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Abstract

In this paper we study the problem of optimal fishing for regime switching, which may be regarded as sequential optimal problem with changes of regimes. The growth dynamics of a given fish species is described by the differential stochastic logistic model in which we take into account two states: prior or during floods and after. The resulting dynamic programming principle leads to a system of variational inequalities, by means of viscosity solutions approach, we prove the existence and uniqueness of the value functions. Then numerical approximation is used to answer the question: what is the optimal fishing effort for a sustainable fishery?

Keywords: : regime switching, floods, crra utility, logistic growth,
mean-reverting prices, viscosity solution, Howard's algorithm .

1 Introduction

The simplest population model commonly used in fisheries is the logistic growth model extended to include catch:

$$\frac{dB}{dt} = rB(1 - \frac{B}{K}) - C \tag{1}$$

where B is the biomass of the stock, r is the intrinsic rate of growth, K(the carrying capacity) is the biomass the stock would tend toward if unfished, and C is the catch rate.

The catch C is constant in quota management. Normally the catch is assumed to be proportional to fishing effort and to stock size, which results in the model proposed by (1) :

$$\frac{dB}{dt} = rB(1 - \frac{B}{K}) - qEB \tag{2}$$

where E is the fishing effort and q is a parameter describing the efficiency of the fishing gear

However, the environment is subject to significant random fluctuations that affect the population per capita natural growth rate. The effect of these fluctuations can be approximated by a white noise $\sigma\epsilon(t)$, where $\epsilon(t)$ is a standard white noise and $\sigma > 0$ measures the strength of environmental fluctuations see (2). Therefore, the above ODE Eq. (2) must be updated to the stochastic differential equation (SDE) which can be written in the standard format:

$$dB(t) = rB(t)(1 - \frac{B(t)}{K})dt - qE(t)B(t)dt + \sigma B(t)dW(t) \tag{3}$$

where $W(t)$ is a standard Wiener process. We will assume that $r - qE > \sigma^2/2$, otherwise the population will rendered extinct (see (3)).

Environmentally driven long-term changes in fish populations, which can play a major role in determining how such populations respond to fishing pressure, are rapidly being recognized as a critical problem in fisheries science ((4)).

The life cycle of African fish species of river is closely related to the seasons - reproduction almost always occurring just prior to, or during, floods ((5), (6), (7), (8)).

Floods appear to be essential for the completion of their reproductive cycle for most species: the absence of floods due to the drought in the Sahel has caused a decline in fish reproduction in the central Niger Delta, the Senegal River and Lake Chad (Stauch, personal communication).

There is some evidence that flood intensity acts in favor of reproduction, as it has been observed that the structured age class related to the high floods in the Kafu were more varied ((9)).

In our study we consider only two seasons: the dry season with intensive fishing and reduced reproduction, the flood period with reduced fishing and intensive reproduction.

A regime switching model provides an alternate approach to capturing non-constant drift and volatility terms for the stochastic process followed by the biomass of fish. Therefore, the above SDE Eq. (3) must be updated to the another stochastic differential equation that captures the regime swithing :

$$dB(t) = r_{\alpha(t)}B(t)(1 - \frac{B(t)}{K})dt - qE_{\alpha(t)}(t)B(t)dt + \sigma_{\alpha(t)}B(t)dW(t) \quad (4)$$

where $\alpha(t)$ refer to regimes and there are 2 regimes, i.e, $\alpha(t) \in \{1, 2\}$

Certainly, there is some evidence that uncertainty in price parameters leads to changes in the optimal policy ((10)), and the number of studies that include uncertainty in both the biological stock dynamics and the price dynamics is steadily increasing. The models in (11) have considered stochastic mean-reverting prices and when compared with the typical geometric Brownian motion model, a mean-reverting price better reflect basic, microeconomic ideas about supply behavior (see (12)).

Let the instantaneous profit from the harvest of the stock biomass $\pi(B_t, h_t)$ be given as:

$$\pi(B_t, h_t) = P_t h_t - c(B_t, h_t) \quad (5)$$

where, h_t denotes the volume of harvest, B_t the stock of the resource, $c(B_t, h_t)$ is the cost function, both at time t and P_t the mean-reverting (actual or spot) price of the harvest at the time of decision making. This can be modeled by the following process:

$$dP_t = \theta(\bar{p}_0 - \bar{p}_1 h - P_t)dt + \sigma_P dW_P(t) \tag{6}$$

The parameters are positve constants, θ is the reversion speed, $\bar{p}_0$ is a maximum price, $\bar{p}_1$ is the slope of the inverse demand curve and σ_P is the volatility of the spot price (see (11)).

Many works set the problem, in the infinite horizon time, as follows:

$$max_{h_t} \int_0^{+\infty} e^{-\beta t} \pi(B_t, h_t)dt \tag{7}$$

Previous work finds, almost without exception, that all fishers are risk-averse ((13), (14), and (15)). Under an expected-utility theory (EUT) specification of choice under uncertainty, we assume a constant relative risk-aversion (CRRA) utility function defined as $U(x) = \dfrac{x^{1-\gamma}}{1-\gamma}$ where x signifies the lottery prize and γ is the CRRA coefficient to be estimated: with $\gamma = 0$ denoting risk neutrality, $\gamma > 0$ indicating risk aversion, and $\gamma < 0$ denoting risk loving ((16)).

In recent years, further emphasis has been put on developing models for optimal management of these stochastic natural resources ((17); (18); (19)). Although the number of studies in bioeconomic modeling that include the stochastic dynamics are increasing, they are still not adequate.

The time horizon also plays a crucial role in optimal policies and the usual infinite horizon framework problem requires the existence of linear growth conditions on drift part of the logistic process for our solution to hold, raising the question of whether another solution may exist or not. In this paper, we consider the finite time horizon T with utility on both profit and remaining biomass.

The outline of this paper is as follows. In Section II we formulate a stochastic optimal control problem. Section III the optimal strategies to the utility maximization problem are derived. In Section IV we present examples to illustrate the results. Finally, in section V we end with some summarizing comments.

2 Mathematical model

2.1 Stochastic logistic growth model

Throughout this paper we let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e. it is increasing and right continuous while $\mathcal{F}_0$ contains all P-null sets). Let $W(t)$ and $W_P(t)$, $t \geq 0$, be scalar independant Brownian motions defined on this probability space.

After specifying a stochastic model for biological growth that we use in this paper. The valuation of the biomass can be described in terms of the following variables:

t is time, $t \in [0; T]$ and T is finite-horizon of time.

$\alpha(t)$ is a right-continuous-time Markov chain, $\mathcal{F}_t$-adapted with finite state space $\mathcal{S} = \{1; 2\}$ and generator $\mathcal{Q} = (q_{ij}) \in \mathrm{R}^2 \times \mathrm{R}^2$ such that $q_{ij} \geq 0$ for $i \neq j$ and $\sum_{j=1}^{2} q_{ij} = 0$. We assume that the Markov chain $\alpha(.)$ is independent of the Brownian motions $W_P(.)$ and $W(.)$

$B(t)$ stock of fish biomass at time t. Whith initial condition $B(0) > 0$

$r_{\alpha(t)}$ intrinsic rate of growth in regime $\alpha(t)$

$E_{\alpha(t)}$ is the fishing effort which depends on the current regime $\alpha(t)$

$\sigma_{\alpha(t)}$ is volatility in regime $\alpha(t)$. In any time $r_{\alpha(t)} - qE > \sigma^2_{\alpha(t)}/2$

We set an SDE under regime switching of the form:

$$dB(t) = f(t, B(t), \alpha(t))dt + g(t, B(t), \alpha(t))dW(t) \tag{8}$$

on $t \geq 0$ with initial value $B(0) = b \in]0; K[$, where

$$f : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathcal{S} \to \mathbb{R} \;\; and \;\; g : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathcal{S} \to \mathbb{R} \tag{9}$$

This equation can be regarded as the result of the following 2 equations:

$$dB(t) = f(t, B(t), i)dt + g(t, B(t), i)dW(t); \quad i = 1, 2 \tag{10}$$

switching from one to the other according to the movement of the Markov chain. Recalling that

$$f(t, B(t), i) = r_i B(t)(1 - \frac{B(t)}{K}) - qE_i(t)B(t) \quad and \quad g(t, B(t), i) = \sigma_i B(t) \tag{11}$$

$B(t)$ is an unknown stochastic process, that is, the solution to Eq.(8) satisfying the initial condition $B(0) = b$ such that $0 < b < K$. The logical requirement is that $B(t)$ must be positive. The resulting stochastic differential equation does not satisfy the standard assumptions for existence and uniqueness of solutions, namely, linear growth and the Lipschitz condition. Nevertheless, for any positive initial condition, the solution exists and is unique under a hypothesis that both f and g satisfy the local Lipschitz condition and in any time $r_{\alpha(t)} - qE > \sigma^2_{\alpha(t)}/2$. The solution of this equation is.(For more details see Appendix A)

$$B_{t,i} = \frac{K \exp[(r_i - qE - (1/2)\sigma_i^2)t + \sigma_i W_t]}{\left[(K/B_0) + r_i \int_0^t \exp[(r_i - qE - (1/2)\sigma_i^2)s + \sigma_i W_s]ds\right]} \qquad i = 1, 2. \tag{12}$$

2.2 The Mean- reverting spot price

The version of the Ornstein-Uhlenbeck (OU) process we employ here is described by

$$dP_t = \theta(\bar{p}_0 - \bar{p}_1 h - P_t)dt + \sigma_P dW_P(t) \tag{13}$$

where the parameters are positve constants, θ is the reversion speed, $\bar{p}_0$ is a maximum price, $\bar{p}_1$ is the slope of the inverse demand curve and σ_p is the volatility of the spot price. Note that the mean (or long-term) price $\bar{p}_0 - \bar{p}_1 h$ may depend upon the harvest level. $W_P(t)$ is standardized Brownian motion as before. Its solution for an initial condition $P(0) = p$ is

$$P_t = pe^{-\theta t} + (\bar{p}_0 - \bar{p}_1 h)(1 - e^{-\theta t}) + \sigma \int_0^t e^{-\theta(t-s)} dW_s \tag{14}$$

One of the most convenient properties is that

$$P_t \sim N\left(\bar{p}_0 - \bar{p}_1 h + (p + \bar{p}_0 - \bar{p}_1 h)e^{-\theta t}, \frac{\sigma^2}{2\theta}(1 - e^{-2\theta t})\right) \tag{15}$$

2.3 The optimization problem

The cost of harvest per unit time is assumed to depend on effort and to have a quadratic form given by

$$c(B_t, E_t) = (c_1 + c_2 E(t))E(t) \tag{16}$$

where $c_1,\ c_2 > 0$ are constants. The quadratic cost structure incorporates the case where the fishermen need to use less efficient vessels and fishing technologies or pay higher overtime wages to implement an extraordinary high effort (see (20), (21)).

By substituting the values in equation 5, the profit function can be rearranged as:

$$\pi(B_t, P_t, E) = (qB_tP_t - c_1 - c_2E(t))E(t) \tag{17}$$

where, c_1 and c_2 are positive parameters.

For a time t in the horizon $[0, T]$, we define a performance criterion for each $i \in \mathcal{S}$ as:

$$V_i(t, b_t, p_t) = \mathrm{E}_{\mathrm{b_t,p_t,i}}\left[\int_t^T e^{-\beta(s-t)}\frac{\pi(B_s, P_s, E_s, i)^{1-\gamma}}{1-\gamma}ds + e^{-\beta(T-t)}V_i(B_T)\right] \tag{18}$$

$$= \mathrm{E}_{\mathrm{b_t,p_t,i}}\left[\int_t^T e^{-\beta(s-t)}l(s, B_s, P_s, E_s, i)ds + e^{-\beta(T-t)}m(T, B_T)\right] \tag{19}$$

We will start the optimization at time $t = 0$. Let $b_0 = b$, $p_0 = p$ with $b,\ p \in]0; +\infty]$, we have

$$V_i(0, b, p) = \mathrm{E}_{\mathrm{b,p,i}}\left[\int_0^T e^{-\beta s}\frac{\pi(B_s, P_s, E_s, i)^{1-\gamma}}{1-\gamma}ds + e^{-\beta T}V_i(B_T)\right] \tag{20}$$

$$= \mathrm{E}_{\mathrm{b,p,i}}\left[\int_0^T e^{-\beta s}l(s, B_s, P_s, E_s, i)ds + e^{-\beta T}m(T, B_T)\right] \tag{21}$$

Here $\mathrm{E}_{\mathrm{b,p,i}}$ is the conditional expectation given $B(0) = b$, $P(0) = p$ and $\alpha(0) = i$ under $\mathbb{P}$, where T is the finite time horizon $\beta > 0$ is a discount factor.

We say that the control process $E(t)$ is admissible if the following tree conditions are satisfied:

1. the SDE (4) for the state process $B(t)$ has a unique strong solution;

2. the SDE (6) for the state process $P(t)$ has a unique strong solution;

3. $\mathrm{E}_{\mathrm{b,p,i}}\left[\int_0^T |e^{-\beta t}\dfrac{\pi(B_t,P_t,E_t,i)^{1-\gamma}}{1-\gamma}|dt + |e^{-\beta T}V(B_T)|\right] < \infty.$

Write $\mathcal{A}$ for the set of admissible controls. The number of tools, gears, hours, vessels and manpower is finite and limited, so we require the set $\mathcal{A}$ in which the controls take values to be bounded. The stochastic control problem is to find an optimal control $E^* \in \mathcal{A}_i$ such that:

$$v_i(b,p) = \sup_{E\in\mathcal{A}_i} V_i(b,p) \tag{22}$$

3 Main results

The Hamilton-Jacobi-Bellman equations associated with this problem is a variational inequality involving, at least heuristically, a nonlinear second order parabolic differential equations :

$$\frac{\partial v_i}{\partial t}(t,b_t,p_t) + \sup_{E\in\mathcal{A}_i}\left\{-\beta v_i(t,b_t,p_t) + \frac{\pi(B_t,P_t,E_t)^{1-\gamma}}{1-\gamma} + \mathcal{L}v_i(t,b_t,p_t)\right\} = 0, \tag{23}$$

$$v_i(T,b_t,p_t) = \kappa^\gamma \frac{B_T^{1-\gamma}}{1-\gamma} \quad for \quad i\in\{0;1\};\ \kappa>0 \tag{24}$$

where $\mathcal{L}$ is an operator defined by:

$$\begin{aligned}\mathcal{L}v_i(t,B,P) &= \theta(\bar{p}_0 - \bar{p}_1 qEB - P)\frac{\partial v_i}{\partial p}(t,B,P) + f(B,E_i)\frac{\partial v_i}{\partial b}(t,B,P)\\ &+ \frac{1}{2}\sigma_P^2\frac{\partial^2 v_i}{\partial p^2}(t,B,P) + \frac{1}{2}g^2(B,E)\frac{\partial^2 v_i}{\partial b^2}(t,B,P) + q_{ij}(v_j(t,B,P) - v_i(t,B,P))\end{aligned} \tag{25}$$

As it is well-known, there is not in general a smooth solution of the equation (23) hence we find the solution in the viscosity sense, as introduced by (22), in subsection 3.2. Recall that E^*_{free} is the optimal solution of equation 23. The optimal harvest rule $E^*(t)$ can be described as follows

$$E^*(t) = \begin{cases} 0 & \text{if } E^*_{\text{free}}(t) < 0 \\ E^*_{\text{free}}(t) & \text{if } 0 \le E^*_{\text{free}}(t) \le E_{\max} \\ E_{\max} & \text{if } E^*_{\text{free}}(t) > E_{\max} \end{cases} \tag{26}$$

In addition to these, we know that the fishery is valueless if the stock goes extinct and therefore add the condition $V_i(0, P_t) = 0$, which must hold for all P_t and i.

3.1 On the regularity of value functions

In this section, we study the growth and continuity properties of the value functions.

We shall make the following assumptions: there exist $\rho > 0$ such that for all $s, t \in [0; T], b, b' \in \mathbb{R}_+, p, p' \in \mathbb{R}_+$ and $E \in \mathcal{A}$

$$|l(t,b,p,E) - l(s,b',p',E)| + |m(b,p) - m(b',p')| \le \rho\,[|t-s| + |b-b'| + |p-p'|] \tag{27}$$

and the global linear growth conditions:

$$|l(t,b,p,E)| + |m(b,p)| \le \rho\,[1 + |b| + |p|] \tag{28}$$

Lemma 3.1. *Let (27) and (28) hold. For any $k \in [0; 2]$ there exists $C = C(k; K; T) > 0$ such that for all $h, t \in [0; T]$, $b, p, b_t, p_t \in \mathbb{R}_+$:*

$$\mathrm{E}|B_h^{t,b_t}|^k \le C(1 + |b_t|^k); \quad \mathrm{E}|P_h^{t,p_t}|^k \le C(1 + |p_t|^k) \tag{29}$$

$$\mathrm{E}|B_h^{t,b_t} - b_t|^k \le C(1 + |b_t|^k)h^{\frac{k}{2}}; \quad \mathrm{E}|P_h^{t,p_t} - p_t|^k \le C(1 + |p_t|^k)h^{\frac{k}{2}} \tag{30}$$

$$\mathrm{E}|B_h^{t,b_t} - B_h^{t,b'_t}|^k \le C|b_t - b'_t|^2; \quad \mathrm{E}|P_h^{t,p_t} - P_h^{t,p'_t}|^k \le C|p_t - p'_t|^2 \tag{31}$$

$$\mathrm{E}\big[\sup_{0\le s\le h} |B_h^{t,b_t}|\big]^k \le C(1 + |b_t|^k)h^{\frac{k}{2}}; \quad \mathrm{E}\big[\sup_{0\le s\le h} |P_h^{t,p_t}|\big]^k \le C(1 + |p_t|^k)h^{\frac{k}{2}} \tag{32}$$

Proof 3.1. *See Appendix B*

Proposition 3.1. *For any $i \in \mathcal{S}$, the value function denoted by $v_i(s,b,p)$ satisfies a linear growth condition and is also Lipschitz in (b,p) uniformly in t. There exists a constant $C > 0$, such that*

$$0 \leq v_i(s, b_s, p_s) \leq C(1 + |b_s| + |p_s|), \qquad \forall (s, b_s, p_s) \in [0;T] \times \mathbb{R}_+ \times \mathbb{R}_+ \tag{33}$$

$$|v_i(s, b_s, p_s) - v_i(s, b'_s, p'_s)| \leq C(|b_s - b'_s| + |p_s - p'_s|), \qquad \forall s \in [0;T], \qquad b_s, b'_s \in \mathbb{R}_+, \qquad p_s, p'_s \in \mathbb{R}_+ \tag{34}$$

Proof 3.2. *We first show that v is Lipschitz in (b,p), uniformly in t and its linear growth condition.*

$$v_i(s, b_s, p_s) = \sup_{E \in \mathcal{A}} \mathrm{E}\left[\int_s^T e^{-\beta(u-s)} l(i, u, B_u^{s,b_s}, P_u^{s,p_s}, E_u)du + e^{-\beta(T-s)} m(B_T^{s,b_s}, P_T^{s,p_s})\right] \tag{35}$$

1. *Using elementary inequality $|\sup A - \sup B| \leq \sup |A - B|$, Lipschitz con-*

dition (27) on l ; m and from estimate (3.1), with k=1,

$$
\begin{aligned}
&|v_i(s,b_s,p_s) - v_i(s,b'_s,p'_s)| \\
&\le \sup_{E\in\mathcal{A}_i} \Bigg| \mathrm{E}\Big[\int_s^T e^{-\beta(u-s)} \Big(l(i,u,B_u^{s,b_s},P_u^{s,p_s},E_u) - l(i,u,B_u^{s,b'_s},P_u^{s,p'_s},E_u) \Big) du \\
&\qquad\qquad + e^{-\beta(T-s)} \Big(m(B_T^{s,b_s},P_T^{s,p_s}) - m(B_T^{s,b'_s},P_T^{s,p'_s}) \Big) \Big] \Bigg| \\
&\le \sup_{E\in\mathcal{A}_i} \mathrm{E}\Big[\int_s^T \Big| \Big(l(i,u,B_u^{s,b_s},P_u^{s,p_s},E_u) - l(i,u,B_u^{s,b'_s},P_u^{s,p'_s},E_u) \Big) \Big| du \\
&\qquad\qquad + \Big| \Big(m(B_T^{s,b_s},P_T^{s,p_s}) - m(B_T^{s,b'_s},P_T^{s,p'_s}) \Big) \Big| \Big] \\
&\le \sup_{E\in\mathcal{A}_i} \mathrm{E}\Big[\int_s^T \Big(|B_u^{s,b_s} - B_u^{s,b'_s}| + |P_u^{s,p_s} - P_u^{s,p'_s}| \Big) du + \Big(|B_T^{s,b_s} - B_T^{s,b'_s}| + |P_T^{s,p_s} - P_T^{s,p'_s}| \Big) \Big] \\
&\le \sup_{E\in\mathcal{A}_i} \Big[\int_s^T \mathrm{E}\Big(|B_u^{s,b_s} - B_u^{s,b'_s}| + \mathrm{E}|P_u^{s,p_s} - P_u^{s,p'_s}| \Big) du + \Big(\mathrm{E}|B_T^{s,b_s} - B_T^{s,b'_s}| + \mathrm{E}|P_T^{s,p_s} - P_T^{s,p'_s}| \Big) \Big]
\end{aligned}
$$

$$
|v_i(s,b_s,p_s) - v_i(s,b'_s,p'_s)| \le C\Big(|b_s - b'_s| + |p_s - p'_s| \Big) \tag{36}
$$

2. *from linear growth condition (28) on l ; m and from estimate (3.1), with k=1,*

$$
|v_i(s,b_s,p_s)| \le \sup_{E\in\mathcal{A}_i} \mathrm{E}\left[\int_s^T \Big| l(i,u,B_u^{s,b_s},P_u^{s,p_s},E_u) \Big| du + \Big| m(B_T^{s,b_s},P_T^{s,p_s}) \Big| \right] \tag{37}
$$

$$
|v_i(s,b_s,p_s)| \le \rho \sup_{E\in\mathcal{A}_i} \mathrm{E}\left[\int_s^T \Big(1 + |B_u^{s,b_s}| + |P_u^{s,p_s}| \Big) du + \Big(1 + |B_T^{s,b_s}| + |P_T| \Big) \right] \tag{38}
$$

$$
|v_i(s,b_s,p_s)| \le \rho \sup_{E\in\mathcal{A}_i} \left[\int_s^T \Big(1 + \mathrm{E}|B_u^{s,b_s}| + \mathrm{E}|P_u^{s,p_s}| \Big) du + \Big(1 + \mathrm{E}|B_T^{s,b_s}| + \mathrm{E}|P_T^{s,p_s}| \Big) \right] \tag{39}
$$

$$
|v_i(s,b_s,p_s)| \le C\,(1 + |b_s| + |p_s|) \tag{40}
$$

Proposition 3.2. *Under assumptions (49) and (28) the value function* $v \in \mathcal{C}^0([0;T] \times \mathbb{R}_+ \times \mathbb{R}_+)$. *More precisely, there exists a constant* $C > 0$ *such that* *for all* $t,\ s \in [0;T],\ b_t, b_s \in \mathbb{R}_+,\ p_t, p_s \in \mathbb{R}_+$,

$$|v_i(t,b_t,p_t) - v_i(s,b_s,p_s)| \leq C\left[(1+|b_t|+|p_t|)|s-t|^{\frac{1}{2}} + |b_t - b_s| + |p_t - p_s|\right] \tag{41}$$

Proof 3.3. *Let* $0 \leq t < s \leq T$*. To prove continuity property in time t, we use the dynamic programming principle.*

$$v_i(t,b,p) = \sup_{E\in\mathcal{A}_i} \mathrm{E}\left[\int_t^T e^{-\beta(u-t)} l(i,u,B_u^{t,b_t},P_u^{t,p_t},E_u)du + e^{-\beta(T-t)} m(B_T^{t,b_t},P_T^{t,p_t})\right] \tag{42}$$

$$= \sup_{E\in\mathcal{A}_i} \mathrm{E}\left[\int_t^s e^{-\beta(u-t)} l(i,u,B_u^{t,b_t},P_u^{t,p_t},E_u)du + e^{-\beta(s-t)} v(s,B_s^{t,b_t},P_s^{t,b_t},i)\right] \tag{43}$$

$$= \sup_{E\in\mathcal{A}_i} \mathrm{E}\left[\int_o^{s-t} e^{-\beta u} l(i,t+u,B_{t+u}^{t,b_t},P_{t+u}^{t,p_t},E_{t+u})du + e^{-\beta(s-t)} v(s,B_{s-t}^{t,b_t},P_{s-t}^{t,p_t},i)\right] \tag{44}$$

$$\begin{aligned} 0 \leq v_i(t,b_t,p_t) - v_i(s,b_s,p_s) = \sup_{E\in\mathcal{A}_i} \mathrm{E}\bigg[&\int_0^{s-t} e^{-\beta(u)} l(i,u,B_u^{t,b_t},P_u^{t,p_t},E_u)du \\ &+ e^{-\beta(s-t)}\left(v(s,B_{s-t}^{t,b_t},P_{s-t}^{t,b_t},i) - v(s,b_s,p_s,i)\right) \\ &+ \left(e^{-\beta(s-t)} - 1\right) v(s,b_s,p_s,i)\bigg] \end{aligned} \tag{45}$$

Applying linear growth condition (27) on l*, noting that* $0 \leq 1 - e^{-\beta(s-t)} \leq \beta(s-t)$

and v satisfies (3.1) , we deduce that:

$$
\begin{aligned}
&|v_i(t,b_t,p_t) - v_i(s,b_s,p_s)| \\
&\leq \sup_{E\in\mathcal{A}_i} \mathrm{E}\Big[\int_0^{s-t} \big|l(i,u,B_u^{t,b_t},P_u^{t,p_t},E_u)\big|du + \Big|e^{-\beta(s-t)}\big(v(s,B_{s-t}^{t,b_t},P_{s-t}^{t,b_t},i) - v(s,b_s,p_s,i)\big)\Big| \\
&\qquad\qquad + \Big|\Big(e^{-\beta(s-t)}-1\Big)v(s,b_s,p_s,i)\Big|\Big] \\
&\leq (s-t)^{\frac{1}{2}}\left(\int_0^{s-t}\sup_{E\in\mathcal{A}_i}\mathrm{E}\left|l(i,u,B_u^{t,b_t},p_u^{t,p_t},E_u)\right|^2 du\right)^{\frac{1}{2}} + \sup_{E\in\mathcal{A}_i}\mathrm{E}\Big|e^{-\beta(s-t)}\Big(v(s,B_{s-t}^{t,b_t},P_{s-t}^{t,p_t},i) - v(s,b_s,p_s,i)\Big)\Big| \\
&\qquad\qquad + \sup_{E\in\mathcal{A}_i}\mathrm{E}\Big|\Big(e^{-\beta(s-t)}-1\Big)v(s,b_s,p_s,i)\Big| \\
&\leq (s-t)^{\frac{1}{2}}\left(\int_0^{s-t}\rho^2\sup_{E\in\mathcal{A}_i}\left(1+\mathrm{E}|B_u^{t,b_t}|+\mathrm{E}|p_u^{t,p_t}|\right)^2 du\right)^{\frac{1}{2}} + \sup_{E\in\mathcal{A}_i}\mathrm{E}\Big|\Big(v(s,B_{s-t}^{t,b_t},P_{s-t}^{t,p_t},i) - v(s,b_s,p_s,i)\Big)\Big| \\
&\qquad\qquad + \beta|s-t|\sup_{E\in\mathcal{A}}\mathrm{E}\left|v(s,b_s,p_s,i)\right| \\
&\leq |s-t|^{\frac{1}{2}}\left(\int_0^{s-t}\rho\sup_{E\in\mathcal{A}_i}\left(1+\mathrm{E}|B_u^{t,b_t}|+\mathrm{E}|P_u^{t,p_t}|\right)du\right) + \sup_{E\in\mathcal{A}_i}\mathrm{E}\Big|\big(v(s,B_{s-t}^{t,b_t},P_{s-t}^{t,p_t},i) - v(s,b_s,p_s,i)\big)\Big| \\
&\qquad\qquad + \beta|s-t|\sup_{E\in\mathcal{A}_i}\mathrm{E}\Big|v(s,b_s,p_s,i)\Big| \\
&\leq C'\left(|s-t|^{\frac{1}{2}}\int_0^{s-t}(1+\mathrm{E}|B_u^{t,b_t}|+\mathrm{E}|P_u^{t,p_t}|)du + (|b_t-b_s|+|p_t-p_s|) + \beta(1+|b_s|+|p_s|)|s-t|\right) \\
&\qquad\qquad \leq C\left[(1+|b_t|+|p_t|)|s-t|^{\frac{1}{2}} + |b_t-b_s|+|p_t-p_s|\right] \quad (46)
\end{aligned}
$$

3.2 Existence of viscosity solution

104 In this section we will first define what we mean by viscosity solutions. Then we will prove that the value function is a viscosity solution.

From the optimization problem (23), we derive the Bellman equations as follows:

$$
\begin{aligned}
&\frac{\partial v_i}{\partial t}(t,B,P) + \sup_{E\in\mathcal{A}_i}\Bigg\{-\beta v_i(t,B,P) + \frac{\pi^{1-\gamma}}{1-\gamma} + \theta(\bar{p}_0 - \bar{p}_1 qEB - P)\frac{\partial v_i}{\partial p}(t,B,P) \\
&+\Big[r_iB\Big(1-\frac{B}{K}\Big) - qEB\Big]\frac{\partial v_i}{\partial b}(t,B,P) + \frac{1}{2}\sigma_P^2\frac{\partial^2 v_i}{\partial p^2}(t,B,P) + \frac{1}{2}\sigma^2B^2\frac{\partial^2 v_i}{\partial b^2}(t,B,P) \\
&\qquad + q_{ij}(v_j(t,B,P) - v_i(t,B,P))\Bigg\} = 0 \quad (47)
\end{aligned}
$$

The corresponding Hamiltonian has the following form:

$$\mathcal{H}\left(i, s, B, P, u_i, \frac{\partial u_i}{\partial s}, \frac{\partial u_i}{\partial b}, \frac{\partial u_i}{\partial p}, \frac{\partial^2 u_i}{\partial b^2}, \frac{\partial^2 u_i}{\partial p^2}\right)$$
$$= \frac{\partial u_i}{\partial s}(s,B,P) + \sup_{E\in\mathcal{A}_i}\left\{-\beta u_i(s,B,P) + \frac{\pi(B_s,P_s,E_s)^{1-\gamma}}{1-\gamma} + \mathcal{L}u_i(s,B,P)\right\} = 0 \tag{48}$$

We have the following systems:

$$\begin{cases} \mathcal{H}\left(i, s, B, P, u_i, \frac{\partial u_i}{\partial s}, \frac{\partial u_i}{\partial b}, \frac{\partial u_i}{\partial p}, \frac{\partial^2 u_i}{\partial b^2}, \frac{\partial^2 u_i}{\partial p^2}\right) = 0 \quad for \quad (i,s,B,P) \in \mathcal{S}\times[0;T_i]\times\mathbb{R}_+\times\mathbb{R}_+ \\ u_i(T,B,) = \kappa^\gamma \frac{B_T^{1-\gamma}}{1-\gamma} \quad for \quad i,j\in\{0;1\}\ \kappa>0. \end{cases} \tag{49}$$

we recall that

$$\pi(B_s,P_s,E_s) = (qB_sP_s - c_1 - c_2E(s))E(s) \tag{50}$$

In order to study the possibility of existence and uniqueness of a solution of (49), we use a notion of viscosity solution introduced by (22).

Let denote the set of measurable functions on $[0;T]\times\mathbb{R}_+\times\mathbb{R}_+$ with polynomial growth of degree $q\geq 0$ as,

$$\mathcal{C}_q([0;T]\times\mathbb{R}_+\times\mathbb{R}_+)$$
$$= \{\phi : [0;T]\times\mathbb{R}_+\times\mathbb{R}_+, \text{measurable} \mid \exists C>0, |\phi(t,b,p)| \leq C(1+|b|^q+|p|^q)\}. \tag{51}$$

Definition 3.1. *We say that $u_i \in \mathcal{C}^0([0;T]\times\mathbb{R}_+\times\mathbb{R}_+)$ is called*

i. a viscosity subsolution of (49) if for any $i\in\mathcal{S}$, $u_i(T,b,p)\leq \kappa^\gamma\frac{b_T^{1-\gamma}}{1-\gamma}$, for all $b\in\mathbb{R}_+$, $p\in\mathbb{R}_+$ and for all functions $\phi\in\mathcal{C}^{1,2,2}([0;T]\times\mathbb{R}_+\times\mathbb{R}_+)\cap\mathcal{C}_2([0;T]\times\mathbb{R}_+\times\mathbb{R}_+)$ and $(\bar{t},\bar{b},\bar{p})$ such that $u_i-\phi$ attains its local maximum at $(\bar{t},\bar{b},\bar{p})$,

$$\mathcal{H}\left(i,\bar{t},\bar{b},\bar{p},\phi(\bar{t},\bar{b},\bar{p}), \frac{\partial\phi(\bar{t},\bar{b},\bar{p})}{\partial s}, \frac{\partial\phi(\bar{t},\bar{b},\bar{p})}{\partial b}, \frac{\partial\phi(\bar{t},\bar{b},\bar{p})}{\partial p}, \frac{\partial^2\phi(\bar{t},\bar{b},\bar{p})}{\partial b^2}, \frac{\partial^2\phi(\bar{t},\bar{b},\bar{p})}{\partial p^2}\right) \geq 0 \tag{52}$$

ii. *a viscosity supersolution of (49) if for any* $i \in \mathcal{S}$, $u_i(T,b,p) \geq \kappa^\gamma \frac{b_T^{1-\gamma}}{1-\gamma}$, *for all* $b \in \mathbb{R}_+$, $p \in \mathbb{R}_+$ *and if for all functions* $\phi \in \mathcal{C}^{1,2,2}([0;T] \times \mathbb{R}_+ \times \mathbb{R}_+) \cap \mathcal{C}_2([0;T] \times \mathbb{R}_+ \times \mathbb{R}_+)$ *and* $(\underline{t}, \underline{b}, \underline{p})$ *such that* $u_i - \phi$ *attains its local minimum at* $(\underline{t}, \underline{b}, \underline{p})$,

$$\mathcal{H}\left(i, \underline{t}, \underline{b}, \underline{p}, \phi(\underline{t}, \underline{b}, \underline{p}), \frac{\partial \phi(\underline{t}, \underline{b}, \underline{p})}{\partial s}, \frac{\partial \phi(\underline{t}, \underline{b}, \underline{p})}{\partial b}, \frac{\partial \phi(\underline{t}, \underline{b}, \underline{p})}{\partial p}, \frac{\partial^2 \phi(\underline{t}, \underline{b}, \underline{p})}{\partial b^2}, \frac{\partial^2 \phi(\underline{t}, \underline{b}, \underline{p})}{\partial p^2}\right) \leq 0 \tag{53}$$

iii. *a viscosity solution of (49) if it is both a viscosity sub- and a supersolution of equation (49)*

Theorem 3.1. *Under assumptions (27), the value function* v *is a viscosity solution of (47).*

Proof 3.4. *We establish the viscosity super- and sub-solution properties, respectively in the following two steps.*

Step 1. $v_i(t, b_t, p_t)$, $i = 1; 2$ *is a viscosity super-solution of (47).*

We already know that $v \in \mathcal{C}^0([0;T] \times \mathbb{R}_+ \times \mathbb{R}_+)$. *We first note that* $v_i(T,b,p) = \kappa^\gamma \frac{B_T^{1-\gamma}}{1-\gamma}$ *so, the boundary condition at time* $t = T$ *is clearly satisfied. Let* $(s, b_s, p_s) \in [0;T] \times \mathbb{R}_+ \times \mathbb{R}_+$, $i \in \mathcal{S}$ *and* $\phi \in \mathcal{C}^{1,2,2}([0;T] \times \mathbb{R}_+ \times \mathbb{R}_+) \cap \mathcal{C}_2([0;T] \times \mathbb{R}_+ \times \mathbb{R}_+)$ *such that* $v_i(.,.,.) - \phi(.,.,.)$ *has a local minimum at* (s, b_s, p_s). *Let* $\mathtt{N}(b_s, p_s)$ *a neighborhood of* (s, b_s, p_s) *where* $v_i(.,.,.) - \phi(.,.,.)$ *take its minimum, we introduce a new test-function* ψ *as follows:*

$$\psi(.,.,.,j) = \begin{cases} \phi(.,.,.) + [v_i(s, b_s, p_s) - \phi(s, b_s, p_s)], & \text{if } j = i \\ v_i(.,.,.), & \text{if } j \neq i. \end{cases} \tag{54}$$

This helps us to suppose without any loss of generality that this minimum is equal to 0.

Let τ_α *be the first jump time of* $\alpha(t)\big(= \alpha(t)^{b_s, p_s, i}\big)$, *i.e.* $\tau_\alpha = \min\{t \geq s : \alpha(t) \neq i\}$. *Then* $\tau_\alpha > s$, *a.s. Let* $\theta_s \in (s, \tau_\alpha)$ *be such that the state*

$(B_t^{b_s,i}, P_t^{p_s,i})$ *starts at* (b_s, p_s) *and stays in* $\mathtt{N}(b_s, p_s)$ *for* $s \le t \le \theta_s$. *Applying the generalized Itô's formula to the switching process* $e^{-\beta t}\psi(t, B_t, P_t, \alpha(t))$, *taking integral from* $t = s$ *to* $t = \theta_s \wedge h$, *where* $h > 0$ *is a positive constant, and then taking expectation we have*

$$\begin{aligned}
&\mathrm{E_{b_s,p_s,i}}\Big[e^{-\beta\theta_s\wedge h}\psi(\theta_s\wedge h, B_{\theta_s\wedge h}, P_{\theta_s\wedge h}, \alpha(\theta_s\wedge h))\Big] \\
&= \psi(s, B_s, P_s, i) + \mathrm{E_{b_s,p_s,i}}\Bigg[\int_s^{\theta_s\wedge h} e^{-\beta t}\Big\{-\beta\psi(t, B_t, P_t, \alpha(t)) + \frac{\partial\psi(t, B_t, P_t, \alpha(t))}{\partial t} \\
&+ \Big[r_i B_t\Big(1 - \frac{B_t}{K}\Big) - qEB_t\Big]\frac{\partial\psi(t, B_t, P_t, \alpha(t))}{\partial b} + \theta(\bar{p}_0 - \bar{p}_1 qEB_t - P_t)\frac{\partial\psi(t, B_t, P_t, \alpha(t))}{\partial p} \\
&\qquad + \frac{1}{2}\sigma^2 B_t^2\frac{\partial^2\psi(t, B_t, P_t, \alpha(t))}{\partial b^2} + \frac{1}{2}\sigma_P^2\frac{\partial^2\psi(t, B_t, P_t, \alpha(t))}{\partial p^2} \\
&\qquad + q_{\alpha(t)j}(\psi(t, B_t, P_t, j) - \psi(t, B_t, P_t, \alpha(t)))\Big\}dt\Bigg]; \quad \alpha(t) \ne j
\end{aligned} \tag{55}$$

From hypothesis, for any $t \in [s, \theta_s \wedge h]$

$$v_i(t, B_t^{b_s}, P_t^{p_s}) \ge \phi(t, B_t^{b_s}, P_t^{p_s}) + v_i(s, b_s, p_s) - \phi(s, b_s, p_s) \ge \psi(t, B_t^{b_s}, P_t^{p_s}, i) \tag{56}$$

Recalling that $(B_s^{b_s}, P_s^{p_s}) = (b_s, p_s)$ *and using Equations (54) and (56), we have*

$$\begin{aligned}
&\mathrm{E_{b_s,p_s,i}}\Big[e^{-\beta\theta_s\wedge h}\psi(\theta_s\wedge h, B_{\theta_s\wedge h}, P_{\theta_s\wedge h}, \alpha(\theta_s\wedge h))\Big] \ge \\
&+ v_i(s, b_s, p_s) + \mathrm{E_{b_s,p_s,i}}\Bigg[\int_s^{\theta_s\wedge h} e^{-\beta t}\Big\{-\beta v_i(t, B_t, P_t) + \frac{\partial\psi(t, B_t, P_t, \alpha(t))}{\partial t} \\
&+ \Big[r_i B_t\Big(1 - \frac{B_t}{K}\Big) - qEB_t\Big]\frac{\partial\psi(t, B_t, P_t, \alpha(t))}{\partial b} + \theta(\bar{p}_0 - \bar{p}_1 qEB_t - P_t)\frac{\partial\psi(t, B_t, P_t, \alpha(t))}{\partial p} \\
&\qquad + \frac{1}{2}\sigma^2 B_t^2\frac{\partial^2\psi(t, B_t, P_t, \alpha(t))}{\partial b^2} + \frac{1}{2}\sigma_P^2\frac{\partial^2\psi(t, B_t, P_t, \alpha(t))}{\partial p^2} \\
&\qquad\qquad + q_{ij}(v_j(t, B_t, P_t) - v_i(t, B_t, P_t))\Big\}dt\Bigg]
\end{aligned} \tag{57}$$

By Bellman's principle

$$
\begin{aligned}
\psi(s,b_s,p_s,i) = v_i(s,b_s,p_s) = \sup_{E\in\mathcal{A}_i} \mathrm{E}_{\mathrm{b_s,p_s,i}}\bigg[\int_s^{\theta_s\wedge h} & e^{-\beta t} l(i,t,B_t^{s,b_s},P_t^{s,p_s},E_t)dt \\
& + e^{-\beta(\theta_s\wedge h)} v_i(\theta_s\wedge h, B^{s,b_s}_{\theta_s\wedge h}, P^{s,p_s}_{\theta_s\wedge h})\bigg] \\
\geq \sup_{E\in\mathcal{A}_i} \mathrm{E}_{\mathrm{b_s,p_s,i}}\bigg[\int_s^{\theta_s\wedge h} & e^{-\beta t} l(i,t,B_t^{s,b_s},P_t^{s,p_s},E_t)dt \\
& + e^{-\beta(\theta_s\wedge h)} \psi(\theta_s\wedge h, B^{s,b_s}_{\theta_s\wedge h}, P^{s,p_s}_{\theta_s\wedge h}, i)\bigg] \quad (58)
\end{aligned}
$$

Setting $\tau = \mathrm{E}(\theta_s \wedge h)$ *combining (57) and (58) and multiplying both sides by* $1/(\tau - s) > 0$ *, we obtain*

$$
\begin{aligned}
& \sup_{E\in\mathcal{A}_i} \mathrm{E}_{\mathrm{b_s,p_s,i}}\Bigg[\frac{1}{\tau-s}\int_s^{\theta_s\wedge h} e^{-\beta t}\bigg\{\beta v_i(t,B_t,P_t) - \frac{\partial\psi(t,B_t,P_t,\alpha(t))}{\partial t} \\
& -\Big[r_iB_t\Big(1-\frac{B_t}{K}\Big)-qEB_t\Big]\frac{\partial\psi(t,B_t,P_t,\alpha(t))}{\partial b} - \theta(\bar{p}_0-\bar{p}_1qEB_t-P_t)\frac{\partial\psi(t,B_t,P_t,\alpha(t))}{\partial p} \\
& - \frac{1}{2}\sigma^2B_t^2\frac{\partial^2\psi(t,B_t,P_t,\alpha(t))}{\partial b^2} - \frac{1}{2}\sigma_P^2\frac{\partial^2\psi(t,B_t,P_t,\alpha(t))}{\partial p^2} \\
& - q_{ij}[v_j(t,B_t,P_t) - v_i(t,B_t,P_t)] - l(i,t,B_t^{s,b_s},P_t^{s,p_s},E_t)\bigg\}dt\Bigg] \geq 0 \quad (59)
\end{aligned}
$$

letting $\tau \downarrow s$ *and using the dominated convergence theorem, it turns out that*

$$
\begin{aligned}
& e^{-\beta s}\Bigg[-\frac{\partial\psi(s,b_s,p_s,i)}{\partial t} + \inf_{E\in\mathcal{A}_i}\bigg\{\beta v_i(s,b_s,p_s) - \\
& \Big[r_ib_s\Big(1-\frac{b_s}{K}\Big) - qEb_s\Big]\frac{\partial\psi(s,b_s,p_s,i)}{\partial b} - \theta(\bar{p}_0-\bar{p}_1qEb_s-p_s)\frac{\partial\psi(s,b_s,p_s,i)}{\partial p} \\
& - \frac{1}{2}\sigma^2b_s^2\frac{\partial^2\psi(s,b_s,p_s,i)}{\partial b^2} - \frac{1}{2}\sigma_P^2\frac{\partial^2\psi(s,b_s,p_s,i)}{\partial p^2} \\
& - q_{ij}[v_j(s,b_s,p_s) - v_i(s,b_s,p_s)] - l(i,s,b_s,p_s,E_s)\bigg\}\Bigg] \geq 0 \quad (60)
\end{aligned}
$$

This shows that the value function $v_i(t,b_t,p_t)$*,* $i = 1;2$ *satisfies the viscosity super-solution property (53).*

Step 2. $v_i(t,b_t,p_t)$*,* $i = 1;2$ *is a viscosity sub-solution of (47).*

We argue by contradiction. Assume that there exist an $i_0 \in \mathcal{S}$*, a point*

$(s,b_s,p_s) \in [0;T]\times\mathbb{R}_+^*\times\mathbb{R}_+^*$ and a testing function $\phi_{i_0}\in\mathcal{C}^{1,2,2}([0;T]\times\mathbb{R}_+^*\times\mathbb{R}_+^*)\cap\mathcal{C}_2([0;T]\times\mathbb{R}_+^*\times\mathbb{R}_+^*)$ such that $v_{i_0}(.,.,.)-\phi_{i_0}(.,.,.)$ has a local maximum at (s,b_s,p_s) in a bounded neighborhood $\mathtt{N}(b_s,p_s)$, $v_{i_0}(s,b_s,p_s)=\phi_{i_0}(s,b_s,p_s)$, and

$$\min\Bigg[-\frac{\partial\phi_{i_0}(s,b_s,p_s)}{\partial t}+\inf_{E\in\mathcal{A}_{i_0}}\Bigg\{\beta v_{i_0}(s,b_s,p_s)-\\ \Big[r_{i_0}b_s\Big(1-\frac{b_s}{K}\Big)-qEb_s\Big]\frac{\partial\phi_{i_0}(s,b_s,p_s)}{\partial b}-\theta(\bar p_0-\bar p_1qEb_s-p_s)\frac{\partial\phi_{i_0}(s,b_s,p_s)}{\partial p}\\ -\frac{1}{2}\sigma^2b_s^2\frac{\partial^2\phi_{i_0}(s,b_s,p_s)}{\partial b^2}-\frac{1}{2}\sigma_P^2\frac{\partial^2\phi_{i_0}(s,b_s,p_s)}{\partial p^2}-q_{i_0j}[v_j(s,b_s,p_s)-v_{i_0}(s,b_s,p_s)]\\ -l(_{i_0},s,b_s,p_s,E_s)\Bigg\};\quad v_{i_0}(T,b_s,p_s)-\kappa^\gamma\frac{B_T^{1-\gamma}}{1-\gamma}\Bigg]>0\quad i_0\neq j \tag{61}$$

By the continuity of all functions involved in (61) $(v_{i_0},\phi'_{i_0},\phi''_{i_0},q_{ij},l,...)$, there exist a $\delta>0$ and an open ball $\mathtt{B}_\delta(b_s,p_s)\subset\mathtt{N}(b_s,p_s)$ such that

$$-\frac{\partial\phi_{i_0}(t,b_t,p_t)}{\partial t}+\inf_{E\in\mathcal{A}_{i_0}}\Bigg\{\beta v_{i_0}(t,b_t,p_t)-\\ \Big[r_{i_0}b_t\Big(1-\frac{b_t}{K}\Big)-qEb_t\Big]\frac{\partial\phi_{i_0}(t,b_t,p_t)}{\partial b}-\theta(\bar p_0-\bar p_1qEb_t-p_t)\frac{\partial\phi_{i_0}(t,b_t,p_t)}{\partial p}\\ -\frac{1}{2}\sigma^2b_s^2\frac{\partial^2\phi_{i_0}(t,b_t,p_t)}{\partial b^2}-\frac{1}{2}\sigma_P^2\frac{\partial^2\phi_{i_0}(t,b_t,p_t)}{\partial p^2}-q_{i_0j}[v_j(t,b_t,p_t)-v_{i_0}(t,b_t,p_t)]\\ -l(_{i_0},t,b_t,p_t,E_t)\Bigg\}>\delta\quad i_0\neq j;\quad (t,b_t,p_t)\in\mathtt{B}_\delta(b_s,p_s) \tag{62}$$

and

$$v_{i_0}(T,b_t,p_t)-\kappa^\gamma\frac{B_T^{1-\gamma}}{1-\gamma}>\delta\quad (t,b_t,p_t)\in\mathtt{B}_\delta(b_s,p_s) \tag{63}$$

Let $\theta_\delta=\min\{t\geq s:(t,B_t,P_t)\notin\mathtt{B}_\delta(b_s,p_s)\}$ be the first exit time of (t,B_t,P_t) $(=(t,B_t^{s,b_s},P_t^{s,p_s}))$ from $\mathtt{B}_\delta(b_s,p_s)$. Let $\theta=\theta_\delta\wedge\tau_\alpha$ where τ_α is the first jump time of $\alpha(t)^{b_s,p_s,i_0}$. Then $\theta>0$, a.s.. For $0\leq t\leq\theta$, we *have*

$$
\begin{aligned}
&\beta v_{i_0}(t,B_t,P_t) - \frac{\partial\phi_{i_0}(t,B_t,P_t)}{\partial t}\\
&-\Big[r_{i_0}b_t\Big(1-\frac{B_t}{K}\Big)-qE_{i_0}B_t\Big]\frac{\partial\phi_{i_0}(t,B_t,P_t)}{\partial b}-\theta(\bar{p}_0-\bar{p}_1 qEB_t-P_t)\frac{\partial\phi_{i_0}(t,B_t,P_t)}{\partial p}\\
&-\frac{1}{2}\sigma^2 b_s^2\frac{\partial^2\phi_{i_0}(t,B_t,P_t)}{\partial b^2}-\frac{1}{2}\sigma_P^2\frac{\partial^2\phi_{i_0}(t,B_t,P_t)}{\partial p^2}-q_{i_0 j}[v_j(t,B_t,P_t)-v_{i_0}(t,B_t,P_t)]\\
&\qquad - l(_{i_0},t,B_t,P_t,E_t) > \delta \quad i_0\neq j; \quad (t,B_t,P_t)\in \mathtt{B}_\delta(b_s,p_s) \qquad (64)
\end{aligned}
$$

and

$$v_{i_0}(T,b_t,p_t) - \kappa^\gamma\frac{B_T^{1-\gamma}}{1-\gamma} > \delta \quad (t,b_t,p_t)\in \mathtt{B}_\delta(b_s,p_s) \qquad (65)$$

As previously, we can replace ϕ_{i_0} by a new test-function ψ defined as follows:

$$\psi(.,.,.,j) = \begin{cases} \phi_{i_0}(.,.,.), & \text{if } j = i_0 \\ v_{i_0}(.,.,.), & \text{if } \ j \neq i_0. \end{cases} \qquad (66)$$

For any stopping time $\tau\in[s;T]$. Applying Itô's formula to the switching process $e^{-\beta t}\psi(t,B_t,P_t,\alpha(t))$, taking integral from $t=s$ to $t=(\theta_s\wedge\tau)-$ and then taking expectation yield

$$
\begin{aligned}
&\mathrm{E}_{\mathrm{b_s,p_s,i}}\Big[e^{-\beta\theta\wedge\tau}\psi(\theta\wedge\tau,B_{\theta\wedge\tau},P_{\theta\wedge\tau},\alpha(\theta\wedge\tau))\Big]\\
&= v_{i_0}(s,b_s,p_s)+\mathrm{E}_{\mathrm{b_s,p_s,i}}\Bigg[\int_s^{(\theta\wedge\tau)-} e^{-\beta t}\Big\{-\beta\psi(t,B_t,P_t,\alpha(t))+\frac{\partial\psi(t,B_t,P_t,\alpha(t))}{\partial t}\\
&+\Big[r_iB_t\Big(1-\frac{B_t}{K}\Big)-qE_iB_t\Big]\frac{\partial\psi(t,B_t,P_t,\alpha(t))}{\partial b}+\theta(\bar{p}_0-\bar{p}_1 qEB_t-P_t)\frac{\partial\psi(t,B_t,P_t,\alpha(t))}{\partial p}\\
&\qquad+\frac{1}{2}\sigma^2B_t^2\frac{\partial^2\psi(t,B_t,P_t,\alpha(t))}{\partial b^2}+\frac{1}{2}\sigma_P^2\frac{\partial^2\psi(t,B_t,P_t,\alpha(t))}{\partial p^2}\\
&\qquad+q_{\alpha(t)j}[v_j(t,B_t,P_t)-\psi(t,B_t,P_t,\alpha(t))]\Big\}dt\Bigg]; \quad \alpha(t)\neq j \qquad (67)
\end{aligned}
$$

in which we used $\mathrm{E}_{\mathrm{b_s,p_s,i}}\Big[e^{-\beta\theta\wedge\tau}\psi(\theta\wedge\tau,B_{\theta\wedge\tau},P_{\theta\wedge\tau},\alpha(\theta\wedge\tau))\Big] = \mathrm{E}_{\mathrm{b_s,p_s,i}}\Big[e^{-\beta\theta_s\wedge\tau}\psi(\theta\wedge\tau,B_{\theta\wedge\tau},P_{\theta\wedge\tau},\alpha(\theta\wedge\tau)-)\Big]$ *due to continuity. Noting that the integrand in the RHS of (67) is continuous in t. Using (64), (65) and that* $v_{i_0}(t,B_t,P_t)\leq$

$\phi_{i_0}(t, B_t, P_t)$ in (67). Also noting that $\alpha(t) = i_0$ for $0 \le t \le \theta$, it follows

$$
\begin{aligned}
&v_{i_0}(s, b_s, p_s)\\
&\quad\ge \mathrm{E}_{\mathrm{b_s,p_s,i_0}}\bigg[e^{-\beta\theta\wedge\tau}\phi_{i_0}(\theta\wedge\tau, B_{\theta\wedge\tau}, P_{\theta\wedge\tau}, \alpha(\theta\wedge\tau))\\
&\quad + \int_s^{(\theta\wedge\tau)} e^{-\beta t}\Big\{\beta v_{i_0}(t, B_t, P_t) - \frac{\partial\phi_{i_0}(t, B_t, P_t)}{\partial t}\\
&- \Big[r_i B_t\Big(1 - \frac{B_t}{K}\Big) - qE_iB_t\Big]\frac{\partial\phi_{i_0}(t, B_t, P_t)}{\partial b} - \theta(\bar{p}_0 - \bar{p}_1 qEB_t - P_t)\frac{\partial\phi_{i_0}(t, B_t, P_t)}{\partial p}\\
&\quad - \frac{1}{2}\sigma^2 B_t^2\frac{\partial^2\phi_{i_0}(t, B_t, P_t)}{\partial b^2} - \frac{1}{2}\sigma_P^2\frac{\partial^2\phi_{i_0}(t, B_t, P_t)}{\partial p^2}\\
&\quad - q_{i_0 j}[v_j(t, B_t, P_t) - v_{i_0}(t, B_t, P_t)]\Big\}dt\bigg]; \quad i_0 \ne j
\end{aligned}
\tag{68}
$$

i.e

$$
\begin{aligned}
&v_{i_0}(s, b_s, p_s)\\
&\ge \mathrm{E}_{\mathrm{b_s,p_s,i_0}}\bigg[e^{-\beta\tau}v_{i_0}(\tau, B_\tau, P_\tau, \alpha(\tau))\mathbf{1}_{\{\tau<\theta\}} + e^{-\beta\theta}v_{i_0}(\theta, B_\theta, P_\theta, \alpha(\theta))\mathbf{1}_{\{\tau\ge\theta\}}\\
&\qquad + \int_s^{(\theta\wedge\tau)} e^{-\beta t}\big\{l(_{i_0}, t, B_t, P_t, E_t) + \delta\big\}dt\bigg]\\
&\ge \mathrm{E}_{\mathrm{b_s,p_s,i_0}}\bigg[e^{-\beta\tau}[\kappa^\gamma\frac{B_T^{1-\gamma}}{1-\gamma} + \delta])\mathbf{1}_{\{\tau<\theta\}} + e^{-\beta\theta}v_{i_0}(\theta, B_\theta, P_\theta, \alpha(\theta))\mathbf{1}_{\{\tau\ge\theta\}}\\
&\qquad + \int_s^{(\theta\wedge\tau)} e^{-\beta t}\big\{l(_{i_0}, t, B_t, P_t, E_t) + \delta\big\}dt\bigg]\\
&\ge \mathrm{E}_{\mathrm{b_s,p_s,i_0}}\bigg[+\int_s^{(\theta\wedge\tau)} e^{-\beta t}\big\{l(_{i_0}, t, B_t, P_t, E_t)\big\}dt + e^{-\beta\theta}v_{i_0}(\theta, B_\theta, P_\theta, \alpha(\theta))\mathbf{1}_{\{\tau\ge\theta\}}\\
&\quad + e^{-\beta\tau}[\kappa^\gamma\frac{B_T^{1-\gamma}}{1-\gamma}]\mathbf{1}_{\{\tau<\theta\}}\bigg] + \delta\mathrm{E}_{\mathrm{b_s,p_s,i_0}}\bigg[\int_s^{(\theta\wedge\tau)} e^{-\beta t}dt + e^{-\beta\tau}\mathbf{1}_{\{\tau<\theta\}}\bigg]
\end{aligned}
\tag{69}
$$

Now the estimate of the term $\mathrm{E}_{\mathrm{b_s,p_s,i_0}}\Big[\int_s^{(\theta\wedge\tau)} e^{-\beta t}dt + e^{-\beta\tau}\mathbf{1}_{\{\tau<\theta\}}\Big]$*. There exist a positive constant C_0 such that*

$$
\mathrm{E}_{\mathrm{b_s,p_s,i_0}}\bigg[\int_s^{(\theta\wedge\tau)} e^{-\beta t}dt + e^{-\beta\tau}\mathbf{1}_{\{\tau<\theta\}}\bigg] \ge C_0\big(1 - \mathrm{E}_{\mathrm{b_s,p_s,i_0}}\big[e^{-\beta\tau_\alpha}\big]\big) \tag{70}
$$

For details see (23). It follows that

$$
\begin{aligned}
& v_{i_0}(s, b_s, p_s) \\
& \qquad \geq \sup_{\tau\in[s;T];E\in\mathcal{A}} \mathrm{E}_{\mathrm{b_s,p_s,i_0}}\Big[+ \int_s^{(\theta\wedge\tau)} e^{-\beta t}\big\{l(_{i_0}, t, B_t, P_t, E_t)\big\}dt \\
& \qquad + e^{-\beta\theta} v_{i_0}(\theta, B_\theta, P_\theta, \alpha(\theta))\mathbf{1}_{\{\tau\geq\theta\}} + e^{-\beta\tau}[\kappa^\gamma \frac{B_T^{1-\gamma}}{1-\gamma}]\mathbf{1}_{\{\tau<\theta\}}\Big] \\
& \qquad\qquad + C_0\delta\big(1 - \mathrm{E}_{\mathrm{b_s,p_s,i_0}}\big[e^{-\beta\tau_\alpha}\big]\big) \qquad (71)
\end{aligned}
$$

which is a contradiction to the DP principle since $\mathrm{E}_{\mathrm{b_s,p_s,i_0}}\big[e^{-\beta\tau_\alpha}\big] < 1$. *Therefore the value function* $v_i(t, b_t, p_t)$, $i = 1; 2$ *is a viscosity sub-solution of the system (2.8).*

This completes the proof of Theorem 3.1

3.3 Comparison principle: uniqueness of the viscosity solution

In this section, we prove a comparison result from which we obtain the uniqueness of the solution of the coupled system of partial differential equations. In proving comparison results for viscosity solutions, the notion of parabolic superjet and subjet defined by Crandall, Ishii and Lions [19] is particularly useful. Thus, we begin by

Definition 3.2. *Given* $v \in \mathcal{C}^o([0;T]\times\mathbb{R}\times\mathbb{R}\times\mathcal{S})$ *and* $(t;b;p;i) \in [0;T)\times\mathbb{R}\times\mathbb{R}\times\mathcal{S}$, *we define the parabolic superjet:*

$$
\begin{aligned}
& \mathcal{P}^{2,+}v(t,b,p,i) = \Big\{(c,q,M) \in \mathbb{R}\times\mathbb{R}^2\times\mathbb{S}^2 : v(s,b',p',i) \leq v(t,b,p,i) \\
& \quad + c(s-t) + q.((b'-b),(p'-p)) + \frac{1}{2}((b'-b),(p'-p)).M((b'-b),(p'-p)) \\
& \qquad\qquad + o(|((b'-b),(p'-p))|^2) \quad as \quad (s;b',p') \to (t;b,p)\Big\} \qquad (72)
\end{aligned}
$$

and its closure:

$$
\begin{aligned}
& \bar{\mathcal{P}}^{2,+}v(t,b,p,i) = \Big\{(c,q,M) = \lim_{n\to\infty}(c_n,q_n,M_n) \\
& \qquad\qquad with \quad (c_n,q_n,M_n) \in \mathcal{P}^{2,+}v(t_n,b_n,p_n,i) \quad and \\
& \qquad\qquad \lim_{n\to\infty}(t_n,b_n,p_n,v(t_n,b_n,p_n,i)) = (t,b,p,v(t,b,p,i))\Big\} \qquad (73)
\end{aligned}
$$

Similarly, we define the parabolic subjet $\bar{\mathcal{P}}^{2,-}v(t,b,p,i) = -\bar{\mathcal{P}}^{2,+}(-v)(t,b,p,i)$ *and its closure* $\bar{\mathcal{P}}^{2,-}v(t,b,p,i) = -\bar{\mathcal{P}}^{2,+}(-v)(t,b,p,i)$

It is proved in (24) that

$$\mathcal{P}^{2,+(-)}v(t,b,p,i) = \Bigg\{ \bigg(\frac{\phi}{\partial t}(t,b,p,i), D_{(b,p)}\phi(t,b,p,i), D^2_{(b,p)}\phi(t,b,p,i) \\ \textit{and } v-\phi \text{ has a global maximum (minimum) at } (t,b,p,i) \bigg) \Bigg\} \quad (74)$$

The previous notions lead to new definition of viscosity solutions.

Definition 3.3. $u_i \in \mathcal{C}^0([0;T] \times \mathbb{R}^*_+ \times \mathbb{R}^*_+)$ *satisfying the polynomial growth condition is a viscosity solution of (49)*

if

(1) *for any test-function* $\phi \in \mathcal{C}^{1,2,2}([0;T] \times \mathbb{R}^*_+ \times \mathbb{R}^*_+)$ *if* (t,b,p) *is a local maximum point of* $u_i(.,.,.) - \phi(.,.,.)$ *and if* $(c,q,L_1) \in \bar{\mathcal{P}}^{2,+}u(t,b,p,i)$ *with* $c = \partial\phi(t,b,p)/\partial t$; $q = D_{(b,p)}\phi(t,b,p)$ *and* $L_1 \leq D^2_{(b,p)}\phi(t,b,p)$, *then*

$$\mathcal{H}\left(i,s,b,p,u_i,\frac{\partial u_i}{\partial s},\frac{\partial u_i}{\partial b},\frac{\partial u_i}{\partial p},\frac{\partial^2 u_i}{\partial b^2},\frac{\partial^2 u_i}{\partial p^2}\right) \leq 0 \quad (75)$$

in this case u is a viscosity subsolution,

(2) *for any test-function* $\phi \in \mathcal{C}^{1,2,2}([0;T] \times \mathbb{R}^*_+ \times \mathbb{R}^*_+)$ *if* (t,b,p) *is a local minimum point of* $u_i(.,.,.) - \phi(.,.,.)$ *and if* $(c,q,L_2) \in \bar{\mathcal{P}}^{2,-}u(t,b,p,i)$ *with* $c = \partial\phi(t,b,p)/\partial t$; $q = D_{(b,p)}\phi(t,b,p)$ *and* $L_2 \geq D^2_{(b,p)}\phi(t,b,p)$, *then*

$$\mathcal{H}\left(i,s,b,p,u_i,\frac{\partial u_i}{\partial s},\frac{\partial u_i}{\partial b},\frac{\partial u_i}{\partial p},\frac{\partial^2 u_i}{\partial b^2},\frac{\partial^2 u_i}{\partial p^2}\right) \geq 0 \quad (76)$$

in this case u is a viscosity supersolution,

It is proved in (25) that this new definition and the previous one are equivalent. We refer the reader to the mentioned paper for a proof. The last definition is particular suitable for the discussion of a maximum principle which is the backbone of the uniqueness problem for the viscosity solutions theory.

Before state next lemma, we first introduce the inf and sup-convolution operations we are going to use.

Definition 3.4. *For any usc (upper semi-continuous) function $U : \mathbb{R}^m \to \mathbb{R}$ and any lsc (lower semi-continuous) function $V : \mathbb{R}^m \to \mathbb{R}$, we set*

$$R^\alpha[U](z,r) = \sup_{|Z-z|\leq 1} \left\{ U(Z) - r\cdot(Z-z) - \frac{|Z-z|}{2\alpha} \right\} \tag{77}$$

$$R_\alpha[V](z,r) = \inf_{|Z-z|\leq 1} \left\{ V(Z) + r\cdot(Z-z) + \frac{|Z-z|}{2\alpha} \right\} \tag{78}$$

$R^\alpha[U](z,r)$ is called the modified sup-convolution and $R_\alpha[V](z,r)$ the modified inf-convolution. Notice that $R_\alpha[V](z,r) = -R^\alpha[-U](z,r)$

Lemma 3.2. *(nonlocal Jensen-Ishii's lemma (25))*

For any $i \in \mathcal{S}$, let $u_i(.,.,.)$ and $v_i(.,.,.)$ be, respectively, a usc and lsc function defined on $[0;T]\times\mathbb{R}_+\times\mathbb{R}_+$ and $\phi \in \mathcal{C}^{1,2,2}([0;T]\times\mathbb{R}^2_+\times\mathbb{R}^2_+)\cap\mathcal{C}_2([0;T]\times\mathbb{R}^2_+\times\mathbb{R}^2_+)$ if $(\hat{t},(\hat{b}_1,\hat{p}_1),(\hat{b}_2,\hat{p}_2)) \in [0;T]\times\mathbb{R}^2_+\times\mathbb{R}^2_+$ is a zero global maximum point of $u_i(t,b,p)-v_i(t,b',p')-\phi(t,(b,p),(b',p'))$ and if $c-d := D_t\phi(\hat{t},(\hat{b}_1,\hat{p}_1),(\hat{b}_2,\hat{p}_2))$, $q := D_{(b,p)}\phi(\hat{t},(\hat{b}_1,\hat{p}_1),(\hat{b}_2,\hat{p}_2))$, $r := -D_{(b',p')}\phi(\hat{t},(\hat{b}_1,\hat{p}_1),(\hat{b}_2,\hat{p}_2))$, then for any $K > 0$, there exists $\alpha(K) > 0$ such that, for any $0 < \alpha < \alpha(K)$, we have: there exist sequences $t_k \to \hat{t}$, $(b_k,p_k) \to (\hat{b}_1,\hat{p}_1)$, $(b'_k,p'_k) \to (\hat{b}_2,\hat{p}_2)$, $q_k \to q$, $r_k \to r$, matrices M_k, N_k and a sequence of functions ϕ_k, converging to the function $\phi_\alpha := R^\alpha[\phi](((b,p),(b',p')),(q,r))$ uniformly in $\mathbb{R}^2_+\times\mathbb{R}^2_+$ and in $C^2(B((\hat{t},(\hat{b}_1,\hat{p}_1),(\hat{b}_2,\hat{p}_2)),K))$, such that

$$u_i(t_k,(b_k,p_k)) \to u_i(\hat{t},(\hat{b}_1,\hat{p}_1)), \qquad v_i(t_k,(b'_k,p'_k)) \to v_i(\hat{t},(\hat{b}_2,\hat{p}_2)) \tag{79}$$

$(t_k,(b_k,p_k),(b'_k,p'_k))$ is a global maximum of $u_i(.,(.,.))-v_i(.,(.,.))-\phi(.,(.,.),(.,.))$

$$(c_k,q_k,M_k) \in \bar{\mathcal{P}}^{2,+}u_i(t_k,(b_k,p_k)) \tag{80}$$

$$(-d_k,r_k,N_k) \in \bar{\mathcal{P}}^{2,-}v_i(t_k,(b'_k,p'_k)) \tag{81}$$

$$-\frac{1}{\alpha}\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} M_k & 0 \\ 0 & -N_k \end{pmatrix} \leq D^2_{(b,p),(b',p')}\phi(t_k,(b_k,p_k),(b'_k,p'_k)) \tag{82}$$

Here $c_k - d_k = \nabla_t \phi(t_k, (b_k, p_k), (b'_k, p'_k))$, $q_k = \nabla_{(b,p)}\phi(t_k, (b_k, p_k), (b'_k, p'_k))$, $r_k = \nabla_{(b',p')}\phi(t_k, (b_k, p_k), (b'_k, p'_k))$ *and* $\phi_\alpha(\hat{t}, (\hat{b}_1, \hat{p}_1), (\hat{b}_2, \hat{p}_2)) = \phi(\hat{t}, (\hat{b}_1, \hat{p}_1), (\hat{b}_2, \hat{p}_2))$, $\nabla\phi_\alpha(\hat{t}, (\hat{b}_1, \hat{p}_1), (\hat{b}_2, \hat{p}_2)) = \nabla\phi(\hat{t}, (\hat{b}_1, \hat{p}_1), (\hat{b}_2, \hat{p}_2))$

Now we can state our comparison result.

Theorem 3.2. *(comparison principle):*

If $u_i(t, b, p)$ *and* $v_i(t, b, p)$ *are continuous in* (t, b, p) *and are, respectively, viscosity subsolution and supersolution of the HJB system* (47) *with at most linear growth then*

$$u_i(t, b, p) \le v_i(t, b, p) \ \textit{for all} \ (t, b, p, i) \in [0; T] \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathcal{S} \tag{83}$$

Proof 3.5. *For* $\varrho, \epsilon, \delta, \lambda > 0$, *we define the auxiliary functions* ϕ : $(0; T] \times \mathbb{R}^2_+ \times \mathbb{R}^2_+ \to \mathbb{R}$ *and* Ξ: $[0; T] \times \mathbb{R}^2_+ \times \mathbb{R}^2_+ \times \mathcal{S}$ *by*

$$\phi(t, (b, p), (b', p')) = \frac{\varrho}{t} + \frac{1}{2\epsilon}|(b, p) - (b', p')|^2 + \delta e^{\lambda(T-t)}(|(b, p)|^2 + |(b', p')|^2) \tag{84}$$

and

$$\Xi(t, (b, p), (b', p'), i) = v_i(t, b, p) - u_i(t, b', p') - \phi(t, (b, p), (b', p')) \tag{85}$$

By using the linear growth of v_i *and* u_i, *we have for each* $i \in \mathcal{S}$

$$\lim_{|(b,p)|+|(b',p')|\to\infty} \Xi(t, (b, p), (b', p'), i) = -\infty \tag{86}$$

Then, since v_i *and* u_i *are uniformly continuous with respect to* (t, b, p) *on each compact subset of* $[0; T] \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathcal{S}$ *and that* $\mathcal{S}$ *is a finite set,* Ξ *attains its global maximum at some finite point belonging to a compact* $K \subset [0; T] \times \mathbb{R}^2_+ \times \mathbb{R}^2_+ \times \mathcal{S}$ *say,* $\big(t_{\delta\epsilon}, (b_{1\delta\epsilon}, p_{1\delta\epsilon}), (b_{2\delta\epsilon}, p_{2\delta\epsilon}), \alpha_{\delta\epsilon}\big)$. *Observing that* $2\Xi\big(t_{\delta\epsilon}, (b_{1\delta\epsilon}, p_{1\delta\epsilon}), (b_{2\delta\epsilon}, p_{2\delta\epsilon}), \alpha_{\delta\epsilon}\big) \ge \Xi\big(t_{\delta\epsilon}, (b_{1\delta\epsilon}, p_{1\delta\epsilon}), (b_{2\delta\epsilon}, p_{2\delta\epsilon}), \alpha_{\delta\epsilon}\big) + \Xi\big(t_{\delta\epsilon}, (b_{1\delta\epsilon}, p_{1\delta\epsilon}), (b_{2\delta\epsilon}, p_{2\delta\epsilon}), \alpha_{\delta\epsilon}\big)$ *and using the uniform continuity of* v_i *and* u_i *on* K *we have*

$$\begin{aligned}
&\frac{1}{\epsilon}|(b_{1\delta\epsilon}, p_{1\delta\epsilon}) - (b_{2\delta\epsilon}, p_{2\delta\epsilon})|^2 \\
&\le v_i\big(t_{\delta\epsilon}, (b_{1\delta\epsilon}, p_{1\delta\epsilon})\big) - v_i\big(\hat{t}_{\delta\epsilon}, (b_{2\delta\epsilon}, p_{2\delta\epsilon})\big) + u_i\big(t_{\delta\epsilon}, (b_{1\delta\epsilon}, p_{1\delta\epsilon})\big) - u_i\big(t_{\delta\epsilon}, (b_{2\delta\epsilon}, p_{2\delta\epsilon})\big) \\
&\qquad \le 2C|(b_{1\delta\epsilon}, p_{1\delta\epsilon}) - (b_{2\delta\epsilon}, p_{2\delta\epsilon})|
\end{aligned} \tag{87}$$

Thus,

$$|(b_{1\delta\epsilon}, p_{1\delta\epsilon}) - (b_{2\delta\epsilon}, p_{2\delta\epsilon})| \leq 2C\epsilon \tag{88}$$

where C is a positive constant independent of $\varrho, \epsilon, \delta, \lambda$. From the inequality,

$$2\Xi\big(T, (0,0), (0,0), \alpha_{\delta\epsilon}\big) \leq 2\Xi\big(t_{\delta\epsilon}, (b_{1\delta\epsilon}, p_{1\delta\epsilon}), (b_{2\delta\epsilon}, p_{2\delta\epsilon}), \alpha_{\delta\epsilon}\big) \tag{89}$$

and the linear growth for v_i and u_i, we have:

$$\begin{aligned}\delta\big(|(b_{1\delta\epsilon}, p_{1\delta\epsilon}))|^2 + |(b_{2\delta\epsilon}, p_{2\delta\epsilon}|^2\big) &\leq e^{-\lambda(T-t_{\delta\epsilon})}\Big[v_i\big(t_{\delta\epsilon}, b_{1\delta\epsilon}, p_{1\delta\epsilon}\big) - v_i\big(T,0,0\big) \\ &\quad + u_i\big(T,0,0\big) - u_i\big(t_{\delta\epsilon}, b_{2\delta\epsilon}, p_{2\delta\epsilon}\big)\Big] \\ &\leq e^{-\lambda(T-t_{\delta\epsilon})}C_2\big(1 + |(b_{1\delta\epsilon}, p_{1\delta\epsilon})| + |(b_{2\delta\epsilon}, p_{2\delta\epsilon})|\big)\end{aligned} \tag{90}$$

It follows that

$$\frac{\delta\big(|(b_{1\delta\epsilon}, p_{1\delta\epsilon}))|^2 + |(b_{2\delta\epsilon}, p_{2\delta\epsilon}|^2\big)}{(1 + |(b_{1\delta\epsilon}, p_{1\delta\epsilon})| + |(b_{2\delta\epsilon}, p_{2\delta\epsilon})|)} \leq C_2 \tag{91}$$

Consequently, there exists $C_\delta > 0$ such that

$$|(b_{1\delta\epsilon}, p_{1\delta\epsilon})| + |(b_{2\delta\epsilon}, p_{2\delta\epsilon})| \leq C_\delta \tag{92}$$

This inequality implies that for any fixed $\delta \in (0,1)$, the sets $\{(b_{1\delta\epsilon}, p_{1\delta\epsilon}), \epsilon > 0\}$ and $\{(b_{2\delta\epsilon}, p_{2\delta\epsilon}), \epsilon > 0\}$ are bounded by C_δ independent of ϵ. It follows from inequalities (90) and (92) that, possibly if necessary along a subsequence, named again $\big(t_{\delta\epsilon}, (b_{1\delta\epsilon}, p_{1\delta\epsilon}), (b_{2\delta\epsilon}, p_{2\delta\epsilon}), \alpha_{\delta\epsilon}\big)$ that there exist $(b_{1\delta 0}, p_{1\delta 0}) \in \mathbb{R}^2_+$, $t_{\delta\epsilon 0} \in (0,T]$ and $\alpha_{\delta\epsilon 0} \in \mathcal{S}$ such that: $\lim_{\epsilon\downarrow 0}(b_{1\delta\epsilon}, p_{1\delta\epsilon}) = (b_{1\delta 0}, p_{1\delta 0}) = \lim_{\epsilon\downarrow 0}(b_{1\delta\epsilon}, p_{1\delta\epsilon})$, $\lim_{\epsilon\downarrow 0} t_{\delta\epsilon} = t_{\delta 0}$, $\lim_{\epsilon\downarrow 0} \alpha_{\delta\epsilon} = \alpha_{\delta 0}$.
If $t_{\delta\epsilon} = T$ then writing that $\Xi\big(t, (b,p), (b,p), \alpha_{\delta\epsilon}\big) \leq \Xi\big(T, (b_{1\delta\epsilon}, p_{1\delta\epsilon}), (b_{2\delta\epsilon}, p_{2\delta\epsilon}), \alpha_{\delta\epsilon}\big)$,

we have

$$
\begin{aligned}
u_i(t,b,p) - v_i(t,b,p) - \frac{\varrho}{t} - 2\delta e^{\lambda(T-t)}(|(b,p)|^2)& \\
\le u_i(T,(b_{1\delta\epsilon},p_{1\delta\epsilon})) - v_i(T,(b_{2\delta\epsilon},p_{2\delta\epsilon})) - \frac{\varrho}{T}& \\
- \frac{1}{2\epsilon}|(b_{1\delta\epsilon},p_{1\delta\epsilon}) - (b_{2\delta\epsilon},p_{2\delta\epsilon})|^2 - \delta(|(b_{1\delta\epsilon},p_{1\delta\epsilon})|^2 + |(b_{2\delta\epsilon},p_{2\delta\epsilon})|^2)& \\
\le u_i(T,(b_{1\delta\epsilon},p_{1\delta\epsilon})) - v_i(T,(b_{2\delta\epsilon},p_{2\delta\epsilon}))& \\
= [u_i(T,(b_{1\delta\epsilon},p_{1\delta\epsilon})) - v_i(T,(b_{1\delta\epsilon},p_{1\delta\epsilon}))]& \\
+ [v_i(T,(b_{1\delta\epsilon},p_{1\delta\epsilon})) - v_i(T,(b_{2\delta\epsilon},p_{2\delta\epsilon}))]& \\
\le C_1|(b_{1\delta\epsilon},p_{1\delta\epsilon}) - (b_{2\delta\epsilon},p_{2\delta\epsilon})|& \quad (93)
\end{aligned}
$$

where the last inequality follows from the uniform continuity of v_i and by assumption that $u_i(T,(b_{1\delta\epsilon},p_{1\delta\epsilon})) = \kappa^\gamma \frac{b_T^{1-\gamma}}{1-\gamma} = v_i(T,(b_{1\delta\epsilon},p_{1\delta\epsilon}))$ Sending $\varrho,\epsilon,\delta \downarrow 0$ and using estimate (88), we have: $u_i(t,b,p) \le v_i(t,b,p)$. Assume now that $t_{\delta\epsilon} < T$.

Applying Lemma 3.2 with u_i, v_i and $\phi(t,(b,p),(b',p'))$ at the point $(t_{\delta\epsilon},(b_{1\delta\epsilon},p_{1\delta\epsilon}),(b_{2\delta\epsilon},p_{2\delta\epsilon}),\alpha_{\delta\epsilon}) \in (0;T) \times \mathbb{R}_+^2 \times \mathbb{R}_+^2 \times \mathbb{S}$, for any $\zeta \in (0,1)$ there are $d \in \mathbb{R}, M_{\delta\epsilon}, N_{\delta\epsilon} \in \mathbb{S}^2$ such that:

$$
\begin{aligned}
&\Bigg(d - \frac{\varrho}{t^2} - \lambda\delta e^{\lambda(T-t)}(|(b_{\delta\epsilon},p_{\delta\epsilon})|^2 + |(b'_{\delta\epsilon},p'_{\delta\epsilon})|^2), \frac{1}{\epsilon}((b_{\delta\epsilon},p_{\delta\epsilon}) - (b'_{\delta\epsilon},p'_{\delta\epsilon})) \\
&\qquad + 2\delta e^{\lambda(T-t)}(b_{\delta\epsilon},p_{\delta\epsilon}), M_{\delta\epsilon} + 2\delta e^{\lambda(T-t)} I\Bigg) \in \bar{\mathcal{P}}^{2,+}v(t,b,p,i) \\
&\left(d, \frac{1}{\epsilon}((b_{\delta\epsilon},p_{\delta\epsilon}) - (b'_{\delta\epsilon},p'_{\delta\epsilon})) - 2\delta e^{\lambda(T-t)}(b'_{\delta\epsilon},p'_{\delta\epsilon}), N_{\delta\epsilon} - 2\delta e^{\lambda(T-t)} I\right) \in \bar{\mathcal{P}}^{2,-}v(t,b,p,i)
\end{aligned}
\tag{94}
$$

and

$$
\begin{aligned}
-\frac{1}{\zeta}\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} &\le \begin{pmatrix} M_{\delta\epsilon} & 0 \\ 0 & -N_{\delta\epsilon} \end{pmatrix} \le D^2_{(b,p),(b',p')}\phi(t_{\delta\epsilon},(b_{\delta\epsilon},p_{\delta\epsilon}),(b'_{\delta\epsilon},p'_{\delta\epsilon})) \\
&\quad + \zeta\Big(D^2_{(b,p),(b',p')}\phi(t_{\delta\epsilon},(b_{\delta\epsilon},p_{\delta\epsilon}),(b'_{\delta\epsilon},p'_{\delta\epsilon}))\Big)^2 \\
&\le \frac{\epsilon+\zeta(2+4\delta\epsilon e^{\lambda(T-t)})}{\epsilon^2}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + (2\delta+4\zeta\delta^2\epsilon e^{\lambda(T-t)})e^{\lambda(T-t)}\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}
\end{aligned} \tag{95}
$$

Letting $\delta \downarrow 0$ and taking $\zeta = \frac{\epsilon}{2}$, we obtain

$$
-\frac{1}{\epsilon}\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \le \begin{pmatrix} M_{\delta\epsilon} & 0 \\ 0 & -N_{\delta\epsilon} \end{pmatrix} \le \frac{2}{\epsilon}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix} \tag{96}
$$

It follows that

$$
\begin{aligned}
&(b_{\delta\epsilon},p_{\delta\epsilon})M_{\delta\epsilon}\begin{pmatrix} b_{\delta\epsilon} \\ p_{\delta\epsilon} \end{pmatrix} - (b'_{\delta\epsilon},p'_{\delta\epsilon})N_{\delta\epsilon}\begin{pmatrix} b'_{\delta\epsilon} \\ p'_{\delta\epsilon} \end{pmatrix} \\
&\qquad = ((b_{\delta\epsilon},p_{\delta\epsilon}),(b'_{\delta\epsilon},p'_{\delta\epsilon}))\begin{pmatrix} M_{\delta\epsilon} & 0 \\ 0 & -N_{\delta\epsilon} \end{pmatrix}\left(\begin{pmatrix} b_{\delta\epsilon} \\ p_{\delta\epsilon} \end{pmatrix} \begin{pmatrix} b'_{\delta\epsilon} \\ p'_{\delta\epsilon} \end{pmatrix}\right) \\
&\qquad \le ((b_{\delta\epsilon},p_{\delta\epsilon}),(b'_{\delta\epsilon},p'_{\delta\epsilon}))\left[\frac{2}{\epsilon}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}\right]\left(\begin{pmatrix} b_{\delta\epsilon} \\ p_{\delta\epsilon} \end{pmatrix} \begin{pmatrix} b'_{\delta\epsilon} \\ p'_{\delta\epsilon} \end{pmatrix}\right) \\
&\qquad \le \frac{2}{\epsilon}|(b_{\delta\epsilon},p_{\delta\epsilon})-(b'_{\delta\epsilon},p'_{\delta\epsilon})|^2
\end{aligned} \tag{97}
$$

Furthermore, the definition of the viscosity subsolution u_i and supersolution v_i

implies that

$$\min\Bigg[\beta u_{i_0}(t_{\delta\epsilon}, b_{\delta\epsilon}, p_{\delta\epsilon}) - \Big(d - \frac{\varrho}{t_{\delta\epsilon}^2} - \lambda\delta e^{\lambda(T-t_{\delta\epsilon})}(|(b_{\delta\epsilon}, p_{\delta\epsilon})|^2 + |(b'_{\delta\epsilon}, p'_{\delta\epsilon})|^2)\Big)$$

$$+ \inf_{E\in\mathcal{A}_{i_0}} \Bigg\{ - \Big[r_{i_0} b_{\delta\epsilon}\Big(1 - \frac{b_{\delta\epsilon}}{K}\Big) - qEb_{\delta\epsilon}\Big]\Big(\frac{1}{\epsilon}(b_{\delta\epsilon} - b'_{\delta\epsilon}) + 2\delta e^{\lambda(T-t)} b_{\delta\epsilon}\Big)$$

$$-\theta(\bar{p}_0 - \bar{p}_1 qEb_{\delta\epsilon} - p_s)\Big(\frac{1}{\epsilon}(p_{\delta\epsilon} - p'_{\delta\epsilon}) + 2\delta e^{\lambda(T-t)} p_{\delta\epsilon}\Big) - \frac{1}{2}(\sigma b_{\delta\epsilon}; \sigma_P)(M_{\delta\epsilon} + 2\delta e^{\lambda(T-t)} I)\begin{pmatrix} \sigma b_{\delta\epsilon} \\ \sigma_p \end{pmatrix}$$

$$- q_{i_0 j}[u_j(t_{\delta\epsilon}, b_{\delta\epsilon}, p_{\delta\epsilon}) - u_{i_0}(t_{\delta\epsilon}, b_{\delta\epsilon}, p_{\delta\epsilon})]$$

$$- l(_{i_0}, t_{\delta\epsilon}, b_{\delta\epsilon}, p_{\delta\epsilon}, E_{t_{\delta\epsilon}})\Bigg\}; \quad u_{i_0}(T, , b_{\delta\epsilon}, p_{\delta\epsilon}) - \kappa^{\gamma}\frac{B_T^{1-\gamma}}{1-\gamma}\Bigg] \leq 0 \quad i_0 \neq j \tag{98}$$

and

$$\min\Bigg[\beta v_{i_0}(t_{\delta\epsilon}, b_{\delta\epsilon}, p_{\delta\epsilon}) - d + \inf_{E\in\mathcal{A}_{i_0}} \Bigg\{ - \Big[r_{i_0} b_{\delta\epsilon}\Big(1 - \frac{b_{\delta\epsilon}}{K}\Big) - qEb_{\delta\epsilon}\Big]\Big(\frac{1}{\epsilon}(b_{\delta\epsilon} - b'_{\delta\epsilon}) + 2\delta e^{\lambda(T-t)} b'_{\delta\epsilon}\Big)$$

$$-\theta(\bar{p}_0 - \bar{p}_1 qEb_{\delta\epsilon} - p_s)\Big(\frac{1}{\epsilon}(p_{\delta\epsilon} - p'_{\delta\epsilon}) + 2\delta e^{\lambda(T-t)} p'_{\delta\epsilon}\Big) - \frac{1}{2}(\sigma b'_{\delta\epsilon}; \sigma_P)(N_{\delta\epsilon} - 2\delta e^{\lambda(T-t)} I)\begin{pmatrix} \sigma b'_{\delta\epsilon} \\ \sigma_p \end{pmatrix}$$

$$- q_{i_0 j}[v_j(t_{\delta\epsilon}, b'_{\delta\epsilon}, p'_{\delta\epsilon}) - v_{i_0}(t_{\delta\epsilon}, b'_{\delta\epsilon}, p'_{\delta\epsilon})]$$

$$- l(_{i_0}, t_{\delta\epsilon}, b'_{\delta\epsilon}, p'_{\delta\epsilon}, E_{t_{\delta\epsilon}})\Bigg\}; \quad v_{i_0}(T, , b_{\delta\epsilon}, p_{\delta\epsilon}) - \kappa^{\gamma}\frac{B_T^{1-\gamma}}{1-\gamma}\Bigg] \geq 0 \quad i_0 \neq j \tag{99}$$

Let us define operators $A^E(x, v, \phi, X, Z)$ and $B^E(x, v)$.

$$A^E(t, b, p, w, X, YZ) = \Big[r_{i_0} b\Big(1 - \frac{b}{K}\Big) - qEb\Big]X + \theta(\bar{p}_0 - \bar{p}_1 qEb - p_s)Y + \frac{1}{2}wZw' \tag{100}$$

$$B^E(t, b, p, v) = q_{i_0 j}[v_j(t, b, p) - v_{i_0}(t, b, p)] \tag{101}$$

by Subtracting these last two inequalities and remarking that $\min(x; y) - \min(z; t) \leq 0$ implies either $x - z \leq 0$ or $y - t \leq 0$, we divide our consideration into two cases:

Case 1

$$\beta\big[u_{i_0}(t_{\delta\epsilon},b_{\delta\epsilon},p_{\delta\epsilon})-v_{i_0}(t_{\delta\epsilon},b_{\delta\epsilon},p_{\delta\epsilon})\big]+\frac{\varrho}{t_{\delta\epsilon}^2}+\lambda\delta e^{\lambda(T-t_{\delta\epsilon})}(|(b_{\delta\epsilon},p_{\delta\epsilon})|^2+|(b'_{\delta\epsilon},p'_{\delta\epsilon})|^2)$$
$$\leq \sup_{E\in\mathcal{A}_{i_0}}\big\{l(_{i_0},t_{\delta\epsilon},b_{\delta\epsilon},p_{\delta\epsilon},E_{t_{\delta\epsilon}})-l(_{i_0},t_{\delta\epsilon},b'_{\delta\epsilon},p'_{\delta\epsilon},E_{t_{\delta\epsilon}})\big\}$$
$$+\sup_{E\in\mathcal{A}_{i_0}}\Bigg\{A^E\Big(t_{\delta\epsilon},b_{\delta\epsilon},p_{\delta\epsilon},(\sigma b_{\delta\epsilon};\sigma_P),\frac{1}{\epsilon}(b_{\delta\epsilon}-b'_{\delta\epsilon})+2\delta e^{\lambda(T-t_{\delta\epsilon})}b_{\delta\epsilon},\frac{1}{\epsilon}(p_{\delta\epsilon}-p'_{\delta\epsilon})+2\delta e^{\lambda(T-t_{\delta\epsilon})}p_{\delta\epsilon},$$
$$M_{\delta\epsilon}+2\delta e^{\lambda(T-t_{\delta\epsilon})}I\Big)$$
$$-A^E\Big(t_{\delta\epsilon},b'_{\delta\epsilon},p'_{\delta\epsilon},(\sigma b'_{\delta\epsilon};\sigma_P),\frac{1}{\epsilon}(b_{\delta\epsilon}-b'_{\delta\epsilon})+2\delta e^{\lambda(T-t)}b'_{\delta\epsilon},\frac{1}{\epsilon}(p_{\delta\epsilon}-p'_{\delta\epsilon})+2\delta e^{\lambda(T-t_{\delta\epsilon})}p'_{\delta\epsilon},$$
$$N_{\delta\epsilon}-2\delta e^{\lambda(T-t_{\delta\epsilon})}I\Big)\Bigg\}$$
$$+\sup_{E\in\mathcal{A}_{i_0}}\Bigg\{B^E(t_{\delta\epsilon},b_{\delta\epsilon},p_{\delta\epsilon}u)-B^E(t_{\delta\epsilon},b'_{\delta\epsilon},p'_{\delta\epsilon}v)]\Bigg\}\equiv\mathcal{I}_1+\mathcal{I}_2+\mathcal{I}_3 \quad (102)$$

In view of condition (27) on l and from estimate (3.1), we have the classical estimates of $\mathcal{I}_1$ and $\mathcal{I}_2$:

$$\mathcal{I}_1 \leq C|(b_{\delta\epsilon},p_{\delta\epsilon})-(b'_{\delta\epsilon},p'_{\delta\epsilon})| \quad (103)$$
$$\mathcal{I}_2 \leq C(\frac{1}{\epsilon}|(b_{\delta\epsilon},p_{\delta\epsilon})-(b'_{\delta\epsilon},p'_{\delta\epsilon})|^2+2\delta e^{\lambda(T-t_{\delta\epsilon})}(1+|(b_{\delta\epsilon},p_{\delta\epsilon})|^2+|(b'_{\delta\epsilon},p'_{\delta\epsilon})|^2) \quad (104)$$

Using the Lipschitz condition for u and v, we have

$$\mathcal{I}_3 \leq 2C|(b_{\delta\epsilon},p_{\delta\epsilon})-(b'_{\delta\epsilon},p'_{\delta\epsilon})| \quad (105)$$

Writing that $\Xi(t,(b,p),(b,p),i) \leq \Xi(t_{\delta\epsilon},(b_{\delta\epsilon},p_{\delta\epsilon}),(b_{\delta\epsilon},p_{\delta\epsilon}),i)$ for $i \in \mathcal{S}$ and using the inequality (102),

$$u_i(t,b,p)-v_i(t,b,p)-\frac{\varrho}{t}-2\delta e^{\lambda(T-t)}|(b,p)|^2 \leq$$
$$v_i(t_{\delta\epsilon},b_{\delta\epsilon},p_{\delta\epsilon})-u_i(t_{\delta\epsilon},b_{\delta\epsilon},p_{\delta\epsilon})-\frac{\varrho}{t_{\delta\epsilon}}-2\delta e^{\lambda(T-t)}|(b_{\delta\epsilon},p_{\delta\epsilon})|^2 \leq$$
$$\frac{1}{\beta}\Big[\mathcal{I}_1+\mathcal{I}_2+\mathcal{I}_3\Big]-\frac{\varrho}{\beta t_{\delta\epsilon}^2}-\frac{\lambda}{\beta}\delta e^{\lambda(T-t_{\delta\epsilon})}(|(b_{\delta\epsilon},p_{\delta\epsilon})|^2+|(b'_{\delta\epsilon},p'_{\delta\epsilon})|^2) \quad (106)$$

this implies

$$u_i(t,b,p)-v_i(t,b,p)-\frac{\varrho}{t}-2\delta e^{\lambda(T-t)}|(b,p)|^2 \leq$$
$$\frac{1}{\beta}\Big[\mathcal{I}_1+\mathcal{I}_2+\mathcal{I}_3\Big]-\frac{\lambda}{\beta}\delta e^{\lambda(T-t_{\delta\epsilon})}(|(b_{\delta\epsilon},p_{\delta\epsilon})|^2+|(b'_{\delta\epsilon},p'_{\delta\epsilon})|^2) \quad (107)$$

Sending $\epsilon \downarrow 0$, with the above estimates of $(\mathcal{I}_1) - (\mathcal{I}_2) - (\mathcal{I}_3)$, we obtain:

$$u_i(t,b,p) - v_i(t,b,p) - \frac{\varrho}{t} - 2\delta e^{\lambda(T-t)}|(b,p)|^2 \leq \frac{2\delta}{\beta} e^{\lambda(T-t_0)} \Big[C(1+2|(b_0,p_0)|^2) - \lambda|(b_0,p_0)|^2 \Big] \tag{108}$$

Choose λ sufficiently large positive ($\lambda \geq 2C$) and send $\varrho, \delta \to 0^+$ to conclude that $u_i(t,b,p) \leq v_i(t,b,p)$

Case 2 the second case occurs if

$$u_{i_0}(T,,b_{\delta\epsilon},p_{\delta\epsilon}) - v_{i_0}(T,,b_{\delta\epsilon},p_{\delta\epsilon}) \leq 0 \tag{109}$$

and

finally that $u_i(t,b,p) \leq v_i(t,b,p)$

This completes the proof.

The following corollary follows from Theorems 3.1 and 3.2.

Corollary 3.1. *The value function v is a uniques viscosity solution of (47) that has at most a linear growth.*

4 Monotone Finite Difference and Simulation

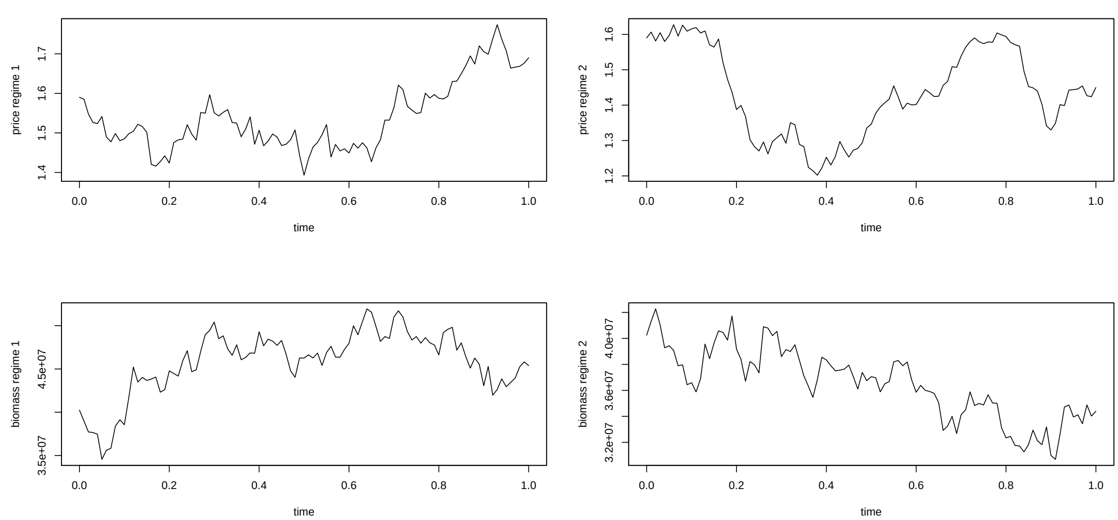
The determination of the effort value requires numerical computations. Thus, instead of arbitrary parameters values, we have decided to use realistic values. We found a quite complete set of parameter values in (2) and (11). The time horizon was set at $T_i = 5$ years. The complete set of parameter values is listed in Table 1.

4.1 Sample Realisations of Price and Stock

We chose the maximum fishing effort value for these sample realisations.

Parameters	Description	Values	Units
r_1; r_2	Intrinsic growth rate	0.71; 0.68	$year^{-1}$
K	Carrying capacity	80.5×10^6	kg
q	Catchability coefficient	3.30×10^{-6}	$SFU^{-1}year^{-1}$
E_{max}	Maximum fishing effort	0.7r/q	SFU
B_o	Initial population size	$0.5K$	kg
β	Discount factor	0.05	$year^{-1}$
p_0	Price per unit yield	1.59	kg^{-1}
c_1	Linear cost parameter	96×10^{-6}	$SFU^{-1}year^{-1}$
c_2	Quadratic cost parameter	0.10×10^{-6}	$SFU^{-2}year^{-1}$
θ	Mean-reversion speed	0.59	
$\bar{p}_0$	Price of the stock	1.211	
$\bar{p}_1$	Strength of demand	0.0001	
σ	Volatility of the stock	0.3	
σ_p	Volatility of the price	0.3	
T	Time horizon	5	years
γ	risk aversion coefficient	0.3	

Table 1: Numerical parameters



4.2 The Numerical Approximation

In this section, we present a numerical solution. We consider the switching process $\alpha(t)$ where $\alpha(t) \in \mathcal{S} = \{1, 2\}$ represents the season. In particular, $\alpha(t) = 1$ stands for the flood period with reduced fishing and intensive reproduction. and $\alpha(t) = 2$ the dry season with intensive fishing and reduced reproduction. The generator of $\alpha(t)$ is given by

$$\begin{pmatrix} 2.0367 & -2.0367 \\ -1.9821 & 1.9821 \end{pmatrix} \tag{110}$$

For our problems we need to ensure that our discretization methods converge to the viscosity solution and determine the optimal effort. Using the basic results of (26) and (27), this ensures that our numerical solutions convergence to the viscosity solution. For this purpose, we use the fully implicit upwind scheme which is unconditionally monotone.

To approximate the solution to (23) we discretize variables t, p and B with stepsizes Δt, Δp and ΔB respectively. The value of v_i at a grid point $(t_n; p_k; B_l)$ in the regime i is denoted by $v^n_{k,l}(i)$. The derivatives of v_i are approximated by

$$\frac{\partial v_i}{\partial t} \approx \frac{v^{n+1}_{k,l}(i) - v^n_{k,l}(i)}{\Delta t},$$

$$\frac{\partial^2 v_i}{\partial b^2} \approx \frac{v^{n+1}_{k,l+1}(i) + v^{n+1}_{k,l-1}(i) - 2v^{n+1}_{k,l}(i)}{(\Delta b)^2},$$

$$\frac{\partial^2 v_i}{\partial p^2} \approx \frac{v^{n+1}_{k+1,l}(i) + v^{n+1}_{k-1,l}(i) - 2v^{n+1}_{k,l}(i)}{(\Delta p)^2},$$

$$\Theta_i \frac{\partial v_i}{\partial p} \approx \begin{cases} \Theta_i \dfrac{v^{n+1}_{k+1,l}(i) - v^{n+1}_{k,l}(i)}{2\Delta p} & \text{if} \quad \Theta_i > 0 \\ \Theta_i \dfrac{v^{n+1}_{k,l}(i) - v^{n+1}_{k-1,l}(i)}{2\Delta p} & \text{if} \quad \Theta_i < 0 \end{cases} \quad \text{and}$$

$$\Phi_i \frac{\partial v_i}{\partial b} \approx \begin{cases} \Phi_i \dfrac{v^{n+1}_{k,l+1}(i) - v^{n+1}_{k,l}(i)}{2\Delta b} & \text{if} \quad \Phi_i > 0 \\ \Phi_i \dfrac{v^{n+1}_{k,l}(i) - v^{n+1}_{k,l-1}(i)}{2\Delta b} & \text{if} \quad \Phi_i < 0 \end{cases} .$$

Discretizing equation (23)

$$
\begin{aligned}
&\frac{v_{k,l}^{n+1} - v_{k,l}^{n}}{\Delta t} + \sup_{E \in \mathcal{A}_i} \Big\{ \Big(-\beta + q_{ij} \Big) v_{k,l}^{n+1} \\
&+ \max\Big(\theta(\bar{p}_0 - \bar{p}_1 qEB_l - P_k); 0\Big) \frac{v_{k+1,l}^{n+1} - v_{k,l}^{n+1}}{\Delta p} - \min\Big(\theta(\bar{p}_0 - \bar{p}_1 qEB_l - P_k); 0\Big) \frac{v_{k,l}^{n+1} - v_{k-1,l}^{n+1}}{\Delta p} \\
&+ \max\Big(r_i B_l (1 - \frac{B_l}{K}) - qEB_l; 0\Big) \frac{v_{k,l+1}^{n+1} - v_{k,l}^{n+1}}{\Delta b} - \min\Big(r_i B_l (1 - \frac{B_l}{K}) - qEB_l; 0\Big) \frac{v_{k,l}^{n+1} - v_{k,l-1}^{n+1}}{\Delta b} \\
&\qquad + \frac{1}{2}\sigma_p^2 \frac{v_{k+1,l}^{n+1} + v_{k-1,l}^{n+1} - 2v_{k,l}^{n+1}}{(\Delta p)^2} + \frac{1}{2}\sigma_B^2 B_l^2 \frac{v_{k,l+1}^{n+1} + v_{k,l-1}^{n+1} - 2v_{k,l}^{n+1}}{(\Delta b)^2} \\
&\qquad\qquad + \frac{(qEB_l P_k - c_1 E - c_2 E^2)^{1-\gamma}}{1-\gamma} - q_{ij} v_{k,l}^{n+1}(j) \Big\} = 0 \qquad (111)
\end{aligned}
$$

and rearranging the terms, we obtain

$$
\begin{aligned}
&\Big(-\beta + \frac{1}{\Delta t} - \frac{\sigma_p^2}{(\Delta p)^2} - \frac{\sigma_B^2 B_l^2}{(\Delta b)^2} + q_{ij} - \frac{1}{\Delta p}\theta(\bar{p}_0 - \bar{p}_1 qEB_l - P_k) - \frac{1}{\Delta b}\big(r_i B_l (1 - \frac{B_l}{K}) - qEB_l\big)\Big) v_{k,l}^{n+1} \\
&\quad + \sup_{E \in \mathcal{A}_i} \Big\{ \Big(\frac{\sigma_p^2}{2(\Delta p)^2} + \frac{\max\big(\theta(\bar{p}_0 - \bar{p}_1 qEB_l - P_k); 0\big)}{\Delta p} \Big) v_{k+1,l}^{n+1} \\
&\qquad + \Big(\frac{\sigma_p^2}{2(\Delta p)^2} + \frac{\min\big(\theta(\bar{p}_0 - \bar{p}_1 qEB_l - P_k); 0\big)}{\Delta p} \Big) v_{k-1,l}^{n+1} \\
&\qquad + \Big(\frac{\sigma_B^2 B_l^2}{2(\Delta b)^2} + \frac{\max\big(r_i B_l (1 - \frac{B_l}{K}) - qEB_l; 0\big)}{\Delta b} \Big) v_{k,l+1}^{n+1} \\
&\qquad + \Big(\frac{\sigma^2 B^2}{2(\Delta b)^2} + \frac{\min\big(r_i B_l (1 - \frac{B_l}{K}) - qEB_l; 0\big)}{\Delta b} \Big) v_{k,l-1}^{n+1} \\
&\qquad\qquad - q_{ij} v_{k,l}^{n+1}(j) + \frac{(qEB_l P_k - c_1 E - c_2 E^2)^{1-\gamma}}{1-\gamma} \Big\} = \frac{v_{k,l}^{n}}{\Delta t} \qquad (112)
\end{aligned}
$$

In addition, we consider that:

- the optimization starts at time $t = 0$ and ends at time $t = T < +\infty$
- the time interval is uniformly partitioned as $0 = t_0 < t_1 < ... < t_N = T$ with
  $t_{n+1} - t_n = \Delta t = \frac{T}{N},\ n = 0, 1, ..., N-1;$
- the state variable of biomass takes values within the interval [0, 2K], which

is uniformly partitioned as $0 = B_0 < B_1 < ... < B_m = 2K$ with $B_{l+1} - B_l = \Delta B = 2K/m, \qquad l = 0, 1, ..., m-1$;

- the state variable of prices takes values within the interval $[0, p_{max}]$, which is uniformly partitioned as $0 = p_0 < p_1 < ... < p_m = p_{max}$ with $p_{k+1} - p_k = \Delta p = \dfrac{p_{max}}{m},\ k = 0, 1, ..., m-1$;
- we have boundary conditions, a terminal conditions $v_i(T, b_t, p_t) = \kappa^\gamma \dfrac{B_T^{1-\gamma}}{1-\gamma}$, and a initial condition $v_i(0, b, p) = 0$.

If we define the constants

$$a_i = 1-\beta\Delta t-\frac{\sigma_p^2\Delta t}{(\Delta p)^2}-\frac{\sigma_B^2 B_l^2\Delta t}{(\Delta B)^2}+q_{ij}\Delta t-\frac{\Delta t}{\Delta p}\theta(\bar{p}_0-\bar{p}_1 qEB_l-P_k)-\frac{\Delta t}{\Delta b}\big(r_i B_l(1-\frac{B_l}{K})-qEB_l\big) \tag{113}$$

$$b_i = \frac{\sigma_p^2\Delta t}{2(\Delta p)^2} + \frac{\max\big(\theta(\bar{p}_0 - \bar{p}_1 qEB_l - P_k); 0\big)}{\Delta p}\Delta t \tag{114}$$

$$c_i = \frac{\sigma_p^2\Delta t}{2(\Delta p)^2} + \frac{\min\big(\theta(\bar{p}_0 - \bar{p}_1 qEB_l - P_k); 0\big)}{\Delta p}\Delta t \tag{115}$$

$$d_i = \frac{\sigma_B^2 B_l^2\Delta t}{2(\Delta b)^2} + \frac{\max\big(r_i B_l(1 - \frac{B_l}{K}) - qEB_l; 0\big)}{\Delta B}\Delta t \tag{116}$$

$$e_i = \frac{\sigma_B^2 B_l^2\Delta t}{2(\Delta b)^2} + \frac{\min\big(r_i B_l(1 - \frac{B_l}{K}) - qEB_l; 0\big)}{\Delta B}\Delta t \tag{117}$$

$$f_i = \frac{(qEB_l P_k - c_1 E - c_2 E^2)^{1-\gamma}}{1-\gamma}\Delta t \tag{118}$$

We can rewrite this difference equation in a more manageable form:

$$\sup_{E\in\mathcal{A}_i}\Big\{a_i v_{k,l}^{n+1} + b_i v_{k+1,l}^{n+1} + c_i v_{k-1,l}^{n+1} + d_i v_{k,l+1}^{n+1} + e_i v_{k,l-1}^{n+1} - q_{ij}\Delta t v_{k,l}^{n+1}(j) + f_i\Big\} = v_{k,l}^n \tag{119}$$

Writing (119) in a appropriate matrix form,

$$\sup_{E\in\mathcal{A}_i}\Big\{\mathbf{A}_i^E\mathbf{v}_i^{n+1} - \mathbf{\Lambda}_{ji}\mathbf{v}_j^{n+1} + \mathbf{F}_i^{n+1} - \mathbf{v}_i^n\Big\} = 0 \tag{120}$$

4.3 Howard's algorithm

We denote by $\mathbf{v}_i^n$ and $\mathbf{v}_i^{n+1}$ the approximations at time n and $n+1$.

Step 0: start with an initial value for the control E^0. Compute the solution v_i^0 of $\mathbf{A}_i^{E^0} w - \mathbf{\Lambda}_{ji}\mathbf{v}_j^{n+1} + \mathbf{F}_i^{n+1} - \mathbf{v}_i^n = 0$.

Step $j \to j+1$: given v_h^j, find $E^{j+1} \in \mathcal{A}_i$ maximizing $\mathbf{A}_i^E w - \mathbf{\Lambda}_{ji}\mathbf{v}_j^{n+1} + \mathbf{F}_i^{n+1} - \mathbf{v}_i^n = 0$. Then compute the solution v_h^{j+1} of $\mathbf{A}_i^{E^{j+1}} w - \mathbf{\Lambda}_{ji}\mathbf{v}_j^{n+1} + \mathbf{F}_i^{n+1} - \mathbf{v}_i^n = 0$.

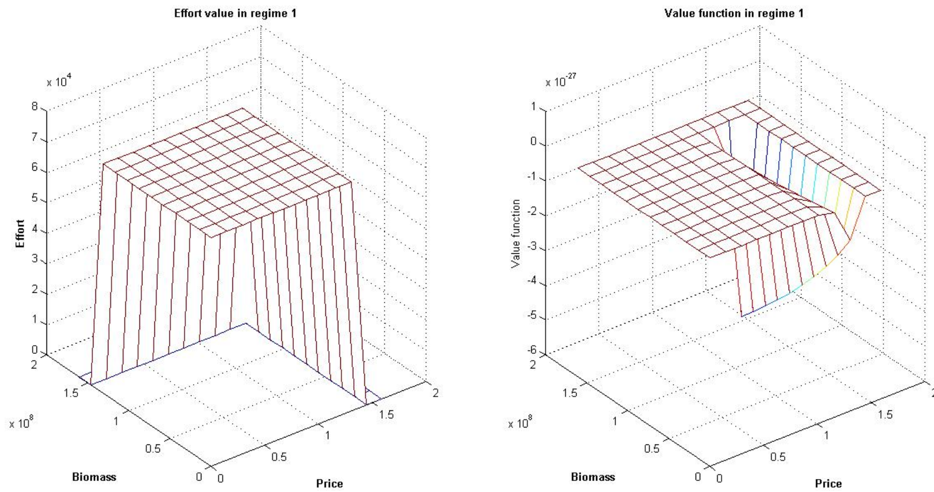
Final step :if $|\mathbf{v}_i^{j+1} - \mathbf{v}_i^j| < \epsilon$, then set $\mathbf{v}_i^{n+1} = \mathbf{v}_i^{j+1}$

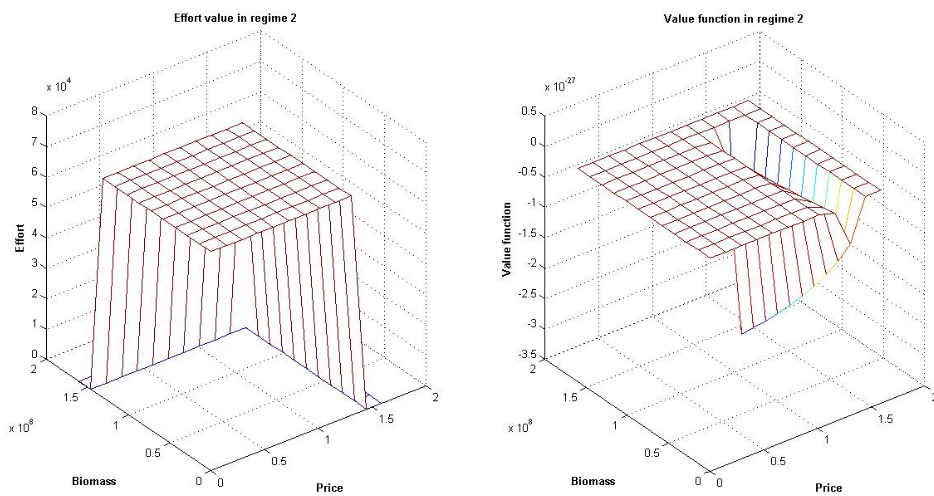
4.4 Optimal effort

We applied the Howard's algorithm: we compute the optimal effort in both regimes. As result:

Regimes	Optimal effort
1	$7.5303 \times 1.0e+04$
2	$7.2121 \times 1.0e+04$

Table 2: Optimal Effort





5 Conclusions

We treat an finite-horizon optimal fishery problem in switching diffusion models. Using the viscosity solution approach, we prove that the value function is the unique viscosity solution of the associated system of HJB equations. As an application, the optimal effort is deduced by using Howard's algorithm.

These methodologies can be applied to similar comparison studies and other fishery models. This will be the subject of a further paper.

Appendix A Stochastic logistic growth with harvesting

$$\begin{aligned} dB_t =& rB_t(1-\frac{B_t}{K})dt - qEB_tdt + \sigma B_t dW(t) && \text{(A.1)}\\ =& rB_t\left(1-\frac{qE}{r}-\frac{B_t}{K}\right)dt + \sigma B_t dW(t) && \text{(A.2)} \end{aligned}$$

Recalling the Itô's chain rules for solving the SDE $dX_t = f(X,t)dt + g(X,t)dW_t$ $t \geq 0$ or

$$dV(X(t),t) = \left(V_t(X(t),t) + f(X(t),t)V_X + \frac{1}{2}g^2(X(t),t)V_{XX}\right)dt + g(X(t),t)V_X dW(t)$$

Let $V(B_t,t) = B_t^{-1}$

$$\frac{\partial V}{\partial t} = 0 \qquad \frac{\partial V}{\partial B_t} = -B_t^{-2} \qquad \frac{\partial^2 V}{\partial B_t^2} = 2B_t^{-3}$$

$$f(B_t,t) = rB_t\left(1-\frac{qE}{r}-\frac{B_t}{K}\right) \qquad g(B_t,t) = \sigma B_t$$

$$dV_t = \left[0 + rB_t\left(1-\frac{qE}{r}-\frac{B_t}{K}\right)(-B_t^{-2}) + \frac{1}{2}\sigma^2 B_t^2(2B_t^{-3})\right]dt + \sigma B_t(-B_t^{-2})dW_t \tag{A.3}$$

$$d(B_t^{-1}) = \left[-r\left(B_t^{-1}(1-\frac{qE}{r}) - \frac{1}{K}\right) + \sigma^2 B_t^{-1}\right]dt - \sigma B_t^{-1}dW_t \tag{A.4}$$

To linearize (A.3), set $B_t^{-1} = y_t$ so that

$$\begin{aligned} dy_t =& \left[-r\left(y_t(1-\frac{qE}{r}) - \frac{1}{K}\right) + \sigma^2 y_t\right]dt - \sigma y_t dW_t && \text{(A.5)}\\ =& \left[\frac{r}{K} + \left(-r + \sigma^2 + qE\right)y_t\right]dt - \sigma y_t dW_t && \text{(A.6)} \end{aligned}$$

We are looking for a solution to (A.5) of the form $y(t) = y_1(t).y_2(t)$ where

$$\begin{aligned} dy_1(t) =& (-r+\sigma^2+qE)y_1dt - \sigma y_1 dW_t, \qquad y_1(0) = 1\\ dy_2(t) =& a_t dt + b_t dW_t, \qquad y_2(0) = y(0) = y_0 \end{aligned}$$

and the process coefficients $a_t; b_t$ are, at this point, unknown.

$$\begin{aligned}
dy_t =& d(y_1.y_2)\\
=& y_1 dy_2 + y_2 dy_1 + dy_1 dy_2\\
=& y_1 dy_2 + y_2 dy_1 + [(-r+\sigma^2+qE)y_1 dt - \sigma y_1 dW_t][a_t dt + b_t dW_t]\\
=& y_1 dy_2 + y_2 dy_1 - \sigma b_t y_1 dt\\
=& y_1(a_t dt + b_t dW_t) + y_2[(-r+\sigma^2+qE)y_1 dt - \sigma y_1 dW_t] - \sigma b_t y_1 dt\\
=& y_1(a_t dt + b_t dW_t) + (-r+\sigma^2+qE)y dt - \sigma y dW_t - \sigma b_t y_1 dt
\end{aligned}$$

Now, select a_t, b_t so that

$$y_1(a_t dt + b_t dW_t) - \sigma b_t y_1 dt = \frac{r}{K} dt$$

Thus

$$b_t = 0 \qquad a_t = \frac{r}{K} y_1^{-1} \tag{A.7}$$

We integrate $dy_1(t)$ and $dy_2(t)$ to

$$\begin{aligned}
y_1(t) =& \exp\left[\int_0^t -\sigma dW_s + \int_0^t \left((-r+\sigma^2+qE) - \frac{1}{2}\sigma^2\right) ds\right]\\
=& \exp\left[-\sigma W_t + \left(-r + qE + \frac{1}{2}\sigma^2\right) t\right]\\
y_2(t) =& y_0 + \int_0^t \frac{r}{K} y_1^{-1}(s) ds
\end{aligned}$$

Thus

$$\begin{aligned}
y(t) =& \exp\left[-\sigma W_t + \left(-r + qE + \frac{1}{2}\sigma^2\right) t\right] \times\\
& \left[y_0 + \frac{r}{K}\int_0^t \exp\left[\sigma W_s + \left(r - qE - \frac{1}{2}\sigma^2\right) s\right] ds\right]
\end{aligned}$$

Hence

$$B_t = \frac{\exp\left[\left(r - qE - (1/2)\sigma^2\right) t + \sigma W_t\right]}{\left[B_0^{-1} + (r/K)\int_0^t \exp\left[(r - qE - (1/2)\sigma^2) s + \sigma W_s\right] ds\right]} \tag{A.8}$$

$$B_t = \frac{K \exp\left[\left(r - qE - (1/2)\sigma^2\right) t + \sigma W_t\right]}{\left[K/B_0 + r\int_0^t \exp\left[(r - qE - (1/2)\sigma^2) s + \sigma W_s\right] ds\right]} \tag{A.9}$$

Appendix B Proof of lemma 3.1

1. Let $k \in [0;2]$ and $h = s - t$.

   According to Hölder inequality $\mathrm{E}\eta^k \leq \left[\mathrm{E}\eta^2\right]^{k/2}$ for $\forall k \geq 0$,

$$\mathrm{E}|P_h^{t,p_t}|^k \leq \left[\mathrm{E}|P_h^{t,p_t}|^2\right]^{k/2} \tag{B.1}$$

   Given that

$$dP(t) = f'(t, P(t))dt + g'(t, P(t))dw(t) \tag{B.2}$$

   According to the elementary inequality $|\sum_{i=1}^M a_i|^2 \leq M \sum_{i=1}^M |a_i|^2$,

$$|P_h^{t,p_t}|^2 \leq 3\left\{|p_t|^2 + \left|\int_0^h f'(u+t, P_u^{t,p_t}, i)du\right|^2 + \left|\int_0^h g'(u+t, P_u^{t,p_t}, i)dw_u\right|^2\right\} \tag{B.3}$$

   Thus, using Ito-isometry and Fubini's theorem

$$\mathrm{E}|P_h^{t,p_t}|^2 \leq 3\left\{|p_t|^2 + \int_0^h \mathrm{E}\left|f'(u+t, P_u^{t,p_t}, i)\right|^2 du + \int_0^h \mathrm{E}\left|g'(u+t, P_u^{t,p_t}, i)\right|^2 du\right\} \tag{B.4}$$

   f' and g' satisfying the growth condition by definition, thus there exists $C_1 \in \mathbb{R}$ such that

$$\mathrm{E}|P_h^{t,b_t}|^2 \leq C_1\left\{1 + |p_t|^2 + \int_0^h \mathrm{E}\left|P_u^{t,p_t}\right|^2 du\right\} \tag{B.5}$$

   Applying Gronwall's inequality we obtain

$$\mathrm{E}|P_h^{t,p_t}|^2 \leq C_1 e^{C_1 h}\left[1 + |p_t|^2\right] \tag{B.6}$$

   i.e

$$\mathrm{E}|P_h^{t,p_t}|^2 \leq C\left[1 + |p_t|^2\right] \tag{B.7}$$

   Using (B.1) and elementary inequalities $(a_1 + a_2)^k \leq 2^{k-1}(|a_1|^k + |a_2|^k)$ and $(\sqrt{|a_1 + a_2|} \leq \sqrt{|a_1|} + \sqrt{|a_2|})$ we deduce

$$\mathrm{E}|P_h^{t,p_t}|^k \leq C\left[1 + |p_t|^k\right] \tag{B.8}$$

   The same reasoning gives....

$$\mathrm{E}|B_h^{t,b_t}|^k \leq C\left[1 + |b_t|^k\right] \tag{B.9}$$

2. We have

$$|P_h^{t,p_t} - p_t|^2 \leq 2\left\{\Big|\int_0^h f'(u+t, P_u^{t,p_t}, i)du\Big|^2 + \Big|\int_0^h g'(u+t, P_u^{t,p_t}, i)dw_u\Big|^2\right\} \tag{B.10}$$

Similar arguments as above we deduce

$$\mathrm{E}|P_h^{t,p_t} - p_t|^2 \leq C_1 \int_0^h \left[1 + \mathrm{E}\Big|P_u^{t,p_t}\Big|^2\right] du \tag{B.11}$$

using (B.7) we deduce

$$\mathrm{E}|P_h^{t,p_t} - p_t|^2 \leq C\big(1 + |p_t|^2\big)h \tag{B.12}$$

Hence

$$\mathrm{E}|P_h^{t,p_t} - p_t|^k \leq C\big(1 + |p_t|^k\big)h^{k/2} \tag{B.13}$$

3. Let us define the process $P_s^{t,p_t} - P_s^{t,p'_t}$. Put $\bar{f}(u+t, P_u^{t,p_t}, P_u^{t,p'_t}, i) = f'(u+t, P_u^{t,p_t}, i) - f'(u+t, P_u^{t,p'_t}, i)$ and $\bar{g}(u+t, P_u^{t,p_t}, P_u^{t,p'_t}, i) = g'(u+t, P_u^{t,p_t}, i) - g'(u+t, P_u^{t,p'_t}, i)$. Then

$$\mathrm{E}|B_h^{t,p_t} - B_h^{t,p'_t}|^2 \leq 3\left(|p_t - p'_t|^2 + \mathrm{E}\Big|\int_0^h \bar{f}(u+t, P_u^{t,p_t}, P_u^{t,p'_t}, i)du\Big|^2 + \mathrm{E}\Big|\int_0^h \bar{g}(u+t, P_u^{t,p_t}, P_u^{t,p'_t}, i)dw_u\Big|^2\right)$$

$$\mathrm{E}|P_h^{t,p_t} - P_h^{t,p'_t}|^2 \leq 3\left(|p_t - p'_t|^2 + \mathrm{E}\int_0^h \Big|\bar{f}(u+t, P_u^{t,p_t}, P_u^{t,p'_t}, i)\Big|^2 du + \mathrm{E}\int_0^h \Big|\bar{g}(u+t, P_u^{t,p_t}, B_u^{t,p'_t}, i)\Big|^2 du\right)$$

$$\mathrm{E}|P_h^{t,p_t} - P_h^{t,p'_t}|^2 \leq C\left(|p_t - p'_t|^2 + \int_0^h \mathrm{E}\Big|P_u^{t,p_t} - P_u^{t,p'_t}\Big|^2 du\right) \tag{B.14}$$

Hence

$$\mathrm{E}|P_h^{t,p_t} - P_h^{t,p'_t}|^2 \leq C|p_t - p'_t|^2 \tag{B.15}$$

Similar arguments as above we deduce

$$\mathrm{E}|B_h^{t,b_t} - B_h^{t,b'_t}|^k \leq C|b_t - b'_t|^2; \quad \mathrm{E}|P_h^{t,p_t} - P_h^{t,p'_t}|^k \leq C|p_t - p'_t|^2 \tag{B.16}$$

4. Using Doob's inequality for submartingale. We get

$$\mathrm{E}\Big[\sup_{0\leq s\leq h} |B_h^{t,b_t}|\Big]^k \leq C(1 + |b_t|^k)h^{\frac{k}{2}}; \quad \mathrm{E}\Big[\sup_{0\leq s\leq h} |P_h^{t,p_t}|\Big]^k \leq C(1 + |p_t|^k)h^{\frac{k}{2}} \tag{B.17}$$

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