ in Definition 5.17.*

It is nothing to check that the parallelogram identity

$$\|u + v\|^2 + \|u - v\|^2 = 2\|u\|^2 + 2\|v\|^2 \tag{43}$$

holds for all $u, v \in \mathscr{E}$.

Let $u, v \in \mathscr{E}$, $\lambda \in \mathbb{R}$, $u' \in cl(u)$ and $v' \in cl(v)$. One has $\|\lambda u - \lambda u'\| = |\lambda| \|u - u'\| = 0$ and

$$\|u + v - u' - v'\| = \|u - u' + v - v'\| \leq \|u - u'\| + \|v - v'\| = 0.$$

Hence $\lambda u' \in cl(\lambda u)$ and $u' + v' \in cl(u + v)$. So laws of $\mathscr{E}$ are compatible with the equivalence relation and then $cls(\mathscr{E})$ is also a pseudovector space with operations

$$\lambda cl(u) = cl(\lambda u) \quad \text{and} \quad cl(u) + cl(v) = cl(u + v).$$

The difference operator $\ominus$ is compatible with the equivalence relation and $cl(u) \ominus cl(v) = cl(u) - cl(v)$ since $\|u \ominus v\| = \|u - v\|$.

Definition 5.18. *A **pre-Hilbert pseudovector space** is a pseudovector space $(\mathscr{E}, \langle \cdot, \cdot \rangle)$ endowed with a scalar product $\langle \cdot, \cdot \rangle$.*

Observe that in a pre-Hilbert space for $u \in \mathscr{E}$,

$$(\langle u, v\rangle = 0 \; \forall v \in \mathscr{E}) \iff u \in cl(\vec{0}). \tag{44}$$

Let $(\mathscr{E}, \langle \cdot, \cdot \rangle)$ be a pre-Hilbert pseudovector space admitting a difference operator with compatible laws. As $\|u \ominus v\| = \|u + (-v)\| \leq \|u\| + \|v\|$, the following is obvious.

Proposition 5.12. *The formula*

$$d(u, v) := \|u \ominus v\| \tag{45}$$

defines a pseudometric d on $\mathscr{E}$.

Observe that when the pseudo-norm is a norm, d becomes a metric. This happens for example when we consider d on $cls(\mathscr{E})$ with the same formula (45). We endow $\mathscr{E}$ with the topology induced by the derived pseudometric d. This is the topology induced by the open balls

$$B(u, r) = \{v \in \mathscr{E}; \; d(u, v) < r\}, \quad u \in \mathscr{E}, r > 0. \tag{46}$$

Definition 5.19. *A **Hilbert pseudovector space** is a pre-Hilbert pseudovector space where any Cauchy sequence converges.*

We assume now that $(\mathscr{E}, \langle, \rangle)$ is a Hilbert pseudovector space.

Proposition 5.13. *Let $\mathscr{E}$ be a Hilbert pseudovector space endowed with a difference operator $\ominus$ and $(u_n)_n$ a sequence that converges to u in $\mathscr{E}$. If $(u_n)_n$ also converges to v then $v \in cl(u)$.*

Proof. As $(u_n)_n$ converges to u and v, for every $\varepsilon > 0$, there exists a large N such that $n > N \implies d(u_n, u) < \varepsilon/2$ and $d(u_n, v) < \varepsilon/2$. This implies that $d(u, v) \leq d(u, u_n) + d(u_n, v) < \varepsilon$ for any $\varepsilon > 0$. Hence $d(u, v) = 0$ which implies that $cl(u) = cl(v)$. $\qquad\square$

Proposition 5.14. *Let $\mathscr{F}$ be a pseudovector subspace of E. Then $\mathscr{F}^\perp$ is a pseudovector subspace of $\mathscr{E}$.*

Theorem 2 (Riesz decomposition). *Let $\mathscr{F} \subset \mathscr{E}$ be a closed pseudovector subspace of a Hilbert pseudovector space $\mathscr{E}$ with compatibles laws and endowed with a difference operator $\ominus$. Every $u \in \mathscr{E}$ can be expressed as*

$$u = v + w \text{ or } u - w = v, \qquad \text{with } v \in \mathscr{F}, w \in \mathscr{F}^\perp. \tag{47}$$

Proof. We assume that $\mathscr{F} \neq \mathscr{E}$. If $u \in \mathscr{F}$ then take $v = u$ and $w = \vec{0}$. We assume now that $u \notin \mathscr{F}$. As $\mathscr{F}$ is closed, $\delta = d(u, \mathscr{F}) := \inf\{d(u, v); \ v \in \mathscr{F}\} > 0$. For a positive integer n, let $v_n \in \mathscr{F}$ such that $d(u, v_n)^2 < \delta^2 + 1/n$. Then (v_n) is a Cauchy sequence. In fact, by applying the parallelogram identity (43) to $u \ominus_\varepsilon v_p$ and $u \ominus_\varepsilon v_q$ one shows that

$$\|(u \ominus_\varepsilon v_p) + (u \ominus_\varepsilon v_q)\|^2 + \|(u \ominus_\varepsilon v_p) \ominus_\varepsilon (u \ominus_\varepsilon v_q)\|^2 = 2d(u, v_p)^2 + 2d(u, v_q)^2. \tag{48}$$

From (31) and (33) one shows that

$$\|(u \ominus_\varepsilon v_p) + (u \ominus_\varepsilon v_q)\|^2 + \|(u \ominus_\varepsilon v_p) \ominus_\varepsilon (u \ominus_\varepsilon v_q)\|^2 = \|v_p \ominus_\varepsilon v_q\|^2 + \|(u + u) \ominus_\varepsilon (v_p + v_q)\|^2.$$

Using the latter and the fact that the laws are compatibles, equation (48) leads to

$$d(v_p, v_q) = 2d(u, v_p)^2 + 2d(u, v_q)^2 - 4d(u, \frac{v_p + v_q}{2})^2 \leq 2\left(\frac{1}{p} + \frac{1}{q}\right). \tag{49}$$

As $\mathscr{E}$ is complete, (v_n) converge let's say to $v \in \mathscr{E}$. In addition as $\mathscr{F}$ is closed and d is continuous, it follows that $v \in \mathscr{F}$ and $d(u, v) = \delta$. By setting $w = u \ominus v$ one has $u = v + w$ or $u - w = v$. It remains to show that $w \in \mathscr{F}^\perp$, ie $\langle w, v' \rangle = 0, \forall v' \in \mathscr{F}$. Let $v' \in \mathscr{F}$ and $\lambda \in \mathbb{R}$. One has

$$\delta^2 \leq \|u \ominus (v + \lambda v')\| = d(u, v)^2 - 2\lambda \langle w, v' \rangle + \lambda^2 \|v'\|^2.$$

Hence $0 \leq -2\lambda \langle w, v' \rangle + \lambda^2 \|v'\|^2$ for every $\lambda \in \mathbb{R}$, which implies that $\langle w, v' \rangle = 0$. $\qquad \square$

In the above Theorem, the pseudovector v is called (orthogonal) projection of u on $\mathscr{F}$ and denoted $P_{\mathscr{F}} u$, and w residual part. Therefore, any pseudovector $u \in \mathscr{E}$ can be expressed as $u = P_{\mathscr{F}} u + (u \ominus P_{\mathscr{F}} u)$ or $u - (u \ominus P_{\mathscr{F}} u) = P_{\mathscr{F}} u$. Let $(x_n)_{n=0}^\infty$ be a sequence in $\mathscr{E}$. We set $\mathscr{F}_n = Span(x_0, \ldots, x_n)$ and $\mathscr{F}_\infty = \overline{Span(\{x_k\}_{k=0}^\infty)}$. These are closed pseudovector subspaces and hence Hilbert pseudospaces.

Lemma 5.5. *Let $u \in \mathscr{F}_\infty$ and u_n his projection on $\mathscr{F}_n$. Then $\lim_{n \to \infty} \|u_n \ominus u\| = 0$.*

Proof. If $u \in Span(\{x_k\}_{k=0}^\infty)$ then there exists $n_0 \in \mathbb{N}$ such that $u \in \mathscr{F}_{n_0}$ and $u_n \in \mathscr{F}_n$ for every $n \geq n_0$. Hence $\lim_{n \to \infty} \|u_n \ominus u\| = 0$ in this case. Let's assume now that $u \notin Span(\{x_k\}_{k=1}^\infty)$. Then there exists a sequence $(v_n)_n$ with $v_n \in \mathscr{F}_n$ such that $\lim_{n \to \infty} \|v_n \ominus u\| = 0$. The result follows from the fact that for any n, $\|u_n \ominus u\| \leq \||v_n \ominus u\||$. $\qquad \square$

Lemma 5.6. *Let $\hat{u} \in \mathscr{E}$, u the projection of $\hat{u}$ on $\mathscr{F}_\infty$ with ε the residual and u_n the projection of $\hat{u}$ on $\mathscr{F}_n$ with ε_n the residual. Then $\lim_{n \to \infty} \|\varepsilon_n\| = \|\varepsilon\|$ and $\lim_{n \to \infty} \|u_n \ominus u\| = 0$.*

Proof. If $u \in Span(\{x_k\}_{k=0}^\infty)$ then there exists $n_0 \in \mathbb{N}$ such that $u \in \mathscr{F}_n$ for every $n \geq n_0$. Hence $u_n = u$ and $\varepsilon_n = \varepsilon$ for every $n \geq n_0$, which implies that $\lim_{n \to \infty} \|\varepsilon_n\| = \|\varepsilon\|$ and $\lim_{n \to \infty} \|u_n \ominus u\| = 0$ in this case. Let's assume now that $u \notin Span(\{x_k\}_{k=0}^\infty)$. Then there exists a sequence $(v_n)_n$ with $v_n \in \mathscr{F}_n$ such that $\lim_{n \to \infty} \|v_n \ominus u\| = 0$. Let $\sigma = \|\varepsilon\| = \|\hat{u} - u\|$ and $\sigma_n = \|\varepsilon_n\| = \|\hat{u} - u_n\|$. As $\mathscr{F}_n \subset \mathscr{F}_\infty$, $\sigma \leq \sigma_n$. Since

$$\sigma_n^2 \leq \|\hat{u} - v_n\|^2 = \|\hat{u} - u + u - v_n\|^2 = \sigma^2 + \|u - v_n\|^2,$$

it follows that $\lim_{n \to \infty} \sigma_n = \sigma$. The result follows from the fact that

$$\|u - u_n\|^2 = \|\hat{u} - u_n - \varepsilon\|^2 = \|\hat{u} - u_n\|^2 - \|\varepsilon\|^2 = \sigma_n^2 - \sigma^2.$$

$\qquad \square$

As $\varepsilon = \hat{u} \ominus u$ and $\varepsilon_n = \hat{u} \ominus u_n$, from the above theorem, one also has $\lim_{n \to \infty} \|\varepsilon_n \ominus \varepsilon\| = 0$.

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Theorem 3. *Let $(x_n)_{n=0}^{\infty}$ be an orthonormal sequence in a Hilbert pseudovector space $\mathscr{E}$ with compatible laws and endowed with a difference operator $\ominus$. Any vector $u \in \mathscr{F}_{\infty} = \overline{Span(\{x_k\}_{k=0}^{\infty})}$ can be expressed as*

$$u = \sum_{k=0}^{\infty} \theta_k x_k \tag{50}$$

in the sense that $\lim\limits_{n \to \infty} \left\| u \ominus \sum\limits_{k=0}^{n} \theta_k x_k \right\| = 0$, with $\theta_k = \langle u, x_k \rangle$ and $\sum\limits_{k=0}^{\infty} \theta^2 < \infty$.

Proof. Let u_n be the orthogonal projection of u on $\mathscr{F}_n = Span(x_0, \ldots, x_n)$ and $v_n \in \mathscr{F}_n^{\perp}$ the residual part, thus $u = u_n + v_n$ or $u - v_n = u_n$. As $u_n \in \mathscr{F}_n$ one has $u_n = \sum\limits_{k=0}^{n} \theta_k x_k$ with $\theta_k = \langle u_n, x_k \rangle = \langle u, x_k \rangle$. By a direct calculation,

$$0 \leq \|v_n\|^2 = \|u\|^2 + \sum_{k=0}^{n} \theta_k^2 - 2\sum_{k=0}^{n} \theta_k \langle x_k, u \rangle = \|u\|^2 - \sum_{k=0}^{n} \theta_k^2.$$

It follows that $\sum\limits_{k=0}^{n} \theta_k^2 \leq \|u\|^2$, which implies that $\sum\limits_{k=0}^{\infty} \theta_k^2 < \infty$. Finally by Lemma 5.5,

$\lim\limits_{n \to \infty} \left\| u \ominus \sum\limits_{k=0}^{n} \theta_k x_k \right\| = 0.$ □

We say that a sequence $(x_n)_{n=0}^{\infty}$ is regular when for any n, $x_n \notin Span(\{x_k\}_{k=n+1}^{\infty})$. It is clear that, any linearly independent sequence is regular.

Theorem 4. *Let $(x_n)_{n=0}^{\infty}$ be a regular sequence in a Hilbert pseudovector space $\mathscr{E}$ with compatible laws and endowed with a difference operator $\ominus$. Any vector $u \in \mathscr{F}_{\infty} = \overline{Span(\{x_k\}_{k=0}^{\infty})}$ can be expressed as*

$$u = \sum_{k=0}^{\infty} \theta_k e_k + w \qquad or \qquad u - w = \sum_{k=0}^{\infty} \theta_k e_k \tag{51}$$

in the sense that $\lim\limits_{n \to \infty} \left\| u - w - \sum\limits_{k=0}^{n} \theta_k e_k \right\| = 0$, where $\{e_k\}_{k=0}^{\infty}$ is an orthonormal sequence in $\mathscr{F}_{\infty}$, $\theta_k = \langle u, e_k \rangle$, $\sum\limits_{k=0}^{\infty} \theta_k^2 < \infty$, $w \in U_{\infty}^{\perp}$ with $U_{\infty} = \overline{Span(\{e_k\}_{k=0}^{\infty})}$.

Proof. Let $S_n = \overline{Span(\{x_k\}_{k=n}^{\infty})} \subset \mathscr{F}_{\infty}$. For $k \geq 0$, let u_k be the orthogonal projection in $\mathscr{F}_{\infty}$ of x_k on S_{k+1} and v_k the residual, thus $x_k = u_k + v_k$ or $x_k - v_k = u_k$. For $j > k$, $v_k \in S_{k+1}^{\perp}$ and $x_j \in S_{k+1}$, hence $0 = \langle v_k, x_j \rangle = \langle v_k, u_j \rangle + \langle v_k, v_j \rangle = \langle v_k, v_j \rangle$ because $u_j \in S_{j+1} \subset S_{k+1}$. Thus $\{v_k\}_k$ is an orthogonal family. Since the sequence $(x_n)_{n=0}^{\infty}$ is regular, $v_k \neq \vec{0}$ and by setting $e_k = v_k/\|v_k\|$, we obtain that $(e_n)_{n=0}^{\infty}$ is an orthonormal sequence. It is nothing to check that $U_{\infty} = \overline{Span(\{e_k\}_{k=0}^{\infty})} = \overline{Span(\{v_k\}_{k=0}^{\infty})} \subset \mathscr{F}_{\infty}$. Let v be the orthogonal projection in $\mathscr{F}_{\infty}$ of u on U_{∞} and w the residual, thus $u = v + w$ or $u - w = v$ with $w \in U_{\infty}^{\perp} \subset \mathscr{F}_{\infty}$. As $v \in U_{\infty}$, by the Theorem 3 one gets

$$u = \sum_{k=0}^{\infty} \theta_k e_k + w \qquad or \qquad u - w = \sum_{k=0}^{\infty} \theta_k e_k \tag{52}$$

with $\theta_k = \langle u, e_k \rangle$ and $\sum\limits_{k=0}^{\infty} \theta_k^2 < \infty$. □

6 Wold decomposition for interval-valued stationary processes

As we have mentioned, $K_{kc}(\mathbb{R})$ is a pseudovector space with compatible laws and admitting a difference operator $\ominus$. $\mathcal{U}[\Omega, K_{kc}(\mathbb{R})]$ inherit from this structure, the difference operator being given by

$$(F_1 \ominus F_2)(\omega) = F_1(\omega) \ominus F_2(\omega), \qquad \forall \omega \in \Omega. \tag{53}$$

The collection $\mathscr{E}$ of all random intervals F such that $E[F] = \vec{0}$ and $Var(F) < \infty$ is a pseudovector subspace of $\mathcal{U}[\Omega, K_{kc}(\mathbb{R})]$. In fact,

$$E[\lambda F] = \lambda E[F] = \vec{0} \quad \text{and} \quad Var(\lambda F) = \lambda^2 Var(F) < \infty, \qquad \forall F \in \mathscr{E}, \forall \lambda \in \mathbb{R},$$

and as $(x + y)^2 \leq 2x^2 + 2y^2$,

$$E[F_1 + F_2] = E[F_1] + E[F_2] = \vec{0}, \ Var(F_1 + F_2) \leq 2Var(F_1) + 2Var(F_2) < \infty, \ \forall F_1, F_2 \in \mathscr{E}.$$

We define on $\mathscr{E}$ the positive symmetric bilinear form

$$\langle F_1, F_2 \rangle = Cov(F_1, F_2). \tag{54}$$

Let F in $\mathscr{E}$ be such that $Cov(F, F) = 0$. Hence, $F = E[F] = \vec{0}$ (the random interval which sends every $\omega \in \Omega$ on $\vec{0} = [0, 0] = \{0\}$), almost everywhere (see Example 3) and so, $Cov(F, G) = 0$ for any $G \in \mathscr{E}$ (look formula (10)). For this positive symmetric bilinear form, two random intervals F_1, F_2 are equivalent with respect to the equivalent relation (39) when they are equal almost everywhere. Hence formula (54) defines a scalar product on the quotient set $L^2[\Omega, K_{kc}(\mathbb{R})]$. We will continue to denote any class in $L^2[\Omega, K_{kc}(\mathbb{R})]$ by a representative $F \in \mathscr{E}$. $L^2[\Omega, K_{kc}(\mathbb{R})]$ inherits from the structure of pseudovector space of $\mathscr{E}$ and the formula (54) defines a scalar product on it. In $L^2[\Omega, K_{kc}(\mathbb{R})]$, one says that a sequence (F_n) converges to $F \in L^2[\Omega, K_{kc}(\mathbb{R})]$ and we denote $F_n \to F$ when

$$\lim_{n \to \infty} Var(F_n \ominus F) = 0.$$

Proposition 6.1. *A sequence $(F_n = [a_n, b_n])$ converges to $F = [a, b]$ in $L^2[\Omega, K_{kc}(\mathbb{R})]$ if and only if (a_n) converges to a and (b_n) converges to b in $L^2[\Omega, \mathscr{A}, P]$.*

Proof. From [YL05], one has $Var(F_n \ominus F) = Var(a_n - a) + Var(b_n - b)$, which implies that $Var(F_n \ominus F) \to 0$ if and only if $Var(a_n - a) \to 0$ and $Var(b_n - b) \to 0$. $\qquad \square$

As direct consequence of the above Proposition and the fact that $L^2[\Omega, \mathscr{A}, P]$ is a Hilbert space [BD13], one has that $L^2[\Omega, K_{kc}(\mathbb{R})]$ is a Hilbert pseudovector space.

Let $(X_t)_{t \in \mathbb{Z}}$ be a zero-mean covariance interval-valued stationary process. The sets $S_t = \overline{Span(\{X_k\}_{k=-\infty}^t)}$ and $S_{-\infty} = \bigcap_{t=-\infty}^{\infty} S_t$ are Hilbert pseudovector subspaces of $L^2[\Omega, K_{kc}(\mathbb{R})]$. For any $j \geq 0$, the projection $P_{S_{t-j}} Xt$ of X_t on S_{t-j} is called the prediction of X_t on S_{t-j}. We shall say that an interval-valued process $(X_t)_{t \in \mathbb{Z}}$ is **deterministic** if for any $t \in \mathbb{Z}$, $X_t \in S_{t-1}$. $X_t \ominus P_{S_{t-1}} X_t$ is called the error in the projection of X_t on S_{t-1} and when $P_{S_{t-1}} X_t = X_t$ and one says that $(X_t)_{t \in \mathbb{Z}}$ is (perfectly) predictable. We shall say that an interval-valued process $(X_t)_{t \in \mathbb{Z}}$ is **deterministic from the past of** another one $(Z_t)_{t \in \mathbb{Z}}$ if for any $t \in \mathbb{Z}$, $X_t \in \overline{Span(\{Z_k\}_{k=-\infty}^{t-1})}$.

Theorem 5. *Let $(X_t)_{t \in \mathbb{Z}}$ be a non-deterministic covariance interval-valued stationary time series process with expectation $\{0\}$ and auto-covariance function $(\gamma(k))$. Then X_t can be expressed as*

$$X_t = \sum_{k=0}^{\infty} \alpha_k \varepsilon_{t-k} + W_t \quad a.s \qquad or \qquad X_t - W_t = \sum_{k=0}^{\infty} \alpha_k \varepsilon_{t-k} \quad a.s \tag{55}$$

where:

(i) $\alpha_k = \frac{1}{\sigma^2} Cov(X_t, \varepsilon_{t-k})$, $\quad \alpha_0 = 1$ *and* $\sum_{k=0}^{\infty} \alpha_k^2 < \infty$;

(ii) $\{\varepsilon_t\} \rightsquigarrow WN(\{0\}, \sigma^2)$, *with* $\sigma^2 = Var(X_t - P_{S_{t-1}} X_t)$;

(iii) $Cov(W_t, \varepsilon_s) = 0$ *for all* $t, s \in \mathbb{Z}$;

(iv) $(W_t)_{t \in \mathbb{Z}}$ *is zero-mean, stationary and deterministic from the past of* $(X_t)_{t \in \mathbb{Z}}$.

Proof. For any $t \in \mathbb{Z}$, application of Theorem 4 to the sequence $(X_{t-k})_{k=0}^{\infty}$ gives that $u = X_t$ can be expressed as

$$X_t = \sum_{k=0}^{\infty} \theta_k e_{t-k} + W_t \quad \text{a.s} \qquad \text{or} \qquad X_t - W_t = \sum_{k=0}^{\infty} \theta_k e_{t-k} \quad \text{a.s} \qquad (56)$$

where $\{e_{t-k}\}_{k=0}^{\infty}$ is an uncorrelated process with $Cov(e_i, e_j) = \delta_{ij}$, $\theta_k = Cov(X_t, e_{t-k})$, $\sum_{k=1}^{\infty} \theta_k^2 < \infty$, $W_t \in U_t^{\perp}$ with $U_t = \overline{Span(\{e_k\}_{k=-\infty}^{t})} \subset S_t$. Since the process $(X_t)_{t \in \mathbb{Z}}$ is non-deterministic, the residual $\varepsilon_t = X_t \ominus P_{S_{t-1}} X_t$ is different from $\vec{0}$ and from the proof of Theorem 4, $\varepsilon_t = \|\varepsilon_t\| e_t$, hence (55) holds with $\alpha_k = \theta_k / \|\varepsilon_{t-k}\|$, and (ε_t) is also uncorrelated. As $W_t, \varepsilon_t \in L^2[\Omega, K_{kc}(\mathbb{R})]$, $E[W_t] = \vec{0} = E[\varepsilon_t]$. $W_t \in U_t^{\perp}$ implies that $Cov(W_t, \varepsilon_s) = 0$ for any $s \leq t$. For $s > t$, taking scalar product of (56) with ε_s one has $Cov(W_t, \varepsilon_s) = Cov(X_t, \varepsilon_s) = 0$ since $\varepsilon_s \in S_{s-1}^{\perp}$ and $X_t \in S_t \subset S_{s-1}$ for $s > t$. This proves **(iii)**. Let $X_{t,n}$ be the projection of X_t on $S_{t,n} = span(\{X_{t-j}\}_{j=1}^{n})$ and $\varepsilon_{t,n}$ the residual. Then $X_{t,n}$ takes the form

$$X_{t,n} = \sum_{j=1}^{n} \beta_{j,n} X_{t-j},$$

where the scalars $\beta_{k,n}$ do not depend on t, since they are solutions of the system of equations

$$\sum_{j=1}^{n} \beta_{j,n} \gamma(j-k) = \gamma(k), \qquad k = 1, \ldots, n.$$

Hence $E[X_{t,n}] = \vec{0}$, $E[\varepsilon_{t,n}] = \vec{0}$. Moreover,

$$Var(\varepsilon_{t,n}) = \|X_t - X_{t,n}\|^2 = \left\| X_t - \sum_{j=1}^{n} \beta_{j,n} X_{t-j} \right\|^2$$

$$= \gamma(0) + \sum_{i,j=1}^{n} \beta_{i,n} \beta_{j,n} \gamma(i-j) - 2 \sum_{j=1}^{n} \beta_{j,n} \gamma(j).$$

Hence $Var(\varepsilon_{t,n}) = \sigma_n$ does not depend on t and so does for $\sigma = \|\varepsilon_t\| = \lim_{n \to \infty} \sigma_n$, where the latter equality comes from Lemma 5.6. Also,

$$Cov(X_{t+k}, \varepsilon_{t,n}) = \gamma(k) - \sum_{j=1}^{n} \beta_{j,n} \gamma(k+j),$$

which does not depend on t. Using Cauchy-Schwarz inequality and Lemma 5.6,

$$\lim_{n \to \infty} |Cov(X_{t+k}, \varepsilon_{t,n} - \varepsilon_t)| \leq \sqrt{\gamma(0)} \lim_{n \to \infty} \|\varepsilon_{t,n} - \varepsilon_t\| = 0,$$

which implies that $Cov(X_{t+k}, \varepsilon_t) = \lim_{n \to \infty} Cov(X_{t+k}, \varepsilon_{t,n})$ and does not depend on t. So,

$$\alpha_k = \frac{1}{\|\varepsilon_t\|} Cov(X_{t+k}, e_k) = \frac{1}{\|\varepsilon_t\|^2} Cov(X_{t+k}, \varepsilon_t)$$

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does not depend on t. Moreover, $\alpha_0 = \frac{Cov(X_t, \varepsilon_t)}{\|\varepsilon_t\|^2} = 1$. All this completes the proof of **(i)** and **(ii)**. For $k \geq 0$,

$$
\begin{aligned}
Cov(W_t, W_{t-k}) &= Cov\left(X_{t-k} - \sum_{j=0}^{\infty} \alpha_j \varepsilon_{t-k-j}, X_t - \sum_{j=0}^{\infty} \alpha_j \varepsilon_{t-j}\right) \\
&= \gamma(k) - \sum_{j=0}^{\infty} \alpha_j Cov(X_t, \varepsilon_{t-k-j}) - \sum_{j=k}^{\infty} \alpha_j Cov(X_{t-k}, \varepsilon_{t-j}) + \sigma^2 \sum_{j=0}^{\infty} \alpha_{j+k} \alpha_j \\
&= \gamma(k) - \sigma^2 \sum_{j=0}^{\infty} \alpha_{j+k} \alpha_j,
\end{aligned}
$$

which does not depend on t. As $W_t \in S_t$, one can write $W_t = \sum_{k=0}^{\infty} a_k X_{t-k}$. Taking covariance with ε_t and using the fact that $\varepsilon_t \perp Span(X_{t-1}, X_{t-2}, \ldots)$ one gets $Cov(W_t, \varepsilon_t) = a_0 Cov(X_t, \varepsilon_t) = a_0 \|\varepsilon_t\|^2$. Since $Cov(W_t, \varepsilon_t) = 0$, one deduces that $a_0 = 0$ hence $W_t \in S_{t-1}$, thus (W_t) is deterministic from the past of (X_t). This completes the proof of **(iv)**. $\qquad\square$

In the above proof, we have seen that, (W_t) is deterministic from the past of (X_t) and W_t is orthogonal to any element of S_s for any s. Contrary to traditional time series, this may not imply that (W_t) is deterministic nor that W_t belong to $S_{-\infty}$. This is due to the fact that in a Hilbert pseudovector space, the orthogonal of a subspace is not necessarily complementary to it. A new challenge shall be to give an adapted definition for purely non-deterministic interval-valued process such that the Wold decomposition of such a process has no deterministic component and then could be expressed as an I-MA(∞)

$$
X_t = \sum_{k=0}^{\infty} \alpha_k \varepsilon_{t-k}. \tag{57}
$$

7 Conclusion

In this work, the definition of set-valued autoregressive moving average (S-ARMA) time series process is given. This definition takes into account an arithmetical difference between random sets and random real variables. More precisely for point-value time series, the equations $X_t - \sum_{i=1}^{p} \phi_i X_{t-i} = K + \varepsilon_t + \sum_{i=1}^{q} \theta_i \varepsilon_{t-i}$ and $X_t = \sum_{i=1}^{p} \phi_i X_{t-i} + K + \varepsilon_t + \sum_{i=1}^{q} \theta_i \varepsilon_{t-i}$ are equivalent, what is not the case for set-valued time series process. A concept of pseudovector space is introduced. A pseudovector space is an almost vector space where opposites of vectors might not exist. This new concept might be studied in pure maths. It is shown that the set of random intervals is a pseudovector space and generalized Hukuhara difference [Ste10] is used to recover a kind of opposite for vectors. Furthermore, a notion of Hilbert pseudovector space is introduced and used to establish Wold decomposition for interval-valued covariance stationary time series process.

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