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On the first moments and semi-moments of fuzzy variables based on a new measure and application for portfolio selection with fuzzy returns

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Abstract

Possibility, Necessity and Credibility measures are used in the literature in order to deal with imprecision. Recently, Yang and Iwamura [11] introduced a new measure as convex linear combination of possibility and necessity measures and they determined some of its axioms. In this paper, we introduce characteristics (parameters) of a fuzzy variable based on that measure, namely, Expected value, Variance, Semi-Variance, Skewness, Kurtosis and Semi-Kurtosis. We determine some properties of these characteristics and we compute them for trapezoidal and triangular fuzzy variables. We display their application for the determination of optimal portfolios when assets returns are described by triangular or trapezoidal fuzzy variables.

Keywords: Fuzzy measure; Fuzzy variable; Expected value; Variance and Semi-Variance; Skewness; Kurtosis and Semi-Kurtosis; Optimal portfolios.

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1 Introduction

Uncertainty can be viewed as an aspect of randomness or ambiguity in real life phenomena. Thereby, one needs a measure on fuzzy events (fuzzy measure) to analyze questions and problems dealing with such uncertainty. After Zadeh's work [12] introducing possibility and necessity measures, a vast literature appeared on fuzzy theory and fuzzy logic based on the aforementioned measures. Twenty years later, Liu [6] introduced credibility measure as the arithmetic mean of the two first measures. This measure has been used to address other questions dealing with uncertainty (see Liu [7], Li et al. [5], Huang et al. [4], Sadefo et al. [9] and Tassak et al. [10]). More recently, Yang and Iwamura [11] introduced a new measure, denoted m_λ, as a convex linear combination of possibility and necessity measures (the weight λ of the possibility measure in the combination is a real parameter in the interval $[0; 1]$) and they determined some of its axioms. This new measure generalized the three previous ones and has been used in fuzzy chance-constrained programming (Yang and Kakuzo [11], Dai et al. [1]) as a more generalized approach compared with credibility constrained programming ones to find optimal strategies for carbon capture. We used it in our recent conference paper to determine Expected value and Variance of a fuzzy variable (Dzuche et al. [2]).

This paper focuses on the recent measure for two major findings that might be useful. First, the weight λ can be considered as the the decision maker confidence on the fuzzy event degree of realization such that degree 1 means a complete confidence and degree 0 means no confidence. The $m_\lambda-$ measure can become either possibility measure,or necessity measure or credibility measure respectively for the decision marker who is self confident, unconfident and neutral (the weight is equal to $\frac{1}{2}$). It is a comprise of the two first measures and this finding is similar to the idea of Hurwicz decision criteria in Microeconomics which is a compromise of maximax and maximin criteria. On the other hand, due to the fact that the new measure generalizes the three previous ones in technical point of view, many theoretical results already solved with one of the three first measures can be generalized with the new measure. In addition, some specific results will be outlined.

The modest contribution of this paper is to study, by means of m_λ, fuzzy variables characteristics and to implement obtained results for portfolio optimization models with fuzzy returns. More specifically, in the continuation of our recent work (Dzuche et al. [2]), we propose some new axioms of m_λ, and we introduce fuzzy variables characteristics w.r.t. to this measure. We determine four first moments (expected value, variance, skewness and kurtosis) and two first semi-moments (semi-variance and semi-kurtosis)of trapezoidal and triangular fuzzy variables. We establish some properties of these characteristics and the obtained theoretical results are applied in portfolio selection with fuzzy returns.

The paper is organized as follows. Section 2 recalls useful notions on fuzzy sets, on four previous usual fuzzy measures. We display some new axioms of the more recent one. In Section 3, we introduce and study fuzzy variables characteristics. More precisely, we focus on the determination of four first moments and two first semi-moments of trapezoidal and triangular fuzzy variables and we establish some properties of these characteristics. Section 4 displays an application of these characteristics in optimal portfolios selection where assets returns proposed by Huang [4] are described by triangular fuzzy variables. We also apply these characteristics to find optimal portfolios by using Tokyo stock exchange data provided by Hasuike et al. [3]. Appendix contains proofs of some results.

2 Preliminaries

Throughout this paper, X is a nonempty set, namely, the universal set. A fuzzy subset A of X is defined by its membership function: $\mu_A : X \to [0,1]$ such that, to each $x \in X$, is associated $\mu_A(x)$ representing the membership grade of x to A. A is denoted by $\{(x, \mu_A(x)), x \in X\}$. If $\forall x \in A, \mu_A(x) \in \{0,1\}$, then A becomes a crisp subset of X.

Let ξ be a mapping from X to $\mathbb{R}$ described by its membership function μ interpreted as: for any $x \in \mathbb{R}$, $\mu(x)$ represents the degree that ξ takes value x. ξ is said to be a fuzzy variable if ξ is measurable. A fuzzy variable ξ is normal if $\exists x_0 \in \mathbb{R}, \mu(x_0) = 1$.
For ξ and η be two fuzzy variables and α a real number, their sum $\xi + \eta$ is a fuzzy variable defined by: $\forall x \in \mathbb{R}, (\xi + \eta)(x) = \xi(x) + \eta(x)$ and $\alpha\xi$ is a fuzzy variable defined by: $\forall x \in \mathbb{R}, (\alpha\xi)(x) = \alpha\xi(x)$.
A fuzzy number ξ is a fuzzy variable that satisfies: $\exists a, b, c, d \in \mathbb{R}$ with $a \leq b \leq c \leq d$ such that (i) μ is upper semi-continuous, (ii) $\forall r \notin [a,d], \mu(r) = 0$, (iii) μ is increasing on $[a,b]$ and decreasing on $[c,d]$ and (iv) $\forall r \in [b,c], \mu(r) = 1$. Thus, we denote it by $\xi = (a,b,c,d)$. In the particular case where μ is a straight line on $[a,b]$ and $[c,d]$, then $\xi = (a,b,c,d)$ is the usual and well-known trapezoidal fuzzy number. Then, we deduce analytical expressions of the membership functions of $\xi : \forall x \in \mathbb{R}, \mu(x) = \begin{cases} \frac{x-a}{b-a} \text{ if } a \leq x \leq b \\ 1, \text{ if } b \leq x \leq c \\ \frac{x-d}{c-d} \text{ if } c \leq x \leq d \\ 0 \text{ elsewhere} \end{cases}$.
If $b = c$, then $\xi = (a,b,d)$ is a triangular fuzzy number.

Figure 1 displays a trapezoidal fuzzy variable and a triangular one.

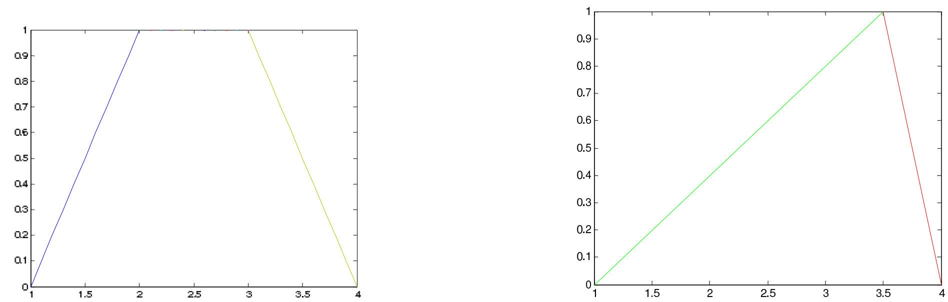

Figure 1: On right trapezoidal fuzzy variable $(1, 2, 3, 4)$ and on left triangular fuzzy variable $(1, 3.5, 4)$.

In the following, we recall classical fuzzy measures proposed in the literature to deal with imprecision.
Let $B \subset X$ and $\xi : X \to \mathbb{R}$ be a fuzzy variable whose membership function μ, that means, each element x of X is associated to a real number $\xi(x)$ by means of ξ with the membership grade $\mu(\xi(x))$. We have the following classical and well-known measures: the Possibility measure defined by $Pos(B) = \sup_{x\in B} \mu(\xi(x))$ (Zadeh [12]); the Necessity measure defined by $Nec(B) = 1 - Pos(B^c) = \inf_{x\in B} \mu(\xi(x))$ (Zadeh [12]) and the Credibility measure defined by $Cred(B) = \frac{1}{2}[Pos(B) + Nec(B)]$ (Liu [6]).
Yang and Iwamura [11] introduced a measure, denoted m_λ and defined by for $\lambda \in [0, 1]$,

$$m_\lambda(B) = \lambda \sup_{x\in X} \mu_B(x)) + (1 - \lambda) \inf_{x\in X} \mu_B(x)) \tag{1}$$

or equivalently, we have:

$$m_\lambda(\{\xi \in B\}) = \lambda Pos(\{\xi \in B\}) + (1 - \lambda)Nec(\{\xi \in B\}).$$

In particular, we have: For $\lambda = 0$ (resp. $\lambda = 1$, resp. $\lambda = \frac{1}{2}$) $m_\lambda = Nec$ (resp. $m_\lambda = Pos$, resp. $m_\lambda = Cr$). They established some axioms of m_λ such as universality, subadditivity for $\lambda \geq \frac{1}{2}$ and monotonicity. Notice that if $\mathcal{P}(X)$ is the power set of X, then the triplet $(X, \mathcal{P}(X), m_\lambda)$ is called a λ-fuzzy space.
In addition, we have the following two new axioms of m_λ.

1.
$$\left(\begin{array}{c} \lambda m_\lambda(B) + (1 - \lambda)m_\lambda(B^c) = \lambda \\ \text{or} \\ (1 - \lambda)m_\lambda(B) + \lambda m_\lambda(B^c) = \lambda \end{array} \right) \tag{2}$$

2. $m_\lambda(\cup_{i\in I} B_i) = \sup_{i\in I} m_\lambda(B_i)$ if $\sup_{i\in I} m_\lambda(B_i) \leq \lambda$ where $(B_i)_{i\in I}$ is a countable family of subsets of X and $I \subset \mathbb{N}$.

When $\lambda = \frac{1}{2}$, the Axiom defined by (2) becomes the duality axiom of credibility measure defined by Liu [6] as $Cr(B) + Cr(B^c) = 1$.

Let us end this Section by evaluating the measure of events $\xi \geq t$ and $\xi \leq t$ for $t \in \mathbb{R}$ when ξ is a trapezoidal fuzzy variable.

Example 1. *$m_\lambda\{\xi \geq t\} = \lambda \sup_{r\geq t} \mu(r) + (1-\lambda)(1 - \sup_{r<t} \mu(r))$ and $m_\lambda\{\xi \leq t\} = \lambda \sup_{r\leq t} \mu(r) + (1-\lambda)(1 - \sup_{r>t} \mu(r))$.*
We obtain:

$$m_\lambda\{\xi \geq t\} = \begin{cases} 0 \quad if\ d \leq t, \\ \dfrac{\lambda(d-t)}{d-c} \quad if\ c \leq t \leq d, \\ \lambda \quad if\ b \leq t < c, \\ \dfrac{\lambda(t-a)+b-t}{b-a} \quad if\ a \leq t < b, \\ 1 \quad if\ t < a \end{cases}$$

$$m_\lambda\{\xi \leq t\} = \begin{cases} 1 \quad if\ d \leq t, \\ \dfrac{\lambda(d-t)+t-c}{d-c} \quad if\ c \leq t \leq d, \\ \lambda \quad if\ b \leq t < c, \\ \dfrac{\lambda(t-a)}{b-a} \quad if\ a \leq t < b, \\ 0 \quad if\ t < a. \end{cases}$$

Throughout this paper, ξ is a normal fuzzy variable.

In the following Section, we introduce four main characteristics of a fuzzy variable by means of $m_\lambda-$ measure. We compute them for trapezoidal and triangular fuzzy variables and determine some of their properties.

3 Characteristics of a fuzzy variable with respect to the m_λ-measure

In the following Subsection, we introduce and study the expected value and the variance of a fuzzy variable with respect to $m_\lambda-$ measure. Results of our recent conference paper (Dzuché et al.[2]) are a part of the findings of the Subsection.

3.1 Expected value and Variance

Definition 1. *Let ξ be a fuzzy variable and $\lambda \in [0,1]$. The expected value of ξ is defined by:*

$$E_\lambda[\xi] = \int_{-\infty}^{0} [m_\lambda\{\xi \geq r\} - 1]dr + \int_{0}^{+\infty} m_\lambda\{\xi \geq r\}dr \tag{3}$$

with the condition that, at least one of the two integrals is finite.

Remark 1. *1) When $\lambda = \frac{1}{2}$, we obtain the expected value defined by Liu [6] by means of the credibility measure as follows:*

$$E_{\frac{1}{2}}[\xi] = \int_{0}^{+\infty} Cr\{\xi \geq r\}dr - \int_{-\infty}^{0} Cr\{\xi \leq r\}dr$$

provided that, at least one of the two integrals is finite.
2) Liu [6] introduced for a normalized fuzzy variable, the upper expected value of ξ and the lower expected value of ξ respectively denoted $\overline{E}[\xi]$ and $\underline{E}[\xi]$ and defined by:

$$\overline{E}[\xi] = \int_{0}^{+\infty} Pos\{\xi \geq r\}dr - \int_{-\infty}^{0} Nec\{\xi \leq r\}]dr \tag{4}$$

and

$$\underline{E}[\xi] = \int_{0}^{+\infty} Nec\{\xi \geq r\}dr - \int_{-\infty}^{0} Pos\{\xi \leq r\}]dr. \tag{5}$$

We have:

$$E_\lambda[\xi] = \lambda\overline{E}[\xi] + (1-\lambda)\underline{E}[\xi] \tag{6}$$

We now compute the expected values of trapezoidal and triangular fuzzy variables.

Example 2. - *The expected value of a trapezoidal fuzzy variable $\xi = (a,b,c,d)$ based on the $m_\lambda-$ measure is defined by:*

$$E_\lambda[\xi] = (1-\lambda)\frac{a+b}{2} + \lambda\frac{c+d}{2}.$$

- *We deduce that the expected value of a triangular fuzzy variable $\xi = (a,b,c)$ is given by:*

$$E_\lambda[\xi] = (1-\lambda)\frac{a}{2} + \lambda\frac{c}{2} + \frac{b}{2}.$$

- *If $\lambda = \dfrac{1}{2}$, we obtain expected values of Liu [6] respectively for trapezoidal and triangular fuzzy variables, that is, $E_{\frac{1}{2}}[\xi] = \frac{a+b+c+d}{4}$ and $E_{\frac{1}{2}}[\xi] = \frac{a+2b+c}{4}$.*

The following result establishes that the expected value of a trapezoidal fuzzy variable is increasing with respect to λ.

Proposition 1. *Let $\xi = (a, b, c, d)$ be a trapezoidal fuzzy variable, $\lambda_1, \lambda_2 \in [0, 1]$.*
If $\lambda_1 \leq \lambda_2$ then $E_{\lambda_1}[\xi] \leq E_{\lambda_2}[\xi]$.

Proof: Let $\xi = (a, b, c, d)$ be a trapezoidal fuzzy variable and let us set: $f(\lambda) = (1-\lambda)\frac{a+b}{2} + \lambda\frac{c+d}{2}$. We have: $f'(\lambda) = \frac{c-a}{2} + \frac{d-b}{2} \geq 0$ since $c \geq a$ and $d \geq b$. So $f = E_\lambda[\xi]$ is increasing with respect to λ. $\square$

We end this Subsection by introducing the Variance of a fuzzy variable and we compute it in some cases.

Definition 2. *Let ξ be a fuzzy variable such that $E[\xi] = e_\lambda$ and $\lambda \in [0, 1]$. The variance of ξ is defined by:*

$$V_\lambda[\xi] = E[(\xi - e_\lambda)^2] = \int_0^{+\infty} m_\lambda\{(\xi - e_\lambda)^2 \geq r\}dr. \tag{7}$$

In the following Example, we compute the variance of a trapezoidal fuzzy variable and a triangular one.

Example 3. 1. *$\xi = (a, b, c, d)$ is a trapezoidal fuzzy variable with expected value e_λ.*
We set: $\alpha = b-a$, $\gamma = c-b$, $\beta = d-c$, $t = \dfrac{\beta(e_\lambda - a) - \alpha(d - e_\lambda)}{\beta - \alpha}$ and without loss of generality, we assume that: $b < e_\lambda < c$, $\gamma < \alpha < \beta$, $c - e_\lambda < e_\lambda - a < d - e_\lambda$ and $c - e_\lambda < t < e_\lambda - a$. We obtain:
$V_\lambda[\xi] = \lambda(c - e_\lambda)^2 + \frac{\lambda}{\alpha}[(e_\lambda - a)(t^2 - (c - e_\lambda)^2)$
$- \frac{2}{3}(t^3 - (c - e_\lambda)^3)] + \frac{\lambda}{\beta}[(d - e_\lambda)((e_\lambda - a)^2 - t^2) - \frac{2}{3}((e_\lambda - a)^3 - t^3)]$
$+ \frac{\lambda}{\beta}[\frac{1}{3}(d - e_\lambda)^3) + \frac{2}{3}(e_\lambda - a)^3 - (e_\lambda - a)^2(d - e_\lambda)]$.

2. *$\xi = (a, b, c)$ is a triangular fuzzy variable with expected value e_λ.*
We set: $\alpha = b - a$, $\beta = c - b$, $t = \dfrac{\beta(e_\lambda - a) - \alpha(c - e_\lambda)}{\beta - \alpha}$ and without loss of generality, we assume that: $b > e_\lambda$ and $\alpha > \beta$. We obtain:
$V_\lambda[\xi] = (b - e_\lambda)^2 - \frac{1-\lambda}{\alpha}[(b - e_\lambda)^2(e_\lambda - a) + \frac{2}{3}(b - e_\lambda)^3]$
$+ \frac{\lambda}{\beta}[(c - e_\lambda)(t^2 - (b - e_\lambda)^2) - \frac{2}{3}(t^3 - (b - e_\lambda)^3)] + \frac{\lambda}{\alpha}[(e_\lambda - a)((c - e_\lambda)^2 - t^2) - \frac{2}{3}((c - e_\lambda)^3 - t^3)]$
$+ \frac{\lambda}{\alpha}[\frac{1}{3}(e_\lambda - a)^3 + \frac{2}{3}(c - e_\lambda)^3 - (c - e_\lambda)^2(e_\lambda - a)]$.

The following result establishes useful properties (linearity and homogeneity) of the expected value and the variance respectively with respect to the $m_\lambda-$ measure.

Proposition 2. *Let ξ and η be two fuzzy variables such that $E_\lambda[\xi] < \infty$ and $E_\lambda[\eta] < \infty$, μ a nonnegative real number, ν a real number and $\lambda \in [0, 1]$.*

1. $\begin{cases} E_\lambda[\xi+\eta] = E_\lambda[\xi] + E_\lambda[\eta] \\ E_\lambda[\mu\xi] = \mu E_\lambda[\xi] \end{cases}$.

2. $V_\lambda[\mu\xi+\nu] = \mu^2 V_\lambda[\xi]$.

Proof: 1) The proof of the linearity is given in the Section Appendix.
2) Since $\alpha\xi+\beta$ is a fuzzy variable; on the other hand, by Definition 2 and the first Proposition 2 , we have:
$V_\lambda[\mu\xi+\nu] = E_\lambda[(\mu\xi+\nu-E_\lambda[\mu\xi+\nu])^2] = E_\lambda[(\mu\xi+\nu-\mu E_\lambda[\xi]-\nu)^2]$. Thus, we have: $V_\lambda[\mu\xi+\nu] = E_\lambda[(\mu\xi-\mu E_\lambda[\xi])^2] = E_\lambda[\mu^2(\xi-E_\lambda[\xi])^2]$. Finally, by the first result of Proposition 2, we have: $V_\lambda[\mu\xi+\nu] = \mu^2 E_\lambda[(\xi-E_\lambda[\xi])^2] = \mu^2 V_\lambda[\xi]$. $\square$

Throughout this paper, we introduce new concepts and establish new results.

In the following Subsection, we introduce and study the Skewness and Kurtosis of a fuzzy variable with respect to the m_λ- measure.

3.2 Skewness and Kurtosis

Definition 3. *Let ξ be a fuzzy variable such that $E_\lambda[\xi] = e_\lambda$ and $\lambda \in [0,1]$.*
1) The skewness of ξ is defined by:

$$SK_\lambda[\xi] = E_\lambda[(\xi-e_\lambda)^3] = \int_{-\infty}^{0} [m_\lambda\{(\xi-e_\lambda)^3 \leq r\}-1]dr + \int_{0}^{+\infty} m_\lambda\{(\xi-e_\lambda)^3 \geq r\}dr \tag{8}$$

2) The kurtosis of ξ is defined by:

$$K_\lambda[\xi] = E_\lambda[(\xi-e_\lambda)^4] = \int_{0}^{+\infty} m_\lambda\{(\xi-e_\lambda)^4 \geq r\}dr. \tag{9}$$

We now compute skewness and kurtosis of a trapezoidal fuzzy variable and a triangular one with expected value e_λ.

Example 4. 1. *For a trapezoidal fuzzy variable $\xi = (a,b,c,d)$.*
The Skewness $SK_\lambda[\xi]$ of ξ is given by:

$$SK_\lambda[\xi] = \frac{1}{4(b-a)}[(1-\lambda)(b-e_\lambda)^4 - \lambda(a-e_\lambda)^4] + \frac{\lambda}{4(d-c)}[(d-e_\lambda)^4-(c-e_\lambda)^4].$$

2. *For a triangular fuzzy variable $\xi = (a,b,c)$.*
The Skewness $SK_\lambda[\xi]$ of ξ is given by:

$$SK_\lambda[\xi] = \frac{1}{4(b-a)}[(1-\lambda)(b-e_\lambda)^4 - \lambda(a-e_\lambda)^4] + \frac{\lambda}{4(c-b)}[(c-e_\lambda)^4-(b-e_\lambda)^4].$$

If $\lambda = \frac{1}{2}$, we obtain Li et al.'s result on Skewness ([5]).

In the following Example, we compute kurtosis of a trapezoidal fuzzy variable and a triangular one.

Example 5. *1. $\xi = (a, b, c, d)$ is a trapezoidal fuzzy variable with expected value e_λ.*
We set: $\alpha = b - a$, $\gamma = c - b$, $\beta = d - c$, $t = \dfrac{\beta(e_\lambda - a) - \alpha(d - e_\lambda)}{\beta - \alpha}$ and without loss of generality, we assume that: $b < e_\lambda < c$, $\gamma < \alpha < \beta$, $c - e_\lambda < e_\lambda - a < d - e_\lambda$ and $c - e_\lambda < t < e_\lambda - a$. We obtain:
$K_\lambda[\xi] = \lambda(c - e_\lambda)^4 + \frac{\lambda}{\alpha}[(e_\lambda - a)(t^4 - (c - e_\lambda)^4)$
$- \frac{4}{5}(t^5 - (c - e_\lambda)^5)] + \frac{\lambda}{\beta}[(d - e_\lambda)((e_\lambda - a)^4 - t^4) - \frac{4}{5}((e_\lambda - a)^5 - t^5)]$
$+ \frac{\lambda}{\beta}[\frac{1}{5}(d - e_\lambda)^5) + \frac{4}{5}(e_\lambda - a)^5 - (e_\lambda - a)^4(d - e_\lambda)]$.

2. $\xi = (a, b, c)$ is a triangular fuzzy variable with expected value e_λ.
We set: $\alpha = b - a$, $\beta = c - b$, $t = \dfrac{\beta(e_\lambda - a) - \alpha(c - e_\lambda)}{\beta - \alpha}$ and without loss of generality, we assume that: $b > e_\lambda$ and $\alpha > \beta$. We obtain:
$K_\lambda[\xi] = (b - e_\lambda)^4 - \frac{1-\lambda}{\alpha}[(b - e_\lambda)^4(e_\lambda - a) + \frac{4}{5}(b - e_\lambda)^5]$
$+ \frac{\lambda}{\beta}[(c - e_\lambda)(t^4 - (b - e_\lambda)^4) - \frac{4}{5}(t^5 - (b - e_\lambda)^5)] + \frac{\lambda}{\alpha}[(e_\lambda - a)((c - e_\lambda)^4 - t^4) - \frac{4}{5}((c - e_\lambda)^5 - t^5)]$
$+ \frac{\lambda}{\alpha}[\frac{1}{5}(e_\lambda - a)^5 + \frac{4}{5}(c - e_\lambda)^5 - (c - e_\lambda)^4(e_\lambda - a)]$.

We end this Subsection by establishing some of their properties.

Proposition 3. *Let ξ be a fuzzy variable, μ and ν two real numbers such that $\mu \geq 0$ and $\lambda \in [0, 1]$. We have:*
1) $S_\lambda[\mu\xi + \nu] = \mu^3 S_\lambda[\xi]$.
2) $K_\lambda[\mu\xi + \nu] = \mu^4 K_\lambda[\xi]$.

Proof: The proof of this Proposition is similar to the proof of the second result of Proposition 2. □

In the following Subsection, we introduce semi-variance and semi-kurtosis of a fuzzy variable ξ such that $E_\lambda[\xi] = e_\lambda$ and $\lambda \in [0, 1]$.

3.3 Semi-variance and semi-kurtosis

Definition 4. *1) The semi-variance of ξ is defined by:*

$$V_\lambda^S[\xi] = E_\lambda[[(\xi - e_\lambda)^-]^2] = \int_0^{+\infty} m_\lambda\{[(\xi - e_\lambda)^-]^2 \geq r\}dr. \qquad (10)$$

2) The semi-kurtosis of ξ is defined by:

$$K_\lambda^S[\xi] = E_\lambda[[(\xi - e_\lambda)^-]^4] = \int_0^{+\infty} m_\lambda\{[(\xi - e_\lambda)^-]^4 \geq r\}dr. \qquad (11)$$

Example 6. *1. For a trapezoidal fuzzy variable $\xi = (a, b, c, d)$, we obtain:*
a)

$$V_\lambda^S[\xi] = \lambda\frac{(e_\lambda - a)^3}{3(b-a)} - \lambda\frac{\max[0; (e_\lambda - b)^3]}{3(b-a)} + (1-\lambda)\frac{\max[0; (e_\lambda - c)^3]}{3(d-c)}.$$

If $\lambda = \dfrac{1}{2}$, we obtain Huang's result on semi-variance [4].
b)

$$K_\lambda^S[\xi] = \lambda\frac{(e_\lambda - a)^5}{5(b-a)} - \lambda\frac{\max[0; (e_\lambda - b)^5]}{5(b-a)} + (1-\lambda)\frac{\max[0; (e_\lambda - c)^5]}{5(d-c)}.$$

If $\lambda = \dfrac{1}{2}$, we obtain Sadefo et al.'s result on semi-kurtosis [9].

2. For a triangular fuzzy variable $\xi = (a, b, c)$, we obtain:
a)

$$V_\lambda^S[\xi] = \lambda\frac{(e_\lambda - a)^3}{3(b-a)} - \lambda\frac{\max[0; (e_\lambda - b)^3]}{3(b-a)} + (1-\lambda)\frac{\max[0; (e_\lambda - b)^3]}{3(c-b)}.$$

If $\lambda = \dfrac{1}{2}$, we obtain Huang's result on semi-variance [4].
b)

$$K_\lambda^S[\xi] = \lambda\frac{(e_\lambda - a)^5}{5(b-a)} - \lambda\frac{\max[0; (e_\lambda - b)^5]}{5(b-a)} + (1-\lambda)\frac{\max[0; (e_\lambda - b)^5]}{5(c-b)}.$$

If $\lambda = \dfrac{1}{2}$, we obtain Sadefo et al.'s result on semi-kurtosis [9].

The following result justifies that the risk's level of a trapezoidal fuzzy variable (defined by semi-variance or semi-kurtosis) depend on the optimistic parameter and on the expected return position with respect to the support of the variable. Its proof is given in the Appendix.

Proposition 4. *Let $\xi = (a, b, c, d)$ be a trapezoidal fuzzy variable with expected value e_λ and $\lambda \in [0, 1]$.*

1. *If $a \le e_\lambda \le b$ or $b \le e_\lambda \le c$ then $V_\lambda^S[\xi]$ and $K_\lambda^S[\xi]$ increases with respect to λ.*

2. *If $c \le e_\lambda \le d$ then:*
   - $V_\lambda^S[\xi]$ increases with respect to λ when $\lambda \le \frac{3}{4}$.
   - $K_\lambda^S[\xi]$ increases with respect to λ when $\lambda \le \frac{5}{6}$.

Remark 2.

For the cases where $a \leq e_\lambda \leq b$ or $b \leq e_\lambda \leq c$, the trapezoidal fuzzy variable describes almost either lower expected returns or average expected returns. That is the reason why the more the investor is confident (λ increases), the more he is exposed to a high risk's level (semi-variance and semi-kurtosis are greater).

For $c \leq e_\lambda \leq d$, the trapezoidal fuzzy variable describes almost either higher expected returns. Thus, the risk remains increasing for an optimistic maximum's level ($\frac{3}{4}$ for semi-variance and $\frac{5}{6}$ for semi-kurtosis).
For $\lambda > \frac{3}{4}$ (resp. $\lambda > \frac{5}{6}$), the monotonicity of the semi-variance of the trapezoidal fuzzy variable (resp. its semi-kurtosis) depends on $\alpha, \beta, \gamma = c-b$. That means, the investor can feel himself secured with high values of returns. In that, risk cannot remain increasing with respect to the optimistic factor, it will necessarily depend on returns' spread (α, β, γ).

In the following, we establish some properties or relationships of characteristics of a fuzzy variable ξ such that $E_\lambda[\xi] = e_\lambda$ and $\lambda \in [0, 1]$.

3.4 Some properties of fuzzy variables characteristics

The following result justifies that variance (respectively kurtosis) is greater than semi-variance (respectively semi-kurtosis). In addition, it justifies that they are equal for symmetric fuzzy variables. Its proof is given in the Appendix.

Proposition 5. *1) We have: $0 \leq V_\lambda^S[\xi] \leq V_\lambda[\xi]$ and $0 \leq K_\lambda^S[\xi] \leq K_\lambda[\xi]$.*
2) Furthermore, if ξ is symmetric with respect to e_λ, that is, $\forall r \in \mathbb{R}, \mu(e_\lambda - r) = \mu(e_\lambda + r)$, then

$$\begin{cases} V_\lambda^S[\xi] = V_\lambda[\xi] \\ K_\lambda^S[\xi] = K_\lambda[\xi] \end{cases}.$$

The following result establishes necessary and sufficient conditions under which Variance and Kurtosis are null for some values of λ.

Proposition 6. *If $\lambda \in [\frac{1}{2}, 1[$, then*

$$\begin{cases} V_\lambda[\xi] = 0 \Leftrightarrow m_\lambda\{\xi = e_\lambda\} = 1 \\ K_\lambda[\xi] = 0 \Leftrightarrow m_\lambda\{\xi = e_\lambda\} = 1 \end{cases}.$$

To prove this result, we need the following Lemma which is proved in the Appendix.

Lemma 1. *Let $A \subset X$ be an event and $\lambda \in [0, 1]$.*
1) $m_\lambda(A) = 0 \Rightarrow m_\lambda(A^c) = \begin{cases} 0 \text{ if } \lambda = 0 \\ \frac{\lambda}{1-\lambda} \text{ or } 1 \text{ if } \lambda \in]0, \frac{1}{2}] \\ 1 \text{ if } \lambda \in]\frac{1}{2}, 1] \end{cases}$
2) Furthermore, for $\lambda \in [\frac{1}{2}, 1[$, we have:

$$m_\lambda(A) = 0 \Leftrightarrow m_\lambda(A^c) = 1.$$

We now justify the Proposition.
Proof of Proposition 6: Let ξ be a fuzzy variable with $E_\lambda[\xi] = e_\lambda$ and $\lambda \in [\frac{1}{2}, 1[$. Let us prove that $V_\lambda[\xi] = 0 \Leftrightarrow m_\lambda\{\xi = e_\lambda\} = 1$.
$(\Leftarrow)$: Let us assume that $m_\lambda\{\xi = e_\lambda\} = 1$.
It is obvious that $m_\lambda\{\xi - e_\lambda = 0\} = 1$ if and only if $m_\lambda\{(\xi - e_\lambda)^2\} = 1$.
By Lemma 1, we obtain $m_\lambda\{(\xi - e_\lambda)^2 \neq 0\} = 0$.
Let $r > 0$, we have:
$m_\lambda\{(\xi - e_\lambda)^2 > r\} \leq \{m_\lambda\{(\xi - e_\lambda)^2 > 0\} \leq \{m_\lambda\{(\xi - e_\lambda)^2 \neq 0\} = 0$.
This implies that $m_\lambda\{(\xi - e_\lambda)^2 > r\} = 0$, $\forall r > 0$.
Therefore, $V_\lambda[\xi] = \int_0^{+\infty} m_\lambda\{(\xi - e_\lambda)^2 \geq r\}dr = 0$.
$(\Rightarrow)$: Let us assume that $V_\lambda[\xi] = 0$.
Since m_λ takes values in [0;1], $\int_0^{+\infty} m_\lambda\{(\xi - e_\lambda)^2 \geq r\}dr = 0$ implies that $m_\lambda\{(\xi - e_\lambda)^2 \geq r\} = 0$, $\forall r > 0$, that is $m_\lambda\{(\xi - e_\lambda)^2 \neq 0\} = 0$.
By Lemma 1, we have: $m_\lambda\{(\xi - e_\lambda)^2 = 0\} = 1$ and we deduce that $m_\lambda\{\xi - e_\lambda = 0\} = 1$, that is, $m_\lambda\{\xi = e_\lambda\} = 1$.
In a similar way, it is easy to prove that $K_\lambda[\xi] = 0 \Leftrightarrow m_\lambda\{\xi = e_\lambda\} = 1$. $\square$
In the following Section, we propose two models and implement them for the determination of optimal portfolios in Finance.

4 Application for portfolio selection

4.1 Optimization models

Let us consider an investor who likes to invest his capital in n securities in the proportion x_1, x_2,..., x_n such that $\forall i \in \{1, 2, ..., n\}$, $x_i \in [0, 1]$ and $\sum_{i=1}^{n} x_i = 1$. It is well-known that an investment of a part x_i of the capital in the i^{th} security generates a return denoted by $x_i\xi_i$ which is not currently known. Making up such investment consists on constituting a portfolio $((x_i, \xi_i))_{1\leq i\leq n}$ where the n fuzzy variables $x_1\xi_1, ..., x_n\xi_n$ are future returns of the n securities and the fuzzy variable $\xi = \xi_1 x_1 + \xi_2 x_2 + ... + \xi_n x_n$ is the total future return or the portfolio future return. With respect to the credibility measure, Huang [4], Li et al. [5], Sadefo et al. [9] assumed that ξ_i is a fuzzy variable and they proposed models based on parameters (mean, variance and semi-variance for Huang; mean, variance and skewness for Li et al.; mean, variance, skewness and kurtosis for Sadefo et al.) in order to determine optimal portfolios. In this Section, we propose two models based on parameters with respect to m_λ to solve the same question of determination of best portfolios. Those new models give a family of solutions which are optimal portfolios and generalize more recent models proposed by Sadefo et al. [9] in the particular case where $\lambda = \frac{1}{2}$. For that, we consider the first family of seven assets returns described by the following triangular fuzzy variables proposed by Huang [4] and used by many authors: $\xi_1 = (-0.3, 1.8, 2.3)$, $\xi_2 = (-0.4, 2.0, 2.2)$, $\xi_3 = (-0.5, 1.9, 2.7)$, $\xi_4 = (-0.6, 2.2, 2.8)$, $\xi_5 = (-0.7, 2.4, 2.7)$, $\xi_6 = (-0.8, 2.5, 3.0)$, $\xi_7 =$

$(-0.6, 1.8, 3.0)$ and the second family of ten assets returns described by the following trapezoidal fuzzy variables proposed by Hasuike et al. [3] (written as (a, b, c, d)): $\xi_1 = (-0.362, -0.123, 0.005, 0.873)$, $\xi_2 = (-0.37, -0.069, 0.069, 0.536)$, $\xi_3 = (-0.329, -0.129, 0.025, 0.738)$, $\xi_4 = (-0.193, 0.005, 0.177, 0.412)$, $\xi_5 = (-0.299, -0.082, 0.114, 0.437)$, $\xi_6 = (-0.342, -0.052, 0.108, 0.49)$,
$\xi_7 = (-0.292, -0.056, 0.067, 0.436)$, $\xi_8 = (-0.25, 0.060, 0.193, 0.649)$, $\xi_9 = (-0.405, -0.093, 0.130, 0.756)$, $\xi_{10} = (-0.554, 0.009, 0.236, 0.5)$. In addition, we set for each family of assets, three targets values t_1, t_2, t_3 considered respectively as the minimum benefit (expected value), the maximum risk (variance) and the minimum skewness that the investor can bear. The two proposed selection models for best portfolios of those assets are:

$$\begin{cases} \text{minimize } K_\lambda[x_{i_1}\xi_{i_1} + x_{i_2}\xi_{i_2} + ... + x_{i_k}\xi_{i_k}] \\ \text{subject to} \\ E_\lambda[x_{i_1}\xi_{i_1} + x_{i_2}\xi_{i_2} + ... + x_{i_k}\xi_{i_k}] \geq t_1 \\ V_\lambda[x_{i_1}\xi_{i_1} + x_{i_2}\xi_{i_2} + ... + x_{i_k}\xi_{i_k}] \leq t_2 \\ S_\lambda[x_{i_1}\xi_{i_1} + x_{i_2}\xi_{i_2} + ... + x_{i_k}\xi_{i_k}] \geq t_3 \\ x_{i_1} + x_{i_2} + ... + x_{i_k} = 1 \\ x_{i_k} \geq 0, k \in \{7; 10\} \end{cases} \tag{12}$$

and

$$\begin{cases} \text{minimize } K^S_\lambda[x_{i_1}\xi_{i_1} + x_{i_2}\xi_{i_2} + ... + x_{i_k}\xi_{i_k}] \\ \text{subject to} \\ E_\lambda[x_{i_1}\xi_{i_1} + x_{i_2}\xi_{i_2} + ... + x_{i_k}\xi_{i_k}] \geq t_1 \\ V_\lambda[x_{i_1}\xi_{i_1} + x_{i_2}\xi_{i_2} + ... + x_{i_k}\xi_{i_k}] \leq t_2 \\ S_\lambda[x_{i_1}\xi_{i_1} + x_{i_2}\xi_{i_2} + ... + x_{i_k}\xi_{i_k}] \geq t_3 \\ x_{i_1} + x_{i_2} + ... + x_{i_k} = 1 \\ x_{i_k} \geq 0, k \in \{7; 10\} \end{cases} \tag{13}$$

In the following paragraph, we implement the two models by using Huang's data.

4.2 Implementation of the two models with Huang's data

In the following, we implement with Huang's data the two previous models for $\lambda = \frac{1}{3}$, $\lambda = \frac{1}{2}$ and $\lambda = \frac{2}{3}$ where the selected target values t_1, t_2, t_3 are given in Table 3.
For that, we set: $\forall i \in \{1, ..., 7\}$, $\xi_i = (a_i, b_i, c_i)$ and the combination of those seven triangular fuzzy variables is also a triangular fuzzy variable denoted by $\xi = \sum_{i=1}^{7} \xi_i = (a, b, c)$ where $a = \sum_{i=1}^{7} a_i$, $b = \sum_{i=1}^{7} b_i$ and $c = \sum_{i=1}^{7} c_i$, $\alpha_i = b_i - a_i$, $\beta_i = c_i - bi$ and $\alpha = \sum_{i=1}^{7} \alpha_i$, $\beta = \sum_{i=1}^{7} \beta_i$.
We recall that Sadefo et al. [9] defined these characteristics in the particular case where $\lambda = \frac{1}{2}$.

By implementing those formulas in Matlab through the previous proposed models, we obtain (i) Table 1 which presents in each line (from the second line) how the best portfolio is made up of in term of percentage of the seven securities for a given value of λ and for either K, either K^S as objective function and, (ii) Table 2 gives characteristics (parameters) of best portfolios of Table 1.

Asset i	1 (%)	2 (%)	3 (%)	4 (%)	5 (%)	6 (%)	7 (%)
$\lambda = \frac{1}{3}$ (K)	0.00	63.39	0.00	0.00	36.61	0.00	0.00
$\lambda = \frac{1}{3}$(K^S)	0.05	0.6471	0.00	0.05	31.33	3.85	0.00
$\lambda = \frac{1}{2}$ (K)	20.04	0.00	0.00	79.89	0.00	0.07	0.00
$\lambda = \frac{1}{2}$ (K^S)	20.00	0.00	0.00	80.00	0.00	0.00	0.00
$\lambda = \frac{2}{3}$ (K)	42.50	0.00	0.00	57.50	0.00	0.00	0.00
$\lambda = \frac{2}{3}$ (K^S)	42.50	0.00	0.00	57.50	0.00	0.00	0.00

Table 1: Optimal selection from each model.

	Mean	Variance	SV	Skewness	Kurtosis	SK
$\lambda = \frac{1}{3}$(K)	1.3005	0.4571	0.2482	-0.0014	0.6593	0.4880
$\lambda = \frac{1}{3}$(K^S)	1.3004	0.4570	0.2482	-1.5336×10^{-5}	0.6594	0.4875
$\lambda = \frac{1}{2}$(K)	1.60	0.7018	0.6140	-0.6823	1.7290	1.6873
$\lambda = \frac{1}{2}$(K^S)	1.60	0.7019	0.6141	-0.6823	1.7291	1.6872
$\lambda = \frac{2}{3}$ (K)	1.80	1.0510	1.0404	-1.6573	3.2213	3.2202
$\lambda = \frac{2}{3}$ (K^S)	1.80	1.0510	1.0404	-1.6573	3.2213	3.2202

Table 2 : Comparison of the characteristics of different optimal portfolios.

Target values	Mean:t_1	Variance: t_2	Skewness: t_3
$\lambda = \frac{1}{3}$	1.3	0.5	0
$\lambda = \frac{1}{2}$	1.6	0.8	-0.6823
$\lambda = \frac{2}{3}$	1.8	1.2	-2

Table 3: Selected target values depending on λ.

Table 1 gives strategies to invest in the seven assets in order to obtain best portfolios for each of the three values of the parameter. More precisely, no investment in ξ_3 and ξ_7, greater investment in ξ_1 and ξ_4 for greater values of λ.
We can observe the impact of the parameter λ variation through the following histogram.

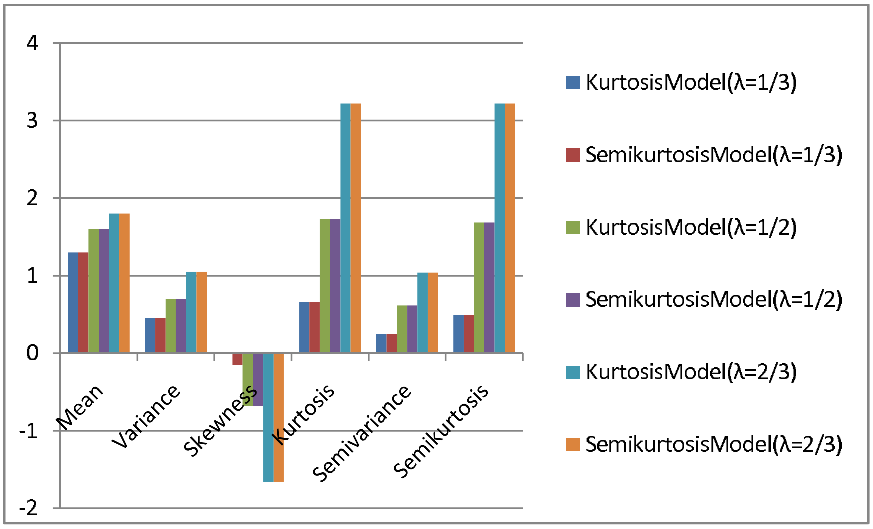

Figure 2: Illustration of fuzzy variables characteristics from different models with Huang's data.

According to the results of Table 2, we can make the following observations:
- The mean increases with respect to λ, it can be interpreted as follows: the more the optimistic parameter is greater (the more self confident the investor is), the greater is the expected return which represents its benefit. In addition, risk measures such as variance, kurtosis, semi-variance, semi-kurtosis increase with respect to λ. It means that, when the investor is self confident in investment and he is looking for greater benefits, meanwhile he is exposed to greater risk.
- The skewness is decreasing with respect to λ. It means that returns' spread is more greater on left of the mean when λ increases. Thus, the investor is exposed to loss (negative returns) when he is more self confident.
- Notice that, for a given value of λ, it is not always possible to get investment proportions with some targets values. By varying λ as in Table 3, one can obtain models that converge to a unique solution with some given target values.
- For $\lambda = \frac{2}{3}$, the two proposed models coincide. For further research, it is interesting to determine different optimistic parameter's values or the minimum optimistic parameter's value such that the two models coincide.

To end this Section, we implement the two proposed models by using real data as Tokyo stock exchange data.

4.3 Implementation of the two models with Tokyo stock exchange data

We implement Tokyo stock exchange data through the two previous models for $\lambda = \frac{1}{4}$, $\lambda = \frac{1}{3}$ and $\lambda = \frac{1}{2}$. The target values t_1, t_2, t_3 are given in Table 6. Different optimal investment proportions and optimal characteristics are respectively given in Tables 4 and 5.

Asset i	1 (%)	2 (%)	3 (%)	4 (%)	5 (%)	6 (%)	7 (%)	8 (%)	9 (%)	10 (%)
$(K_{\lambda=\frac{1}{4}})$	0.00	0.00	0.00	45.30	0.00	0.00	0.00	54.69	0.00	0.00
$(K^S_{\lambda=\frac{1}{4}})$	0.05	0.04	0.05	43.77	0.06	0.05	0.07	55.84	0.04	0.03
$(K_{\lambda=\frac{1}{3}})$	0.00	0.00	0.00	16.83	0.00	0.00	0.00	83.16	0.00	0.00
$(K^S_{\lambda=\frac{1}{3}})$	0.00	0.00	0.00	16.83	0.00	0.00	0.00	83.16	0.00	0.00
$(K_{\lambda=\frac{1}{2}})$	48.51	0.00	0.00	28.78	0.00	0.00	18.87	3.75	0.00	0.00
$(K^S_{\lambda=\frac{1}{2}})$	48.37	0.00	0.00	31.71	0.00	0.00	17.56	2.25	0.00	0.00

Table 4: Optimal selection from each model.

	Mean	Variance	SV	Skewness	Kurtosis	SK
$\lambda = \frac{1}{4}$(K)	0.02	0.0262	0.0047	0.012	0.0049	1.6745×10^{-4}
$\lambda = \frac{1}{4}$(K^S)	0.0201	0.0264	0.0047	0.0122	0.0050	1.7094×10^{-4}
$\lambda = \frac{1}{3}$(K)	0.07	0.0285	0.0114	0.0141	0.0058	6.6001×10^{-4}
$\lambda = \frac{1}{3}$(K^S)	0.07	0.0286	0.0114	0.0142	0.0058	6.6330×10^{-4}
$\lambda = \frac{1}{2}$ (K)	0.09	0.04	0.0389	0.01	0.0083	0.0037
$\lambda = \frac{1}{2}$ (K^S)	0.09	0.04	0.0385	0.01	0.0081	0.0036

Table 5 : Comparison of the characteristics of different optimal portfolios.

Target values	Mean:t_1	Variance: t_2	Skewness: t_3
$\lambda = \dfrac{1}{4}$	0.02	0.04	0.01
$\lambda = \dfrac{1}{3}$	0.07	0.04	0.01
$\lambda = \dfrac{1}{2}$	0.09	0.04	0.01

Table 6: Selected target values depending on λ.

As we can observe from Tables 5 and 2, optimal portfolios characteristics except the skewness (skewness neither describes benefit nor risk), vary in the same manner with respect to the parameter λ whatever data are described.

The following histogram provides for Tokyo stock exchange data, clear observations of those characteristics variations according to the model and the parameter λ.

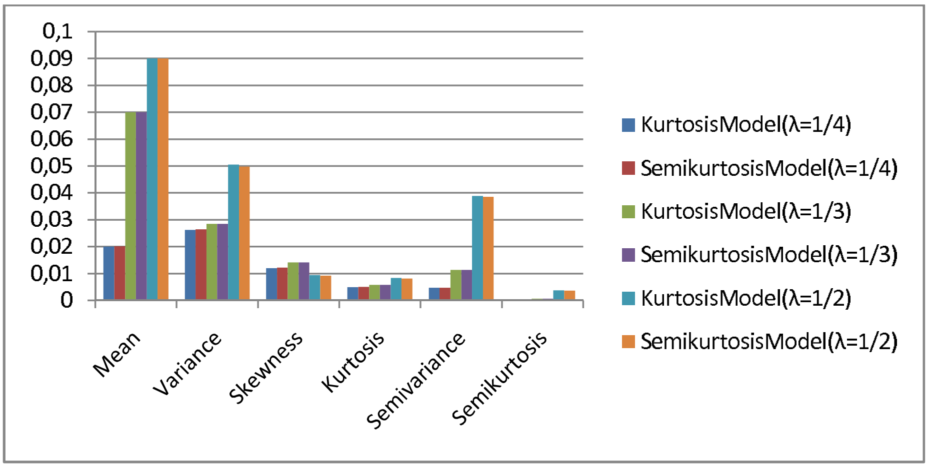

Figure 3: Illustration of fuzzy variables characteristics from different models with Tokyo stock exchange data.

5 Concluding Remarks

In this paper, we have determined two useful axioms of the $m_\lambda-$ measure of fuzzy events combining possibility and necessity measures with an optimistic parameter which describes investor attitude to make a decision in uncertain situations. By means of the m_λ-measure and its axioms, we have defined and determined properties of some characteristics (four first moments and two first semi-moments) of a fuzzy variable. These moments and semi-moments have been determined and analyzed for trapezoidal and triangular fuzzy variables. The obtained results generalize those obtained earlier by Liu [6], Li et al. [5], Huang et al. [4] and Sadefo et al. [9]. Those theoretical results have been illustrated in portfolio selection with fuzzy returns in Finance. We have proposed and implemented two models which led to a huge variety of solutions in portfolio selection according to some given target values. We have displayed the impact of the variation of the optimistic parameter on the investment's risk and the benefit: the benefit and the risk increase with respect to the optimistic parameter.
For further research, we propose to tackle some interesting questions among

which: (i) the determination of k-moments and $2k$-moments (for $k \in \mathbb{N}$) of a fuzzy variable (ii) the study of dominance on fuzzy variables based on the new measure and (iii) the study of partial lower moments of a fuzzy variable based on the new measure.

6 Appendix

Proof of Proposition 2 (linearity of the mean): According to the expressions recalled in relations (4) and (5) and the notions of optimistic and pessimistic functions respectively given by:
$X_{\sup}(\alpha) = \sup\{r/Pos\{x \in X/\xi(x) \geq r\} \geq \alpha\}$ and $X_{\sup}(\alpha) = \sup\{r/Pos\{x \in X/\xi(x) \geq r\} \geq \alpha\}$ with $\alpha \in (0, 1]$, Liu and Liu [7] proved that:

1. $\overline{E}[\xi] = \int_0^1 X_{\sup}(\alpha)d\alpha$ and $\underline{E}[\xi] = \int_0^1 X_{\inf}(\alpha)d\alpha$.
2. $\overline{E}[\xi + \eta] = \overline{E}[\xi] + \overline{E}[\eta]$ and $\underline{E}[\xi + \eta] = \underline{E}[\xi] + \underline{E}[\eta]$.
3. For $a \geq 0$, $\overline{E}[a\xi] = a\overline{E}[\xi]$ and $\underline{E}[a\xi] = a\underline{E}[\xi]$.
4. For $a \leq 0$, $\overline{E}[a\xi] = a\underline{E}[\xi]$ and $\underline{E}[a\xi] = a\overline{E}[\xi]$.

Furthermore, according to relation (6), we have:
For $\lambda \in \mathbb{R}$, $E_\lambda[\xi] = \lambda\overline{E}[\xi] + (1 - \lambda)\underline{E}[\xi]$ and thus:
$E_\lambda[\xi + \eta] = \lambda\overline{E}[\xi + \eta] + (1 - \lambda)\underline{E}[\xi + \eta]$, that is,
$E_\lambda[\xi + \eta] = [\lambda\underline{E}[\xi] + (1 - \lambda)\underline{E}[\xi]] + [\lambda\underline{E}[\eta] + (1 - \lambda)\underline{E}[\eta]] = E_\lambda[\xi] + E_\lambda[\eta]$.
On the other hand, we have for a nonnegative real number α:
$E_\lambda[\alpha\xi] = \lambda\overline{E}[\alpha\xi] + (1 - \lambda)\underline{E}[\alpha\xi]$=$\alpha(\lambda\underline{E}[\xi] + (1 - \lambda)\underline{E}[\xi]]) = \alpha E_\lambda[\xi]$.

Proof of Proposition 4: Let $\xi = (a, b, c, d)$ be a trapezoidal fuzzy variable with expected value e_λ and $\lambda \in [0, 1]$.

- If $a \leq e_\lambda \leq b$ then $V^S_\lambda[\xi] = \lambda\frac{(e_\lambda - a)^3}{3\alpha}$ and $K^S_\lambda[\xi] = \lambda\frac{(e_\lambda - a)^5}{5}$.
  We have: $(V^S)'_\lambda[\xi] = \frac{(e_\lambda - a)^3}{3\alpha} + \lambda\frac{e'_\lambda(e_\lambda - a)^2}{\alpha} \geq 0$ and $(K^S)'_\lambda[\xi] = \frac{(e_\lambda - a)^5}{5\alpha} + \lambda\frac{e'_\lambda(e_\lambda - a)^4}{\alpha} \geq 0$.

- If $b \leq e_\lambda \leq c$ then $V^S_\lambda[\xi] = \lambda\frac{(e_\lambda - a)^3}{3\alpha} - \lambda\frac{(e_\lambda - b)^3}{3\alpha}$ and $K^S_\lambda[\xi] = \lambda\frac{(e_\lambda - a)^5}{5} - \lambda\frac{(e_\lambda - b)^5}{5\alpha}$.
  We have: $(V^S)'_\lambda[\xi] = \frac{(e_\lambda - a)^3}{3\alpha} - \frac{(e_\lambda - b)^3}{3\alpha} + \lambda\frac{e'_\lambda(e_\lambda - a)^2}{\alpha} - \lambda\frac{e'_\lambda(e_\lambda - b)^2}{\alpha} \geq 0$ and $(K^S)'_\lambda[\xi] = \frac{(e_\lambda - a)^5}{5\alpha} - \frac{(e_\lambda - a)^5}{5\alpha} + \lambda\frac{e'_\lambda(e_\lambda - a)^4}{\alpha} - \lambda\frac{e'_\lambda(e_\lambda - b)^4}{\alpha} \geq 0$.

- If $c \leq e_\lambda \leq d$ then $V^S_\lambda[\xi] = \lambda\frac{(e_\lambda - a)^3}{3\alpha} - \lambda\frac{(e_\lambda - b)^3}{3\alpha} + (1 - \lambda)\frac{(e_\lambda - c)^3}{3\beta}$ and $K^S_\lambda[\xi] = \lambda\frac{(e_\lambda - a)^5}{5} - \lambda\frac{(e_\lambda - b)^5}{5\alpha} + (1 - \lambda)\frac{(e_\lambda - c)^5}{5\beta}$.
  We have: $(V^S)'_\lambda[\xi] = \frac{(e_\lambda - a)^3}{3\alpha} - \frac{(e_\lambda - b)^3}{3\alpha} + \lambda\frac{e'_\lambda(e_\lambda - a)^2}{\alpha} - \lambda\frac{e'_\lambda(e_\lambda - b)^2}{\alpha} -$

$\frac{(e_\lambda-c)^3}{3\beta}+(1-\lambda)\frac{e'_\lambda(e_\lambda-c)^2}{\beta}$ and $(K^S)'_\lambda[\xi]=\frac{(e_\lambda-a)^5}{5\alpha}-\frac{(e_\lambda-a)^5}{5\alpha}+\lambda\frac{e'_\lambda(e_\lambda-a)^4}{\alpha}-\lambda\frac{e'_\lambda(e_\lambda-b)^4}{\alpha}-\frac{(e_\lambda-c)^5}{5\alpha}+(1-\lambda)\frac{e'_\lambda(e_\lambda-c)^4}{\beta}$.
We easily check that:
$\lambda\leq\frac{3}{4}\Rightarrow(V^S)'_\lambda[\xi]\geq 0$ and $\lambda\leq\frac{5}{6}\Rightarrow(K^S)'_\lambda[\xi]\geq 0$. $\square$

Proof of Proposition 5 1) Let us prove that $0\leq V^S_\lambda[\xi]\leq V_\lambda[\xi]$. Let $x\in X$ and $r\in\mathbb{R}$. We have:
$[(\xi-e_\lambda)^-]^2=\begin{cases}(\xi-e_\lambda)^2 \text{ if } \xi\leq e_\lambda\\ 0 \text{ if } \xi>e_\lambda\end{cases}$. This definition leads us to two cases:
i) First case: $\xi(x)\leq e_\lambda$.
We have: $[(\xi(x)-e_\lambda)^-]^2=(\xi(x)-e_\lambda)^2$.
Therefore, $[(\xi(x)-e_\lambda)^-]^2\geq r\Leftrightarrow(\xi(x)-e_\lambda)^2\geq r$.
ii) Second case: $\xi(x)>e_\lambda$.
We have: $[(\xi(x)-e_\lambda)^-]^2=0$ which implies that $(\xi(x)-e_\lambda)^2\geq[(\xi(x)-e_\lambda)^-]^2$.
Thus, $[(\xi(x)-e_\lambda)^-]^2\geq r$ implies $(\xi(x)-e_\lambda)^2\geq r$.
It follows that: $\forall(x,r)\in X\times\mathbb{R},\{x/[(\xi(x)-e_\lambda)^-]^2\geq r\}\subseteq\{x/(\xi(x)-e_\lambda)^2\geq r\}$.
By the fact that m_λ is monotone, we obtain:
$\forall r\in\mathbb{R}, m_\lambda\{[(\xi(x)-e_\lambda)^-]^2\geq r\}\leq m_\lambda\{(\xi(x)-e_\lambda)^2\geq r\}$.
Finally, $V_\lambda[\xi]=\int_0^{+\infty}m_\lambda\{(\xi-e_\lambda)^2\geq r\}dr\geq\int_0^{+\infty}m_\lambda\{[(\xi-e_\lambda)^-]^2\geq r\}dr=V^S_\lambda[\xi]$.
In the same manner, one can prove that $0\leq k^S_\lambda[\xi]\leq K_\lambda[\xi]$.

2) Let ξ be a symmetric fuzzy variable. Let us prove that $V^S_\lambda[\xi]=V_\lambda[\xi]$.
Let us write: $V^S_\lambda[\xi]=E_\lambda[[(\xi-e_\lambda)^-]^2]=\int_0^{+\infty}\Updownarrow_\lambda\{[(\xi-e_\lambda)^-]^2\geq r\}dr=\int_0^{+\infty}\Updownarrow_\lambda\{(\xi-e_\lambda)^-\in]-\infty;-\sqrt{r}]\cup[\sqrt{r};+\infty[\}dr$ and
$V_\lambda[\xi]=E_\lambda[(\xi-e_\lambda)^2]=\int_0^{+\infty}\Updownarrow_\lambda\{(\xi-e_\lambda)^2\geq r\}dr=\int_0^{+\infty}\Updownarrow_\lambda\{(\xi-e_\lambda)\in]-\infty;-\sqrt{r}]\cup[\sqrt{r};+\infty[\}dr$.
If μ and ν are respectively membership functions of fuzzy variables $\xi-e_\lambda$ and $(\xi-e_\lambda)^-$, then we have $\nu=\begin{cases}\mu \text{ if } \xi-e_\lambda\leq 0\\ 0 \text{ otherwise}\end{cases}$. Let us observe that ξ is symmetric with respect to e_λ if and only if $\xi-e_\lambda$ is symmetric with respect to 0. We have, $\forall r>0$:
$\mathcal{M}_\lambda\{(\xi-e_\lambda)\in]-\infty;-\sqrt{r}]\cup[\sqrt{r};+\infty[\}=\lambda\sup_{x\in]-\infty;-\sqrt{r}]\cup[\sqrt{r};+\infty[}\mu(x)+(1-\lambda)(1-\sup_{x\in]-\sqrt{r};-\sqrt{r}[}\mu(x))=\lambda\max(\sup_{x\in]-\infty;-\sqrt{r}]}\mu(x),\sup_{x\in[\sqrt{r};+\infty[}\mu(x))+(1-\lambda)[1-\max(\sup_{x\in]-\sqrt{r};0]}\mu(x),\sup_{x\in[0;\sqrt{r}[}\mu(x))]$.
Let us notice that by fact that ξ is symmetric, we have: $\sup_{x\in]-\infty;-\sqrt{r}]}\mu(x)=\sup_{x\in[\sqrt{r};+\infty[}\mu(x)$ and $\sup_{x\in]-\sqrt{r};0]}\mu(x)=\sup_{x\in[0;\sqrt{r}[}\mu(x)$. so, we have:
$\Updownarrow_\lambda\{(\xi-e_\lambda)\in]-\infty;-\sqrt{r}]\cup[\sqrt{r};+\infty[\}=\lambda\sup_{x\in]-\infty;-\sqrt{r}]}\mu(x)+(1-\lambda)[1-\sup_{x\in]-\sqrt{r};0]}\mu(x))$.
However, $\sup_{x\in]-\infty;-\sqrt{r}]}\mu(x)=\sup_{x\in]-\infty;-\sqrt{r}]}\nu(x)$ and $\sup_{x\in]-\sqrt{r};0]}\mu(x)=\sup_{x\in]-\sqrt{r};0]}\nu(x)$, that is:
$\Updownarrow_\lambda\{(\xi-e_\lambda)\in]-\infty;-\sqrt{r}]\cup[\sqrt{r};+\infty[\}=\lambda\sup_{x\in]-\infty;-\sqrt{r}]}\nu(x)+(1-\lambda)[1-\sup_{x\in]-\sqrt{r};0]}\nu(x))$.

By the fact that ν is null on the intervals $[\sqrt{r};+\infty[$ and $[0;\sqrt{r}[$, we finally have:
$\Updownarrow_\lambda\{(\xi-e_\lambda)\in]-\infty;-\sqrt{r}]\cup[\sqrt{r};+\infty[\} = \lambda\sup_{x\in]-\infty;-\sqrt{r}]\cup[\sqrt{r};+\infty[}\nu(x)+(1-\lambda)[1-\sup_{x\in]-\sqrt{r};-\sqrt{r}[}\nu(x)) = \Updownarrow_\lambda\{(\xi-e_\lambda)^-\in]-\infty;-\sqrt{r}]\cup[\sqrt{r};+\infty[\}$.
Therefore, we get $V_\lambda^S[\xi]=V_\lambda[\xi]$.
By a similar way, one can prove that $K_\lambda^S[\xi]=K_\lambda[\xi]$. $\square$

Proof of Lemma 1: 1) Let us consider $A\subset X$ such that $M_\lambda(A)=0$ and $\lambda\in[0,1]$. Let us recall the third axiom of the measure $M_\lambda(A)$:
$$\begin{cases}\lambda m_\lambda(A)+(1-\lambda)m_\lambda(A^c)=\lambda\\ or\\ (1-\lambda)m_\lambda(A)+\lambda m_\lambda(A^c)=\lambda\end{cases}$$

- If $\lambda=0$, then we obtain by Axiom 3:
  $\begin{cases}m_\lambda(A^c)=0\\ or\\ 0=0\end{cases}$. That is, $\lambda m_\lambda(A^c)=0$.
- If $\lambda\in]0,\frac{1}{2}]$, then we obtain by axiom 4:
  $\begin{cases}(1-\lambda)m_\lambda(A^c)=\lambda\\ or\\ \lambda m_\lambda(A^c)=\lambda\end{cases}$ which implies that $\begin{cases}m_\lambda(A^c)=\frac{\lambda}{1-\lambda}\\ or\\ m_\lambda(A^c)=1\end{cases}$.
  Furthermore, $\lambda\in]0,\frac{1}{2}]\Rightarrow 0<m_\lambda(A^c)=\frac{\lambda}{1-\lambda}\leq 1$.
- If $\lambda\in]\frac{1}{2},1[$, then we obtain by Axiom 4:
  $\begin{cases}(1-\lambda)m_\lambda(A^c)=\lambda\\ or\\ \lambda m_\lambda(A^c)=\lambda\end{cases}$ which implies that $\begin{cases}m_\lambda(A^c)=\frac{\lambda}{1-\lambda}\\ or\\ m_\lambda(A^c)=1\end{cases}$.
  However, $\lambda\in]\frac{1}{2},1[\Rightarrow m_\lambda(A^c)=\frac{\lambda}{1-\lambda}>1$, which is wrong. So, we have $m_\lambda(A^c)=1$.
  Finally, for $\lambda=1$, we obtain by Axiom 3:
  $\begin{cases}0=1\\ or\\ m_\lambda(A^c)=1\end{cases}$. However, $0=1$ is wrong. So, $\lambda m_\lambda(A^c)=1$.

2) Let us consider $A\subset X$ and $\lambda\in[\frac{1}{2},1[$.
$\Rightarrow$) If $m_\lambda(A)=0$, then according to Lemma 1, we have $m_\lambda(A^c)=1$.
$\Leftarrow$) If $m_\lambda(A^c)=1$, then by Axiom 3, we have:
$$\begin{cases}\lambda m_\lambda(A)+(1-\lambda)=\lambda\\ or\\ (1-\lambda)m_\lambda(A)+\lambda=\lambda\end{cases}\text{ which implies that }\begin{cases}m_\lambda(A)=\frac{2\lambda-1}{\lambda}\\ or\\ m_\lambda(A)=0\end{cases}.$$
However, $\lambda\in[\frac{1}{2},1[\Rightarrow m_\lambda(A)=\frac{2\lambda-1}{\lambda}<0$, which is wrong. So, we have $m_\lambda(A)=0$. Hence the result. $\square$

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