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Université de Montpellier - MUSE « Montpellier Université d'Excellence »
Contact : alain.marciano@umontpellier.fr

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Fuzzy lower partial moment and Mean-risk dominance: An application for poverty measurement

Christian Deffo Tassak[a], Louis Aimé Fono[b,†] and Jules Sadefo Kamdem[c]

[a]Laboratoire de Mathématiques Appliquées aux Sciences Sociales

Université de Yaoundé I, B.P. 812 Yaoundé, Cameroon, Email: dtassak@yahoo.com

[b]Laboratoire de Mathématiques et Faculté des Sciences

Université de Douala, B.P. 24157 Douala

[c] LAMETA CNRS UMR 5474, DFR SJE, Campus de Troubiran, Université de Guyane, France.

[†] Corresponding author: lfono2000@yahoo.fr; Tel. +(237) 696-02-29-15

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Abstract

A more general concept of risk in economics consists on the chance of getting an income or a return less than a threshold one. Risk has been studied and generalized more earlier by Fishburn [8] through Mean Partial Lower Moment specially when income can be described by a random variable. In this paper, we present a new concept of partial moment, namely Fuzzy Lower Partial Moment (FLPM) based on credibility measure, to quantify risk of getting a return described by a fuzzy variable and we study its properties. Based on FLPM, we introduce mean risk dominance for fuzzy variables, we characterize the dominance for some specific cases and we determine some of its properties. Furthermore, we study the consistency of mean-risk models with respect to first and second order dominances. We display one application of FLPM by introducing a new poverty index for poverty measurement in the context of fuzzy environment and we examine some of its properties.

Key Words: Credibility measure; Fuzzy variable; Fuzzy lower partial moment; Mean-Risk dominance; Poverty index.

1 Introduction

The concept of downside risk has been for a long time a meaningful topic in portfolio selection and management. Since the seminal work of Markowitz [16] where he proposed below-mean semi-variance and more generally below-target semi-variance as risk measures which take into account investors'attitude towards risk, downside risk has been developed more later by Bawa [1] and Fishburn [8] in their respective frameworks based on Lower Partial Moment (LPM) as below-target risk. The first proved that LPM measure is mathematically related to stochastic dominance for risk tolerance values equal to one or two and the second proved the equivalence of the LPM measure to stochastic dominance for all risk tolerance values as strictly nonnegative integers. Kaplan and Knowles [13], Brogan and Stidham [3], Sunoj and Maya [24] also contributed to develop downside risk measure in their recent frameworks. Let us notice that all precede results rely on the fact that assets' returns are described by random variables. However, according to the recent litterature (Zadeh [26], Liu [15], Sadefo et al. [20], Tassak et al. [25]), some situations in real life are related to fuzzy phenomena which induces imprecise and vague concepts. For instance, assets' returns can be expressed as follows: "around 15 F", "between 20F and 30 F ", etc..., and fuzzy variables are used to deal with those concepts. Thus, an open question is to study downside risk for fuzzy variable and its related concepts and applications. The aim of this paper is three folds: introduce and study the Fuzzy Lower Partial Moment (FLPM) based on credibility measure, define and study the mean-risk dominance for two fuzzy variables based on FLPM and, apply FLPM to analyze poverty in the case where source of fuzziness for poverty is due to imprecise estimation of income.

Before we display the roadmap of our article, let us review some frameworks on poverty in order to specify the guideline of our third specific objective. In fact, literature's review on poverty started with Sen [22] who argued that the first issue when measuring poverty is to identify the poor among the total population by using a poverty line which allows to distinguish poor and non poor, and the second issue is the construction of an index to measure poverty's level. Other scholars (Foster [9], Hagenaars [11], Seidls [21], Chakravarty [5] , Ravallion [19] and Zhen ([27], [28], [29]) brought a useful contribution with their respective reviews on poverty measure construction issue. More recently, Gallardo [10] offered an analysis of the state of the art in the conceptualization of vulnerability to poverty through an approach based on the mean-risk dominance criterion which used the downside mean semi-deviation as the risk parameter. However, poverty line research has longer been discussed in poverty measurement literature and led to introduce fuzzi-

ness by the fact people don't always agree on an individual's poverty status. Following that, Zheng [29] indicated three sources of fuzziness for poverty which have been analyzed in literature: fuzziness due to imprecise estimation of income; fuzziness due to the fact that poverty is essentially a multidimensional concept and, fuzziness due to fact that different people may have different ideas about the poverty line which makes an individual poor or non poor. Those sources of fuzziness have been studied by Chakravarty [6], Shorrocks and Subramanian [23], Beliakov [2] and Dombi [7], Qizilbah [18], Cerioli and zani [4], Zheng [29]. Recently, Zheng [29] explored the third source of fuzziness by introducing a meaningful "density function" for a poverty membership function and he proposed axioms for the characterization of additively separable fuzzy poverty measures. In this paper, we do not consider this approach and we focus only on the first source of fuzziness. This can be justified by the fact that in some societies, people have many jobs in order to improve welfare and overcome bad conditions life. More concretely, it has been observed in many african societies that one individual may have several economic activities besides his official and recognized activity with respect to his qualification or competence. Thus, some unknown incomes are undeclared and it is not often possible to know the exact income of one individual.
The paper is organized as follows. Section 2 gives some preliminaries on fuzzy variable and its characteristics. In Section 3, we define Fuzzy Lower Partial Moment (FLPM) based on credibility measure for measuring investment risk of returns described by fuzzy variables. We characterize the FLPM for a particular class of fuzzy variables whose distribution functions belong to family of exponential functions. Based on FLPM, we define the mean-risk dominance for two fuzzy variables and characterize it in some particular cases. Some properties of the new dominance are established and we justify that it is not complete binary relation on fuzzy variables. Moreover, we study the consistency of mean-risk models and justify their optimal portfolios belong to the set of best portfolios with respect to the first or second order dominances based on credibility measure developped earlier by Tassak et al. [25]. Section 4 presents an application of FLPM for poverty measurement. Section 5 gives some concluding remarks.

2 Preliminaries

Let ξ be a fuzzy variable with membership function μ where for any $x \in \mathbb{R}$, $\mu(x)$ represents the possibility that ξ takes value x. ξ is normal if $\exists x_0/\mu(x_0) = 1$. ξ is non-negative fuzzy variable if $\forall x < 0, \mu(x) = 0$. The support of ξ is the crisp set defined by $Supp(\xi) = \{x \in \mathbb{R}, \mu(x) > 0\}$.

Throughout this paper, we will assume that all fuzzy variables are normal, that is, $\exists x_0 \in \mathbb{R}, \mu(x_0) = 1$.

Note that for ξ taking values in B, Zadeh [26] has defined the possibility measure of B by $Pos(\{\xi \in B\}) = \sup_{x\in B} \mu(x)$ and the necessity measure of ξ by $Nec(\{\xi \in B\}) = 1 - \sup_{x\in B^c} \mu(x)$ respectively. But neither of these measures are self-dual. Therefore, Liu and Liu [15] introduced the credibility measure as the average of possibility measure and necessity measure as follows: for any set B,

$$Cr(\{\xi \in B\}) = \frac{1}{2}\left(\sup_{x\in B} \mu(x) \ - \ \sup_{x\in B^c} \mu(x) + 1\right). \tag{1}$$

Credibility measure is self-dual, that is, $Cr(\{\xi \in B\}) + Cr(\{\xi \in B^c\}) = 1$.

Liu and Liu [15] introduced the expected value of ξ defined as follows

$$E[\xi] = e = \int_0^{+\infty} Cr\{\xi \geq r\}\, dr \ - \ \int_{-\infty}^0 Cr\{\xi \leq r\}\, dr \tag{2}$$

provided that at least one of the above integrals is finite. Note that, expected value is one of the most important concept of fuzzy variable which gives the center of its distribution.

Liu [14] also defined the credibility distribution $\Phi : \mathbb{R} \to [0, 1]$ of a fuzzy variable ξ as follows:

$$\forall t \in \mathbb{R}, \Phi(t) = Cr\{\xi \leq t\} = \frac{1}{2}[1 + \sup_{x\in]-\infty;t]} \mu(x) - \sup_{x\in]t;+\infty[} \mu(x)]. \tag{3}$$

When Φ is absolutely continuous, we have the credibility density function $\phi : \mathbb{R} \to [0, \infty[$ such that

$$\forall t \in \mathbb{R}, \Phi(t) = \int_{-\infty}^t \phi(u)du. \tag{4}$$

Obviously, we have $\int_{-\infty}^{+\infty} \phi(u)du = 1$.

ϕ belongs to exponential family if there exists $\gamma > 0$, $K(\cdot)$ an arbitrary function and $D(\cdot)$ a differentiable function such that

$$\phi(x) = e^{\gamma\, x + K(x) + D(\gamma)}, \qquad x \in (0, \infty). \tag{5}$$

Notice that, $\Phi(x^-) = \sup\{\Phi(y), y < x\}$ is the credibility of getting a value of ξ not exceeding x. A distribution function Φ of ξ is a non-degenerate distribution function if $\forall t \in \mathbb{R}$, $t \neq E(\xi) \Rightarrow \Phi(t) \neq 0$.

The distribution function Φ of a fuzzy number $\xi = (a, b, c, d)$ is defined by:

$$\forall r \in \mathbb{R}, \Phi(r) = \begin{cases} 0 \text{ if } r < a \\ \frac{1}{2}\mu(r) \text{ if } a \leq r < b \\ \frac{1}{2} \text{ if } b \leq r < c \\ 1 - \frac{1}{2}\mu(r) \text{ if } c \leq r < d \\ 1 \text{ if } d \leq r \end{cases} . \tag{6}$$

It is an increasing function, that means, if $x \in [a, b], y \in [b, c]$, and $z \in [c, d]$, then $\Phi(x) \leq \Phi(y) \leq \Phi(z)$.

In the next Section, we will introduce and study the lower partial moment of a fuzzy variable ξ, namely, the Fuzzy Lower Partial Moment (FLPM) of ξ and the Mean risk dominance between fuzzy variables.

3 Fuzzy Lower Partial Moments and Mean-risk dominance of fuzzy variables

3.1 Definitions, specific cases and examples

Definition 1. *Let ξ be a fuzzy variable, $\tau \in \mathbb{R}$ and $n \in \mathbb{N}^*$. The Fuzzy Lower Partial Moment (FLPM) of ξ with order n and target value τ is defined as*

$$FLPM_{n,\tau}[\xi] = E[\max(\tau - \xi, 0)^n] \tag{7}$$

where the expectation operator E is defined by (2).

Throughout this paper, we will simply say "FLPM of ξ" instead of "The Fuzzy Lower Partial Moment (FLPM) of ξ with order n and target value τ."

Let us deduce some values of a target value or of the order for which the FLPM of ξ becomes some well-known statistics parameters of ξ and some new fuzzy counterparts of well-known notions of the Probability Theory. In addition, we express the FLPM of a fuzzy variable by means of its credibility distribution function.

Remark 1. *Let ξ be a fuzzy variable, $\tau \in \mathbb{R}$ and $n \in \mathbb{N}^*$.*

1. *If the target value $\tau = E[\xi]$, then we obtain the following notions:*

- *For* $n \in 2\mathbb{N} - \{0\}$, $FLPM_{n,\tau}[\xi]$ *is the semi-moment of order* n *of* ξ *introduced earlier by Sadefo et al.[20].*
- *For* $n = 4$, $FLPM_{n,\tau}[\xi]$ *becomes the fuzzy semi-kurtosis of* ξ *introduced by Sadefo et al.[20].*
- *For* $n = 2$, $FLPM_{n,\tau}[\xi]$ *becomes the fuzzy semi-variance of* ξ *introduced by Huang [12].*

2. *For some specific values of* n, *we obtain fuzzy counterparts of some well-known notions of the downside risk under probability theory:*

   - *We introduce the so-called credibility of loss equals to the* 0^{th} *order* $FLPM_{0,\tau}$ *of* ξ.
   - *In the case where* $n = 1$, $FLPM_{1,\tau} = E[\max(\tau - \xi, 0)]$ *is called the expected loss of* ξ. *Here the constant target value* τ *can be considered as the threshold point separating returns in two parts: downside returns and upside returns depending of the threshold.*
   - *For* $n = 2$, $FLPM_{2,\tau} = E[\max(\tau - \xi, 0)^2]$ *is the Credibilistic Fuzzy Target Semi-Variance (CFTS).*
   - *By setting* $n = 3$ *(resp.* $n = 4$), *we obtain* $FLPM_{3,\tau}$ *(resp.* $FLPM_{4,\tau}$*) which is the Credibilistic Fuzzy Target Semi-Skewness (CFTSS) (resp. the Credibilistic Fuzzy Target Semi-Kurtosis (CFTSV)).*

3. *The expression of* $FLPM_{n,\tau}$ *with respect to credibility distribution is:*

$$FLPM_{n,\tau}[\xi] = \int_0^{+\infty} Cr\{\max(\tau - \xi, 0)^n \geq r\}\, dr = n \int_{-\infty}^{\tau} (\tau - u)^{n-1}\Phi(u)du. \tag{8}$$

4. *If* Φ *has a derivative and* ξ *has a lower bounded support then:*

$$FLPM_{n,\tau}[\xi] = \int_{-\infty}^{\tau} (\tau - u)^n d\Phi(u) \tag{9}$$

$$= \int_{-\infty}^{\tau} (\tau - u)^n \phi(u). \tag{10}$$

   The previous definition of $FLPM_{n,\tau}$ *justifies that it is a function of the credibility distribution function and, it is a non-decreasing function of its target* τ *return. As* τ *increases,* $FLPM_{n,\tau}$ *also increases.*

We end this Section with the expressions of FLPM of trapezoidal and triangular fuzzy variables.

Corollary 1. *1. The FLPM of the trapezoidal fuzzy variable $\xi = (a, b, c, d)$ is:*

$$FLPM_{n,\tau}[\xi] = \begin{cases} 0 \;\; if\; \tau < a \\ \frac{(\tau-a)^{n+1}}{2(\alpha+1)(b-a)} \;\; if\; a \leq \tau < b \\ \frac{[(\tau-a)^{n+1}-(\tau-b)^{n+1}]}{2(n+1)(b-a)} \;\; if\; b \leq \tau < c \\ \frac{[(\tau-a)^{n+1}-(\tau-b)^{n+1}]}{2(n+1)(b-a)} + \frac{(\tau-c)^{n+1}}{2(n+1)(d-c)} \;\; if\; c \leq \tau < d \\ \frac{[(\tau-a)^{n+1}-(\tau-b)^{n+1}]}{2(n+1)(b-a)} + \frac{[(\tau-c)^{n+1}-(\tau-d)^{n+1}]}{2(n+1)(d-c)} \;\; if\; \tau \geq d \end{cases}. \tag{11}$$

2. The FLPM of the triangular fuzzy number $\xi = (a, b, d)$ is:

$$FLPM_{n,\tau}[\xi] = \begin{cases} 0 \;\; if\; \tau < a \\ \frac{(\tau-a)^{n+1}}{2(n+1)(b-a)} \;\; if\; a \leq \tau < b \\ \frac{[(\tau-a)^{n+1}-(\tau-b)^{n+1}]}{2(n+1)(b-a)} + \frac{(\tau-b)^{n+1}}{2(n+1)(d-b)} \;\; if\; b \leq \tau < d \\ \frac{[(\tau-a)^{n+1}-(\tau-b)^{n+1}]}{2(n+1)(b-a)} + \frac{[(\tau-b)^{n+1}-(\tau-d)^{n+1}]}{2(n+1)(d-b)} \;\; if\; \tau \geq d \end{cases}. \tag{12}$$

In the next Subsection, we establish some properties of the $FLPM$ of a fuzzy variable ξ based on its absolutely continuous credibility distribution function.

3.2 Some results on $\mathbf{FLPM}_{n,\tau}$

The following result determines the credibility distribution function Φ of ξ in term of derivatives of its $FLPM$ when Φ has a compact support. More precisely, it suggests that we can determine the credibility distribution $\Phi(\tau)$ uniquely given $\mathrm{FLPM}_{n,\tau}$ with $n \in \mathbb{N}^*$.

Proposition 1. *The credibility distribution function Φ of a fuzzy variable ξ with a lower bounded support satisfies the following relation:*

$$\frac{d^n}{d\tau^n} FLPM_{n,\tau} = n!\, \Phi(\tau), \;\; that\ is, \;\; \Phi(\tau) = \frac{1}{n!} \frac{d^n}{d\tau^n} FLPM_{n,\tau}. \tag{13}$$

Proof: Let Φ be the credibility distribution function with compact support of the fuzzy variable ξ. We have: $\frac{d^n}{d\tau^n}\mathrm{FLPM}_{n,\tau} = \frac{d^n}{d\tau^n}[\int_{-\infty}^{\tau}(\tau-u)^n d\Phi(u)] = \int_{-\infty}^{\tau} \frac{d^n}{d\tau^n}[(\tau-u)^n d\Phi(u)]$. It is easy to check that $\forall n \in \mathbb{N}^*$, $\frac{d^n}{d\tau^n}(\tau-u)^n = n!$

and finally, we have:

$$\frac{d^n}{d\tau^n}\mathrm{FLPM}_{n,\tau} = \int_{-\infty}^{\tau} n! d\Phi(u) = n! \int_{-\infty}^{\tau} d\Phi(u) = n![\Phi(\tau) - \lim_{u \to -\infty} \Phi(u)]$$
$$= n![\Phi(\tau) - 0] = n!\Phi(\tau).$$

Hence the result. □

The following result determines necessary and sufficient conditions on a FLPM under which the density function ϕ of ξ, satisfying a particular inequality, belongs to exponential family.

Proposition 2. *Let ϕ_γ be the credibility density function of a nonnegative fuzzy variable ξ satisfying the following condition:*

$$\frac{d}{d\gamma}\phi_\gamma(u) \geq \phi_\gamma(u)(u + D'(\gamma)), \forall u \in (0, \infty) \tag{14}$$

where $D'(\gamma)$ is the derivative of $D(\cdot)$ with respect to γ.
ϕ_γ belongs to exponential family, that means,

$$\phi_\gamma(u) = e^{\gamma u + K(u) + D(\gamma)}, \qquad u \in (0, \infty), \quad \gamma > 0, \tag{15}$$

where $K(\cdot)$ is an arbitrary function, if and only if, its $FLPM_{\alpha,\tau}$ satisfy the following recurrence relationship:

$$FLPM_{n+1,\tau} = (\tau + D^{'}(\gamma))FLPM_{n,\tau} - \frac{d}{d\gamma}FLPM_{n,\tau}. \tag{16}$$

Proof: ($\Rightarrow$) Assume that the credibility density function ϕ_γ is defined by: $\phi_\gamma(u) = e^{\gamma u + K(u) + D(\gamma)}$ where $u \in (0, \infty)$, $\gamma > 0$, K and D two arbitrary functions.
By computing the derivative of ϕ_γ given by relation (15) with respect to γ, one can easily check that ϕ_γ satisfies relation (14).
Let us prove that $\mathrm{FLPM}_{n+1,\tau} = (\tau + D^{'}(\gamma))\mathrm{FLPM}_{n,\tau} - \frac{d}{d\gamma}\mathrm{FLPM}_{n,\tau}$.

We have:

$$\begin{aligned}
\frac{d}{d\gamma}\text{FLPM}_{n,\tau} &= \frac{d}{d\gamma}[\int_0^\tau (\tau-u)^\alpha e^{\gamma\, u+K(u)+D(\gamma)}du] \\
&= \int_0^\tau (\tau-u)^n \frac{d}{d\gamma} e^{\gamma\, u+K(u)+D(\gamma)}du \\
&= \int_0^\tau (\tau-u)^n (u+D'(\gamma)) e^{\gamma\, u+K(u)+D(\alpha)}du \\
&= \int_0^\tau u(\tau-u)^n e^{\gamma\, u+K(u)+D(\gamma)}du + \int_0^\tau D'(\gamma)(\tau-u)^n e^{\gamma\, u+K(u)+D(\gamma)}du \\
&= \int_0^\tau (u-\tau+\tau)(\tau-u)^n e^{\gamma\, u+K(u)+D(\gamma)}du + D'(\gamma)\text{FLPM}_{n,\tau} \\
&= -\int_0^\tau (\tau-u)^{n+1} e^{\gamma\, u+K(u)+D(\gamma)}du + \tau\int_0^\tau (\tau-u)^n e^{\gamma\, u+K(u)+D(\gamma)}du + D'(\gamma)\text{FLPM}_{n,\tau} \\
&= -\text{FLPM}_{n+1,\tau} + (\tau+D'(\gamma))\text{FLPM}_{n,\tau}.
\end{aligned}$$

Hence the result.

($\Leftarrow$) Now we prove the sufficient condition.
By means of relation (**??**), relation (16) can be expressed as follows:
$\int_0^\tau (\tau-u)^{\alpha+1}\phi_\gamma(u)\, du = (\tau+D'(\gamma))\int_0^\tau (\tau-u)^n\phi_\gamma(u)\, du - \frac{d}{d\gamma}\int_0^\tau (\tau-u)^{\alpha+1}\phi_\gamma(u)\, du$
which implies that:

$$\int_0^\tau [(\tau-u)^n\phi(u)(-u-D'(\gamma)) + \frac{d}{d\gamma}((\tau-u)^n\phi_\gamma(u))]\, du = 0 \qquad (17)$$

By using the fact that ϕ_γ satisfies relation (14), relation (17) traduces the nullity of the integration of a positive function.
Thus, we obtain:

$$\frac{d}{d\gamma}((\tau-u)^n\phi_\gamma(u)) = (\tau-u)^n\phi_\gamma(u)(u+D'(\gamma)) \qquad (18)$$

Finally, by integrating each side of relation (18) with respect to γ, we obtain:
$(\tau-u)^n\phi_\gamma(u) = ke^{\gamma u+D(\gamma)}$, $k>0$, which leads to:
$\phi_\gamma(u) = e^{\gamma u+D(\gamma)+K(u)}$ with $K(u) = \ln(\frac{k}{((\tau-u)^n)})$, $u \in]0;\tau[$.
It suffices to consider the function ϕ_γ defined as:
$\phi_\gamma(u) = e^{\gamma u+D(\gamma)+K(u)}$ with $K(u) = \ln|(\frac{k}{((\tau-u)^n)}|$, $u \in (0;+\infty)\setminus\{\tau\}$ and $K(\tau)=0$. $\square$

We end this Subsection with the following useful result establishing necessary and sufficient condition under which the FLPM of ξ is null. In addition, it gives expression of the expected loss of ξ (its FLPM for $n = 1$) by means of the distribution function of ξ.

Proposition 3. *Let ξ be a fuzzy variable, Φ its credibility distribution function, $n \in \mathbb{N}^*$ and $\tau \in \mathbb{R}$.*

1.

$$FLPM_{n,\tau}[\xi] = 0 \Leftrightarrow \Phi(\tau^-) = 0. \tag{19}$$

2.

$$If \;\; n = 1, \quad then \;\; \Phi(\tau) = FLPM_{n,\tau}[\xi]. \tag{20}$$

Proof: 1) ($\Rightarrow$) Assume that $\mathrm{FLPM}_{n,\tau}[\xi] = 0$, then (8) implies $\forall r \in \mathbb{R}, r < \tau \implies \Phi(r) = 0$, that means, $\Phi(\tau^-) = \sup\{\Phi(r), r < \tau\} = 0$.
($\Leftarrow$) If $\Phi(\tau^-) = 0$, then the inequality $\Phi(r) \geq 0$ implies $\forall r \in \mathbb{R}, r < \tau \implies \Phi(r) = 0$. According to the relation (8), the previous implication $\mathrm{FLPM}_{n,\tau}[\xi] = 0$.

2) Assume that $n = 1$.
According to the relation (8), we have: $\mathrm{FLPM}_{n,\tau}[\xi] = n\int_{-\infty}^{\tau} \phi(u)du = n\Phi(\tau) = \Phi(\tau)$. $\square$

In the following Subsection, we introduce and study Mean Risk dominance of fuzzy variables and the consistency of mean-risk models.

3.3 Mean-risk dominance based on $\mathrm{FLPM}_{n,\tau}$

3.3.1 Definition and properties

We now define the fuzzy mean-risk dominance relation based on FLPM.

Definition 2. *Let $n \in \mathbb{N}^*$ and $\tau \in \mathbb{R}$.*
The fuzzy mean-risk dominance is the binary relation on the set of fuzzy variables denoted by $\succeq_{n,\tau}$ and defined as follows: For two ξ_1 and ξ_2 be two fuzzy variables,

$$\xi_1 \succeq_{n,\tau} \xi_2 \; if \; \begin{cases} E[\xi_1] \geq E[\xi_2] \\ FLPM_{n,\tau}[\xi_1] \leq FLPM_{n,\tau}[\xi_2] \end{cases}. \tag{21}$$

The strict dominance denoted by $\succ_{n,\tau}$ is defined with at least one strict inequality in (21).

The following result characterizes the new dominance relation $\succeq_{n,\tau}$ in the three following cases: (1) the two fuzzy variables have disjoint supports and

τ is less than the minimum of the lower bounds of the two supports, (2) the two fuzzy variables are symmetric and τ is between the lower bounds of the two supports and (3) one of the two fuzzy variables is a crisp number and the other one is a fuzzy variable with τ as its upper bound.
Notice that the three results of this theorem can be interpreted as follows:

1. The first case means that, in absence of risk, the "best" fuzzy variable is the one with greater expected return.

2. According to the second case, when two distributions have equal means, it is more suitable to choose the less risky one.

3. The third case reveals that: if two distributions have the same expected return value which is below to the target, in the most case , the "best" distribution is the one which make "certain" to get this value.

We now state our result.

Theorem 1. *Let ξ_1 and ξ_2 be two fuzzy variables with distribution functions Φ_1 and Φ_2. Assume that (21) holds. Then:*

1. *If $\Phi_1(\tau^-) = \Phi_2(\tau^-) = 0$, then $\xi_1 \succ_{n,\tau} \xi_2$ if and only if $E[\xi_1] > E[\xi_2]$.*

2. *If* $\begin{cases} E[\xi_1] = E[\xi_2] \\ \Phi_1(\tau^-) = 0 \\ \Phi_2(\tau^-) > 0 \end{cases}$, *then $\xi_1 \succ_{n,\tau} \xi_2$.*

3. *If $E[\xi_1] = E[\xi_2] = \tau - r$ (with $r > 0$), Φ_1 is a degenerate distribution that assigns credibility 1 to $\tau - r$ with $r > 0$, and Φ_2 is a non-degenerate distribution with $\Phi_2(\tau) = 1$, then:*

$$\xi_1 \succ_{n,\tau} \xi_2 \text{ if and only if } n > 1.$$

To establish this proof, we recall the Jensens' Inequality for fuzzy variable introduced earlier by Liu [14] (Theorem 1.59, page 68): "Let ξ be a fuzzy variable and $f : \mathbb{R} \to \mathbb{R}$ a strictly convex function. If $E[\xi]$ and $E[f(\xi)]$ are finite, then $f(E[\xi]) < E[f(\xi)]$."

We now establish the proof of the Theorem.
Proof: 1) Let us assume that $\Phi_1(\tau^-) = \Phi_2(\tau^-) = 0$.
By relation (19), we have $\mathrm{FLPM}_{n,\tau}[\xi_1] = \mathrm{FLPM}_{n,\tau}[\xi_2] = 0$.
($\Rightarrow$) Assume on the contrary that $\xi_1 \succ_{\alpha,\tau} \xi_2$ and $E[\xi_1] \leq E[\xi_2]$. This inequality and the equality imply that there is not any strict inequality between the means or the fuzzy lower partial moments of the fuzzy variables ξ_1 and ξ_2.

This contradicts $\xi_1 \succeq_{n,\tau} \xi_2$. Therefore, we have: $E[\xi_1] > E[\xi_2]$.
($\Leftarrow$) Assume that $E[\xi_1] > E[\xi_2]$. Thus, the equality $\text{FLPM}_{n,\tau}[\xi_1] = \text{FLPM}_{n,\tau}[\xi_2] = 0$ and the definition of $\succ_{n,\tau}$ imply $\xi_1 \succ_{n,\tau} \xi_2$.
2) Assume that $E[\xi_1] = E[\xi_2], \Phi_1(\tau^-) = 0, \Phi_2(\tau^-) > 0$.
That means $\text{FLPM}_{n,\tau}[\xi_1] = 0$ and $\text{FLPM}_{n,\tau}[\xi_2] > 0$, according to relation (19).
3) Let us assume that Φ_1 is a degenerate distribution that assigns credibility 1 to $\tau - r$ with $r > 0$, and Φ_2 is a non-degenerate distribution that has $\Phi_2(\tau) = 1$ and $E[\xi_1] = E[\xi_2] = \tau - r$.
Let us set $f(y) = (\tau - y)^n$ for $y \leq \tau$, and $r > 0$.
According to the fact that Φ_1 is a degenerate distribution function that assigns credibility 1 to $\tau - r$, we have $\int_{-\infty}^{\tau} (\tau - y)^n d\Phi_1(y) = r^n$ and $f(E[\xi_1]) = r^n$.
f is strictly convex as $n > 1$. By the Inequality of Jensens and the fact that $E[\xi_1] = E[\xi_2]$, we have: $E[f(\xi_2)] = \int_{-\infty}^{\tau} (\tau - y)^n d\Phi_2(y) > f(E[\xi_1]) = r^n$.
Finally, we have $\int_{-\infty}^{\tau} (\tau - y)^n d\Phi_2(y) > \int_{-\infty}^{\tau} (\tau - y)^n d\Phi_1(y)$. Thus $\xi_1 \succ_{n,\tau} \xi_2$.
We can prove the converse case in the same way. $\square$

The following example compares two trapezoidal fuzzy variables by means of the mean-risk dominance for the target value $\tau = \frac{1}{2}$ and the order $n = 2$.

Example 1. *Let $\xi_1 = (-1, -\frac{1}{2}, \frac{3}{2}, 2)$ and $\xi_2 = (-2, 0, 1, 3)$ be two trapezoidal fuzzy variables.*
We have $E[\xi_1] = E[\xi_2] = \frac{1}{2}$. By taking $\tau = \frac{1}{2}$ and $n = 2$. We have: $FLPM_{n,\tau}[\xi_1] = \frac{19}{24} \leq FLPM_{n,\tau}[\xi_2] = \frac{31}{24}$. It follows that $\xi_1 \succeq_{2,\frac{1}{2}} \xi_2$.

Let us justify that $\succeq_{n,\tau}$ is not a complete relation on the set of fuzzy variables.

Remark 2. *Let $\xi_1 = (1, 4, 5)$ and $\xi_2 = (2, 3, 4)$ be two fuzzy variables, $n = 2$ and $\tau = 4$. We have $E[\xi_1] = \frac{7}{2}$, $E[\xi_2] = 3$, $FLPM_{2,4}[\xi_1] = \frac{3}{2}$ and $FLPM_{2,4}[\xi_2] = \frac{4}{3}$. Thus, $E[\xi_1] > E[\xi_2]$ and $FLPM_{2,4}[\xi_1] > FLPM_{2,4}[\xi_2]$. Hence $\xi_1 \not\succeq_{2,4} \xi_2$ and $\xi_2 \not\succeq_{2,4} \xi_1$. Thereby, $\succeq_{n,\tau}$ is not a complete relation.*

Let us end this Subsection by establishing some properties satisfied by the mean-risk dominance relation on the set of fuzzy variables. More precisely, the two first properties justify that $\succeq_{n,\tau}$ is a pre-order on the set of fuzzy variables. The third property establishes indifference part of $\succeq_{n,\tau}$ and the fourth property establishes sufficient condition on supports of two trapezoidal fuzzy variables in order to compare them.

Proposition 4. *Let $n \in \mathbb{N}^*$ and $\tau \in \mathbb{R}$, ξ, η, ζ three given fuzzy variables. The fuzzy mean-risk dominance $\succeq_{n,\tau}$ satisfies the following properties:*

1. $\xi \succeq_{n,\tau} \xi$.

2. $\xi \succeq_{n,\tau} \eta$ *and* $\eta \succeq_{n,\tau} \zeta \Rightarrow \xi \succeq_{n,\tau} \zeta$.

3. If $\xi \succeq_{n,\tau} \eta$ *and* $\eta \succeq_{n,\tau} \xi$*, then* $\xi \sim_{n,\tau} \eta$.

4. If ξ *and* η *are trapezoidal fuzzy variables, then* $\inf supp(\xi) > \sup supp(\eta)$ $\Rightarrow \xi \succeq_{n,\tau} \eta$.

Proof: Let $n \in \mathbb{N}^*$ and $\tau \in \mathbb{R}$, ξ, η, ζ three given fuzzy variables.
1) We have: $\begin{cases} E[\xi] \geq E[\xi] \\ \text{FLPM}_{n,\tau}[\xi] \leq \text{FLPM}_{n,\tau}[\xi] \end{cases}$. Thus, $\xi \succeq_{n,\tau} \xi$.
2) Let us assume that $\xi \succeq_{n,\tau} \eta$ and $\eta \succeq_{n,\tau} \zeta$.
We have: $\begin{cases} E[\xi] \geq E[\eta] \\ \text{FLPM}_{n,\tau}[\xi] \leq \text{FLPM}_{n,\tau}[\eta] \end{cases}$ and $\begin{cases} E[\eta] \geq E[\zeta] \\ \text{FLPM}_{n,\tau}[\eta] \leq \text{FLPM}_{n,\tau}[\zeta] \end{cases}$;
by the transitivity of inequalities, that leads to $\begin{cases} E[\xi] \geq E[\zeta] \\ \text{FLPM}_{n,\tau}[\xi] \leq \text{FLPM}_{n,\tau}[\zeta] \end{cases}$
. Hence, $\xi \succeq_{n,\tau} \zeta$.
3) Let us assume that $\xi \succeq_{n,\tau} \eta$ and $\eta \succeq_{n,\tau} \xi$.
We have: $\begin{cases} E[\xi] \geq E[\eta] \\ \text{FLPM}_{n,\tau}[\xi] \leq \text{FLPM}_{n,\tau}[\eta] \end{cases}$ and $\begin{cases} E[\eta] \geq E[\xi] \\ \text{FLPM}_{n,\tau}[\eta] \leq \text{FLPM}_{n,\tau}[\xi] \end{cases}$,
that leads to $\begin{cases} E[\xi] = E[\eta] \\ \text{FLPM}_{n,\tau}[\xi] = \text{FLPM}_{n,\tau}[\eta] \end{cases}$. Therefore, $\xi \sim_{n,\tau} \eta$.
4) Let us assume that $\xi = (a, b, c, d)$, $\eta = (a', b', c', d')$ and $\inf supp(\xi) > \sup supp(\eta)$, that is, $a > d'$. Necessarily, we have: $a > a'$, $b > b'$, $c > c'$ and $d > d'$. By the fact that, $E[\xi] = \frac{a+b+c+d}{4}$ and $E[\eta] = \frac{a'+b'+c'+d'}{4}$, we have $E[\xi] \geq E[\theta]$.
On the other hand, we have: $\text{FLPM}_{n,\tau}[\xi] = \int_{-\infty}^{\tau} (\tau - u)^{n-1} \Phi_1(u) du$ and $\text{FLPM}_{\alpha,\tau}[\eta] = \int_{-\infty}^{\tau} (\tau - u)^{n-1} \Phi_2(u) du$. By the fact that $a > a'$, $b > b'$, $c > c'$ and $d > d'$, we have $\Phi_1(r) \leq \Phi_2(r)$, $\forall r \in \mathbb{R}$, according to Tassak et al. [25]. That leads to $\text{FLPM}_{n,\tau}[\xi] \leq \text{FLPM}_{n,\tau}[\eta]$. As $E[\xi] \geq E[\eta]$ and $\text{FLPM}_{n,\tau}[\xi] \leq \text{FLPM}_{n,\tau}[\eta]$, we conclude that $\xi \succeq_{n,\tau} \eta$. $\square$

The mean risk-dominance can generate a mean-risk model of the form

$$\text{maximize } E[\xi] - \lambda \text{FLPM}_{n,\tau}[\xi] \tag{22}$$

in the sense of a trade-off analysis where $\lambda > 0$, is a trade-off coefficient. Following the work of Ogryczak and Ruszczynski [17], in the following paragraph, we introduce the two notions of consistency and λ-consistency of a mean-risk model with the first or the second order dominance of fuzzy variables, introduced and studied by Tassak et al.[25], defined by:

- $\xi_1 \succeq_1 \xi_2$ if $\forall r \in \mathbb{R}, \Phi_1(r) \leq \Phi_2(r)$.
- $\xi_1 \succeq_2 \xi_2$ if $\forall t \in \mathbb{R}, \int_{-\infty}^{t} [\Phi_2(r) - \Phi_1(r)] dr \geq 0$.

We then establish some results on these two notions.

3.3.2 Consistency of mean-risk models with respect to first and second order dominances

Let us define those notions.

Definition 3. *Let $k, n \in \{1, 2\}$, $\lambda \in \mathbb{R}_+^*$, $\tau \in \mathbb{R}$, ξ_1 and ξ_2 be two fuzzy variables.*

1. *A mean-risk model is said to be consistent with a relation dominance of order k if:*
$$\xi_1 \succeq_k \xi_2 \Rightarrow \xi_1 \succeq_{n,\tau} \xi_2.$$
2. *A mean-risk model is said to be λ-consistent with a relation dominance of order k if:*
$$\xi_1 \succeq_k \xi_2 \Rightarrow \begin{cases} E[\xi_1] \geq E[\xi_2] \\ E[\xi_1] - \lambda FLPM_{n,\tau}[\xi_1] \geq E[\xi_2] - \lambda FLPM_{n,\tau}[\xi_2] \end{cases}$$

The following result establishes consistency of the mean-risk model with each order dominance.

Proposition 5. 1. *A mean-risk model is consistent with the first order dominance.*

2. *A mean-risk model is consistent with the second order dominance if the downside risk is the expected loss.*

Proof: Let $k \in \{1; 2\}$, $n \in \mathbb{N}^*$, $\tau \in \mathbb{R}$, ξ_1 and ξ_2 be two fuzzy variables with credibility distributions Φ_1 and Φ_2 such that $\xi_1 \succeq_k \xi_2$. Let us prove that $\xi_1 \succeq_{n,\tau} \xi_2$.
1) For $k = 1$.
a) Assume that $\xi_1 \succeq_1 \xi_2$ and we prove that $\xi_1 \succeq_{\alpha,\tau} \xi_2$.
$\xi_1 \succeq_1 \xi_2 \Rightarrow \forall r \in \mathbb{R}, \Phi_1(r) \leq \Phi_2(r)$, that is,

$$\forall r \in \mathbb{R}, Cr\{\xi_1 \leq r\} \leq Cr\{\xi_2 \leq r\} \tag{23}$$

and

$$\forall r \in \mathbb{R}, Cr\{\xi_1 \geq r\} \geq Cr\{\xi_2 \geq r\} \tag{24}$$

According to the definition of $\succeq_1$.
On the other hand, we have: $E[\xi_i] = \int_0^{+\infty} Cr\{\xi_i \geq r\}\ dr \ - \ \int_{-\infty}^{0} Cr\{\xi_i \leq$

$r\}\ dr\ \forall i \in \{1;2\}$
According to (23) and (24), we conclude that $E[\xi_1] \geq E[\xi_2]$.
In the same manner, we have: $\text{FLPM}_{n,\tau}[\xi_i] = \alpha \int_{-\infty}^{\tau} (\tau - x)^{n-1} Cr\{\xi_i \leq x\} dx$ $\forall i \in \{1;2\}$
These last relations lead to $\text{FLPM}_{n,\tau}[\xi_1] < \text{FLPM}_{n,\tau}[\xi_2]$. Finally, we obtain $\xi_1 \succeq_{n,\tau} \xi_2$.
b) Since $\forall r \in \mathbb{R}, \Phi_1(r) \leq \Phi_2(r)$ then $\forall t \in \mathbb{R}, \int_{-\infty}^{t} [\Phi_2(r) - \Phi_1(r)]\ \ dr \geq 0$. We easily obtain the proof.
2) For $k = 2$.
Let us assume that $\xi_1 \succeq_2 \xi_2$. The following equality
$E[\xi_i] = \int_0^{+\infty} (1 - \Phi_i(r))\ \ dr - \int_{-\infty}^{0} \Phi_i(r)\ \ dr$, $\forall i \in \{1, 2\}$, leads to:
$E[\xi_1] - E[\xi_2] = \int_{-\infty}^{+\infty} [\Phi_2(r) - \Phi_1(r)]\ \ dr$. By using the characterization of $\succeq_2$ and by the fact that $\xi_1 \succeq_2 \xi_2$, we obtain $\int_{-\infty}^{+\infty} [\Phi_2(r) - \Phi_1(r)]\ \ dr \geq 0$, that means $E[\xi_1] \geq E[\xi_2]$.
On the other hand, by using relation (8), we get:
$\text{FLPM}_{1,\tau}[\xi_1] - \text{FLPM}_{1,\tau}[\xi_2] = \int_{-\infty}^{\tau} (\tau - u)^1\ \ (\Phi_1 - \Phi_2)(u)$. By using the definition of $\succeq_2$ and by the fact that $\xi_1 \succeq_2 \xi_2$, we obtain $\int_{-\infty}^{\tau} (\Phi_2 - \Phi_1)(u)\ \ du \geq 0$, that is, $\text{FLPM}_{1,\tau}[\xi_1] \leq \text{FLPM}_{1,\tau}[\xi_2]$.
Finally, by the fact that $E[\xi_1] \geq E[\xi_2]$ and $\text{FLPM}_{1,\tau}[\xi_1] \leq \text{FLPM}_{1,\tau}[\xi_2]$, we conclude that $\xi_1 \succeq_{1,\tau} \xi_2$. □ The following result establishes the 1-consistency of the mean-risk model with each order dominance.

Corollary 2. 1. *A mean-risk model is 1-consistent with the first order dominance.*

2. *A mean-risk model is 1-consistent with the second order dominance if the downside risk is the expected loss.*

Proof: The proof is easily deduced from Proposition 5 by taking $\lambda = 1$. □

As a consequence of the previous results, the next result determines optimal solutions of mean-risk models as non dominated elements of order dominances. In this view, the determination of such optimal solutions becomes simple since Tassak et al.[25] characterized such non dominated fuzzy variables.

Proposition 6. 1. *Optimal solutions of model (22) are non dominated fuzzy variables with respect to first order dominance $\succeq_1$.*

2. *Optimal solutions of model (22) are non dominated fuzzy variables with respect to second order dominance $\succeq_2$ if the downside risk is the expected loss.*

Proof: 1) Let be $n \in \mathbb{N}^*$, $\tau \in \mathbb{R}$.
Let us consider an optimal solution ξ_0 of model (22) which is not efficient with respect to first dominance $\succeq_1$, that is, there exist a fuzzy variable η_0 such that $\eta_0 \succ_1 \xi_0$. According to Proposition 5, we have:

$$\eta_0 \succ_1 \xi_0 \Rightarrow \eta_0 \succ_{n,\tau} \xi_0 \tag{25}$$

By (21), we have: (25)$\Rightarrow \begin{cases} E[\eta_0] \geq E[\xi_0] \\ \mathrm{FLPM}_{n,\tau}[\eta_0] \leq \mathrm{FLPM}_{n,\tau}[\xi_0] \end{cases}$ with at least one strict inequality, which leads to $E[\eta_0] - \lambda \mathrm{FLPM}_{n,\tau}[\eta_0] > E[\xi_0] - \lambda \mathrm{FLPM}_{n,\tau}[\xi_0]$, $\forall \lambda > 0$. Therefore, this last inequality contradicts the fact that ξ_0 is an optimal solution of model (22).
2) The proof is the same for optimal solutions with respect to second order dominance $\succeq_2$ and $n = 1$. $\square$

By considering model (22) as a general form of models taking into consideration investors preferences through targets values [1], Proposition 6 justifies that optimal portfolios obtained through optimization models defined in Sadefo et al. [20] belong to the sets of best portfolios respectively with first and order dominance, defined in Tassak et al. [25].

In the last Section of this paper, we present an application of a FLPM as a tool for poverty measurement in fuzzy context. The obtained poverty index is analyzed.

4 Application of FLPM for poverty measurement

In many situations, an individual's income can be given without any precision or can not completely be known, it is reasonable to describe that income by a fuzzy variable ξ. For example, in many african countries, people carry out one or several economic's activities to improve their welfare. In other words, besides their professional permanent job, most individuals practise other ones to face to increasing real life difficulties or expensive level of life. We assume that permanent jobs generate incomes which are currently or officially known and can vary during a time period and we describe such type of income by an equipossible fuzzy variable $\xi = (a, b)$ (on left of Figure 1). Meanwhile, secondary activities namely are not generally taken into account

[1] In model (22), the trade-off coefficient λ can illustrate in a particular case, an investor's preference level, of mean to risk or inversely.

Figure 1: Two fuzzy variables describing fuzzy incomes

in an individual's official salary: this may involve a wrong analysis in measuring a poverty level's population in such context. In order to handle with an individual activities's incomes, we use the semi-trapezoidal fuzzy variable $\xi = (a, b, c)$ (on right of Figure 1) to describe the total income where the interval $[b, c]$ contains unknown or undeclared incomes generated by "secondary activities".

From formula (11), we have the following FLPMs:

- For $\xi = (a, b, c)$, becomes

$$FLPM_{n,z}[\xi] = \begin{cases} 0 \text{ if } z \leq a \\ \frac{1}{2}(z-a)^n \text{ if } a \leq z \leq b \\ \frac{1}{2}(z-a)^n + \frac{1}{2(n+1)(c-b)}(z-b)^{n+1} \text{ if } b \leq z \leq c \\ \frac{1}{2}(z-a)^n + \frac{1}{2(n+1)(c-b)}[(z-b)^{n+1} - (z-c)^{n+1}] \text{ if } z \geq c \end{cases}. \tag{26}$$

- For $\xi = (a, b)$,

$$FLPM_{n,z}[\xi] = \begin{cases} 0 \text{ if } z \leq a \\ \frac{1}{2}(z-a)^n \text{ if } a \leq z \leq b \\ \frac{1}{2}[(z-a)^n + (z-b)^n] \text{ if } z \geq b \end{cases}. \tag{27}$$

In the following, we will use FLPM to evaluate poverty in such population where active population have income described by previous trapezoidal fuzzy variable. More specifically, the expected loss (expected gap income) with respect to a threshold z is given by $FLPM_{1,z}[\xi] = E[\max(z-\xi, 0)]$. Thereby, we introduce the following notions of poverty on individuals.

Definition 4. *Let i and j be two individuals with incomes respectively described by fuzzy numbers $\xi_i = (a_i, b_i, c_i)$ and $\xi_j = (a_j, b_j, c_j)$ and z a nonnegative real number.*

1. *i is said to be poor with respect to the poverty line z if $E[\xi_i] < z$.*
2. *i is said to be almost more poor than j if $E[\xi_i] \leq E[\xi_j]$.*
3. *i is said to be more poor than j if $a_i \leq a_j$, $b_i \leq b_j$ and $c_i \leq c_j$.*

We now introduce our poverty index based on the expected loss. Thus, we assume that an individual is symbolized by a positive integer.

Definition 5. *Let $\mathcal{P}$ be a population of N individuals, $z \in (0, \infty)$, $\mathcal{Q}_z = \{i \in \mathcal{P}, E[\xi_i] < z\}$ the set of poor individuals with respect to the poverty line z, $q_z = |\mathcal{Q}_z|$ the number of poor individuals with respect to the poverty line z. The poverty index I with respect to the poverty line z is given by:*

$$I(z) = \frac{1}{N} \sum_{k=1}^{q_z} \frac{FLPM_{1,z}^k}{z}.$$

Remark 3. *1. If the individual i is more poor than the individual j, then i is almost more poor than j.*

2. The generic term of the index $\frac{FLPM_{1,z}^i}{z}$ is the expected income gap ratio of individual i with respect to the poverty line z and $FLPM_{1,z}^i$ is his expected gap income.

Let us establish the variation of the index with respect to the poverty line.

Lemma 1. *The expected income gap ratio is nondecreasing with respect to the poverty line z.*

Proof: According to (26), we have:

$$(\frac{FLPM_{1,z}[\xi]}{z})' = \begin{cases} 0 \text{ if } z \leq a \\ \frac{a}{2z^2} \text{ if } a \leq z \leq b \\ \frac{a}{2z^2} + \frac{1}{4(c-b)z^2}(z-b)(z+b) \text{ if } b \leq z \leq c \\ \frac{a}{2z^2} + \frac{1}{4z^2}(c+b) \text{ if } z \geq c \end{cases}.$$

We conclude that $FLPM_{1,z}[\xi]$ is nondecreasing with respect to z by the fact that its derivative is a nonnegative function. $\square$

In the following, we examine if the new index satisfies some intuitive properties among which two meaningful and well-known axioms of a poverty index proposed by Sen [22] and defined by:

- Poverty Monotonicity (PM):
  For all $x, y \in [0, \infty)^n$ and $z \in (0, \infty)$, if $\mathcal{Q}_{x,z} = \mathcal{Q}_{y,z}$ and $x = y$ except for $x_i > y_i$ with $i \in \mathcal{P}$, then $I(x, z) < I(y, z)$.

- Transfer Sensitivity (TM):
  For all $x, y \in [0, \infty)^n$ and $z \in (0, \infty)$, if y is obtained from x by a progressive transfer among the poor, then $I(y, z) < I(x, z)$.

We start with two first properties.

Proposition 7. *1. The poverty index is nondecreasing with respect to the poverty line z.*

2. The poverty index satisfies PM.

Proof: 1) Let z_1 and z_2 be two poverty lines such that $z_1 \leq z_2$, ξ a fuzzy variable that describes an unknown income with its distribution function Φ. As $q_{z_1} \leq q_{z_2}$, we have:
$I(z_2) - I(z_1) = \frac{1}{N}(\sum_{k=1}^{q_{z_1}} \frac{FLPM_{1,z_2}^k}{z_2} - \frac{FLPM_{1,z_1}^k}{z_1}) + \frac{1}{N}\sum_{k=q_{z_1}}^{q_{z_2}} \frac{FLPM_{1,z_2}^k}{z_2}$.
According to Lemma 1, we have: $\frac{FLPM_{1,z_2}^k}{z_2} \geq \frac{FLPM_{1,z_1}^k}{z_1}$, therefore $I(z_2) \geq I(z_1)$.
2) Let us consider an individual j whose income is described by $\xi_j = (a_j, b_j, c_j)$, z a poverty line and θ a nonnegative real number that generates the new income $\xi_j' = (a_j + \theta, b_j + \theta, c_j + \theta)$. Without loss of generality, let us assume that i is the only one in the population $\mathcal{P}$ who has an improved income. Let us prove that the new poverty index $I'(z)$ is less than $I(z)$.
We distinguish two cases:
(i) If $E[\xi_j'] \geq z$, then j is not poor any longer with respect to z and we have: $I'(z) = \frac{1}{N}\sum_{k=1}^{q_z-1} \frac{FLPM_{1,z}^k}{z}$. Obviously, we get $I'(z) \leq I(z)$.
(ii) If $E[\xi_j'] < z$, then j remains poor with respect to z and we have $I'(z) = \frac{1}{N}(\sum_{k\in\mathcal{Q}_z\setminus\{j\}} \frac{FLPM_{1,z}^k}{z} + \frac{FLPM_{1,z}^{'j}}{z})$. We only need to prove that $FLPM_{1,z}^{'j} \leq FLPM_{1,z}^j$.
Without loss of generality, let us assume that $a_j \leq z \leq b_j$. We have two cases: (ii1) If $a_j + \theta \leq z \leq b_j + \theta$, then $FLPM_{1,z}^{'j} = \frac{1}{2}(z - a_j - \theta) \leq \frac{1}{2}(z - a_j)$. Thus $FLPM_{1,z}^{'j} \leq FLPM_{1,z}^j$.
(ii2) If $z < a_j + \theta$, then $FLPM_{1,z}^{'j} = 0$ and $FLPM_{1,z}^{'j} \leq FLPM_{1,z}^j$.
In all cases, we get: $I'(z) \leq I(z)$. $\square$

Most poverty index on random variables that exist in the literature violate the Transfer Sensitivity axiom. It is easy to justify that our new index also violates it. Thereby, we characterize the set of poverty lines on which this axiom is satisfied. For that, we propose a new version of TS axiom, namely the Restricted Transfer Sensitivity (RTS) axiom, that is satisfied by the index. Let us define the Transfer Sensitivity (RTS):
$\exists z_0 > 0$, for all $x, y \in [0, \infty)^n$ and $z \in (0, z_0]$, if y is obtained from x by a progressive transfer among the poor, then $I(y, z) < I(x, z)$.
Thus we have the following result.

Proposition 8. *Restricted Transfer Sensitivity axiom of the new index*
Let us consider N individuals ($N \in \mathbb{N}$) whose incomes are respectively de-

scribed by semi-trapezoidal fuzzy variables $\xi_k = (a_k, b_k, c_k)$, $k \in \{1, ..., N\}$ and z a poverty line.
The poverty index $I(z)$ satisfies the RTS axiom between two given individuals such that one is poor than another if $z_0 = \min_{1\leq k\leq N} b_k]$.

Proof: Let be N individuals ($N \in \mathbb{N}$) whose incomes are respectively described by semi-trapezoidal fuzzy variables $\xi_k = (a_k, b_k, c_k)$ and z a real number such that $z \in (0, \min_{1\leq k\leq N} b_k]$. Without loss of generality, let us assume that i and j are the only two individuals whose incomes are transferred such that i is more poor than j before the transfer ($a_j \geq a_i$, $b_j \geq b_i$) and the two individuals remain poor after the transfer with respect to the poverty line z. Let us consider a nonnegative real number θ such that $\theta \leq \min(a_j - a_i, b_j - b_i)$(that means, j remains less poor than i after the transfer) and the new incomes after the transfer are described by $\xi'_i = (a_i + \theta, b_i + \theta)$ and $\xi'_j = (a_j - \theta, b_j - \theta)$.
The new poverty index is:
$I'(z) = \frac{1}{N}(\sum_{k\in\mathcal{Q}_z\setminus\{i,j\}} \frac{FLPM^k_{1,z}}{z} + \frac{FLPM'^i_{1,z}}{z} + \frac{FLPM'^j_{1,z}}{z})$. We have to prove that $I'(z) \leq I(z)$. It suffices to prove that $FLPM'^i_{1,z} + FLPM'^j_{1,z} \leq FLPM^i_{1,z} + FLPM^j_{1,z}$.
Furthermore, without loss of generality, we assume that $z \in [a_j, b_i] = [a_j \vee a_i, b_j \wedge b_i]$. Let us set: $T = FLPM'^i_{1,z} + FLPM'^j_{1,z} - FLPM^i_{1,z} - FLPM^j_{1,z}$.
We have: $T = \frac{1}{2}(2\theta(z - a_j) + \theta^2 - 2\theta(z - a_i) + \theta^2) = \theta(a_i - a_j + \theta)$.
As $\theta \leq \min(a_j - a_i, b_j - b_i)$, we have $T \leq 0$.
Finally, we obtain: $I'(z) \leq I(z)$. $\square$

Corollary 3. *The poverty index satisfies the Transfer Sensitivity axiom between two individuals such that one is poor than another if their incomes are described by equipossible fuzzy variables.*

Interpretation 1. *The previous result seems more natural to transfer incomes between two poor individuals in order to reduce poverty when their incomes are completely known or declared in a time period.*

Let us end this Section by examining what can be the critical poverty line.
The concept of poverty is a relative and subjective's concept, in fact:
- a person who gets a car is rich if he belongs to a population where most of individuals move on foot or by bicycle.
- a person who gets a car is not necessary rich if he belongs to a population where most of individuals get a car.
Thus, it seems natural that people do not agree on the poverty's status of an individual. So, it is more suitable and interesting to classify individuals

with respect to a critical poverty level rather than looking for a target value which can help to check whether an individual is really poor or rich.
As it seems more natural, the poverty line that induces a minimum poverty level is the one that could guarantee welfare to a large number of individuals in the population. Therefore, the critical poverty line is obtained by minimizing the poverty index $I(z)$ and we propose the following model:

$$\begin{cases} \text{minimize I(z)} \\ \text{w.r.t} \\ z \in (0, +\infty) \end{cases} \tag{28}$$

Following that idea, individuals whose incomes are less than the critical poverty line can really be considered as poor individuals in the population.

In the following, we propose a numerical example where model (28) is applied to determine the critical poverty line with ten given fuzzy incomes.

Example 2. *Let us consider for a time period of four years the following incomes*[2] *from ten employees in Cameroon provided by the ministry of Finance and described by semi-trapezoidal fuzzy variables* [3]*:*

Employee	*Fuzzy income*
1	$\xi_1 = (215.5, 238.6, 301.2)$
2	$\xi_2 = (250.4, 276.3, 315.6)$
3	$\xi_3 = (226.9, 258.6, 330.5)$
4	$\xi_4 = (306.9, 338.6, 406.2)$
5	$\xi_5 = (349.2, 370.4, 425.3)$

Employee	*Fuzzy income*
6	$\xi_6 = (190.8, 207.2, 285.9)$
7	$\xi_7 = (233.9, 245.7, 328.2)$
8	$\xi_8 = (181.4, 197.8, 283.7)$
9	$\xi_9 = (219, 226.8, 296.2)$
10	$\xi_{10} = (174.3, 189.6, 275.1)$

By using Matlab with "fmincon" command at the initial point $z_0 = 306$*, the implementation of model (28) gives the following results: the critical poverty line is obtained at* $z^* = 296.8$ *and the corresponding poverty index is* $I(z^*) = 0.16$*.*
On the other hand, one can evaluate different expected incomes in order to identify those which are below the critical poverty line. We observe that except employees 4 and 5, whose expected incomes are respectively 339.7 and 373.5 (beyond the critical poverty line), the other employees can be considered as poor in this group. It seems that "secondary activities" influence weakly on an employee's status: it seems evident because those activities don't generate regular or certain incomes.

[2]Those incomes are expressed in thousand of francs CFA

[3]The source of different "official incomes" is: "MINFI-Direction de la solde" and "unknown incomes" are obtained through further investigations.

5 Concluding remarks

In this paper, we introduce Fuzzy Lower Partial Moment (FLPM) of a fuzzy variable which evaluates downside risk of a fuzzy risk and generalizes semi-moment of a fuzzy variable. It preserves, on fuzzy variables, some well-known properties of Lower Partial Moment for ramdom variables. Moreover, we supplemented the two (first and second) order dominances on fuzzy variables by defining and studying a third dominance relation for binary comparisons of fuzzy variables through their benefits and risks by means of the FLPM. We establish conditions under which the set of best portfolios of mean-risk model, an optimization program to determine optimal portfolios by means of the new dominance, belong to the set of best portfolios with respect to the first or second order dominance. Those obtained properties stipulate consitency and λ-consistency (λ is the trade off coefficient between expected return and FLPM of a given portfolio defining the mean-risk model) of the new dominance with the two known dominances. By considering individuals' incomes as fuzzy variables, we define a new poverty index based on FLPM and studied its properties. More precisely, we established that the new index satisfies the poverty monotonicity and violates Transfered sensitivity. We then deduce the set of individuals' incomes under which the index satisfies the given property. We propose a model to generate a critical poverty line useful to identify poor and non poor individuals. We implement a proposed model for getting the critical poverty line in a numerical example based on a typical sample of ten employees in Cameroon.

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