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# Fishery Management in a Regime Switching Environment: Utility Based Approach

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## Abstract

In this paper we study the problem of optimal fishing for regime switching, which may be regarded as sequential optimal problem with changes of regimes. The growth dynamics of a given fish species is described by the differential stochastic logistic model in which we take into account two states: prior or during floods and after. The resulting dynamic programming principle leads to a system of variational inequalities, by means of viscosity solutions approach, we prove the existence and uniqueness of the value functions. Then numerical approximation is used to answer the question: what is the optimal fishing effort for a sustainable fishery?

*Keywords:* : regime switching, floods, crra utility, logistic growth, mean-reverting prices, viscosity solution, Howard's algorithm .

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## 1 Introduction

The simplest population model commonly used in fisheries is the logistic growth model extended to include catch:

$$\frac{dB}{dt} = rB\left(1 - \frac{B}{K}\right) - C \quad (1)$$

2 where  $B$  is the biomass of the stock,  $r$  is the intrinsic rate of growth,  $K$  (the  
3 carrying capacity) is the biomass the stock would tend toward if unfished, and  
4  $C$  is the catch rate.

The catch  $C$  is constant in quota management. Normally the catch is assumed to be proportional to fishing effort and to stock size, which results in the model proposed by (1) :

$$\frac{dB}{dt} = rB\left(1 - \frac{B}{K}\right) - qEB \quad (2)$$

5 where  $E$  is the fishing effort and  $q$  is a parameter describing the efficiency of the  
6 fishing gear

However, the environment is subject to significant random fluctuations that affect the population per capita natural growth rate. The effect of these fluctuations can be approximated by a white noise  $\sigma\epsilon(t)$ , where  $\epsilon(t)$  is a standard white noise and  $\sigma > 0$  measures the strength of environmental fluctuations see (2). Therefore, the above ODE Eq. (2) must be updated to the stochastic differential equation (SDE) which can be written in the standard format:

$$dB(t) = rB(t)\left(1 - \frac{B(t)}{K}\right)dt - qE(t)B(t)dt + \sigma B(t)dW(t) \quad (3)$$

7 where  $W(t)$  is a standard Wiener process. We will assume that  $r - qE > \sigma^2/2$ ,  
8 otherwise the population will be rendered extinct (see (3) ).

9 Environmentally driven long-term changes in fish populations, which can  
10 play a major role in determining how such populations respond to fishing pressure, are rapidly being recognized as a critical problem in fisheries science ((4)).

11 The life cycle of African fish species of river is closely related to the seasons  
12 - reproduction almost always occurring just prior to, or during, floods ((5), (6),  
13 (7), (8)).  
14

Floods appear to be essential for the completion of their reproductive cycle  
 16 for most species: the absence of floods due to the drought in the Sahel has  
 caused a decline in fish reproduction in the central Niger Delta, the Senegal  
 18 River and Lake Chad (Stauch, personal communication).

There is some evidence that flood intensity acts in favor of reproduction, as  
 20 it has been observed that the structured age class related to the high floods in  
 the Kafu were more varied ((9)).

In our study we consider only two seasons: the dry season with intensive  
 22 fishing and reduced reproduction, the flood period with reduced fishing and  
 intensive reproduction.  
 24

A regime switching model provides an alternate approach to capturing non-  
 26 constant drift and volatility terms for the stochastic process followed by the  
 biomass of fish. Therefore, the above SDE Eq. (3) must be updated to the  
 28 another stochastic differential equation that captures the regime switching :

$$dB(t) = r_{\alpha(t)}B(t)\left(1 - \frac{B(t)}{K}\right)dt - qE_{\alpha(t)}(t)B(t)dt + \sigma_{\alpha(t)}B(t)dW(t) \quad (4)$$

where  $\alpha(t)$  refer to regimes and there are 2 regimes, i.e,  $\alpha(t) \in \{1, 2\}$

30 Certainly, there is some evidence that uncertainty in price parameters leads  
 to changes in the optimal policy ((10)), and the number of studies that in-  
 32 clude uncertainty in both the biological stock dynamics and the price dynamics  
 is steadily increasing. The models in (11) have considered stochastic mean-  
 34 reverting prices and when compared with the typical geometric Brownian motion  
 model, a mean-reverting price better reflect basic, microeconomic ideas  
 36 about supply behavior (see (12)).

Let the instantaneous profit from the harvest of the stock biomass  $\pi(B_t, h_t)$   
 be given as:

$$\pi(B_t, h_t) = P_t h_t - c(B_t, h_t) \quad (5)$$

where,  $h_t$  denotes the volume of harvest,  $B_t$  the stock of the resource,  
 $c(B_t, h_t)$  is the cost function, both at time t and  $P_t$  the mean-reverting (ac-  
 tual or spot) price of the harvest at the time of decision making. This can be

modeled by the following process:

$$dP_t = \theta(\bar{p}_0 - \bar{p}_1 h - P_t)dt + \sigma_P dW_P(t) \quad (6)$$

The parameters are positive constants,  $\theta$  is the reversion speed,  $\bar{p}_0$  is a maximum price,  $\bar{p}_1$  is the slope of the inverse demand curve and  $\sigma_P$  is the volatility of the spot price (see (11)).

Many works set the problem, in the infinite horizon time, as follows:

$$\max_{h_t} \int_0^{+\infty} e^{-\beta t} \pi(B_t, h_t) dt \quad (7)$$

Previous work finds, almost without exception, that all fishers are risk-averse ((13), (14), and (15)). Under an expected-utility theory (EUT) specification of choice under uncertainty, we assume a constant relative risk-aversion (CRRA) utility function defined as  $U(x) = \frac{x^{1-\gamma}}{1-\gamma}$  where  $x$  signifies the lottery prize and  $\gamma$  is the CRRA coefficient to be estimated: with  $\gamma = 0$  denoting risk neutrality,  $\gamma > 0$  indicating risk aversion, and  $\gamma < 0$  denoting risk loving ((16)).

In recent years, further emphasis has been put on developing models for optimal management of these stochastic natural resources ((17); (18); (19)). Although the number of studies in bioeconomic modeling that include the stochastic dynamics are increasing, they are still not adequate.

The time horizon also plays a crucial role in optimal policies and the usual infinite horizon framework problem requires the existence of linear growth conditions on drift part of the logistic process for our solution to hold, raising the question of whether another solution may exist or not. In this paper, we consider the finite time horizon  $T$  with utility on both profit and remaining biomass.

The outline of this paper is as follows. In Section II we formulate a stochastic optimal control problem. Section III the optimal strategies to the utility maximization problem are derived. In Section IV we present examples to illustrate the results. Finally, in section V we end with some summarizing comments.

## 2 Mathematical model

### 2.1 Stochastic logistic growth model

Throughout this paper we let  $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$  be a complete probability space with a filtration  $\{\mathcal{F}_t\}_{t \geq 0}$  satisfying the usual conditions (i.e. it is increasing and right continuous while  $\mathcal{F}_0$  contains all P-null sets). Let  $W(t)$  and  $W_P(t)$ ,  $t \geq 0$ , be scalar independent Brownian motions defined on this probability space.

After specifying a stochastic model for biological growth that we use in this paper. The valuation of the biomass can be described in terms of the following variables:

$t$  is time,  $t \in [0; T]$  and  $T$  is finite-horizon of time.

$\alpha(t)$  is a right-continuous-time Markov chain,  $\mathcal{F}_t$ -adapted with finite state space  $\mathcal{S} = \{1; 2\}$  and generator  $\mathcal{Q} = (q_{ij}) \in \mathbb{R}^2 \times \mathbb{R}^2$  such that  $q_{ij} \geq 0$  for  $i \neq j$  and  $\sum_{j=1}^2 q_{ij} = 0$ . We assume that the Markov chain  $\alpha(\cdot)$  is independent of the Brownian motions  $W_P(\cdot)$  and  $W(\cdot)$

$B(t)$  stock of fish biomass at time  $t$ . Whith initial condition  $B(0) > 0$

$r_{\alpha(t)}$  intrinsic rate of growth in regime  $\alpha(t)$

$E_{\alpha(t)}$  is the fishing effort which depends on the current regime  $\alpha(t)$

$\sigma_{\alpha(t)}$  is volatility in regime  $\alpha(t)$ . In any time  $r_{\alpha(t)} - qE > \sigma_{\alpha(t)}^2/2$

We set an SDE under regime switching of the form:

$$dB(t) = f(t, B(t), \alpha(t))dt + g(t, B(t), \alpha(t))dW(t) \quad (8)$$

on  $t \geq 0$  with initial value  $B(0) = b \in ]0; K[$ , where

$$f : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathcal{S} \rightarrow \mathbb{R} \text{ and } g : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathcal{S} \rightarrow \mathbb{R} \quad (9)$$

This equation can be regarded as the result of the following 2 equations:

$$dB(t) = f(t, B(t), i)dt + g(t, B(t), i)dW(t); \quad i = 1, 2 \quad (10)$$

switching from one to the other according to the movement of the Markov chain.

Recalling that

$$f(t, B(t), i) = r_i B(t) \left(1 - \frac{B(t)}{K}\right) - qE_i(t)B(t) \quad \text{and} \quad g(t, B(t), i) = \sigma_i B(t) \quad (11)$$

$B(t)$  is an unknown stochastic process, that is, the solution to Eq.(8) satisfying the initial condition  $B(0) = b$  such that  $0 < b < K$ . The logical requirement is that  $B(t)$  must be positive. The resulting stochastic differential equation does not satisfy the standard assumptions for existence and uniqueness of solutions, namely, linear growth and the Lipschitz condition. Nevertheless, for any positive initial condition, the solution exists and is unique under a hypothesis that both  $f$  and  $g$  satisfy the local Lipschitz condition and in any time  $r_{\alpha(t)} - qE > \sigma_{\alpha(t)}^2/2$ . The solution of this equation is.(For more details see Appendix A )

$$B_{t,i} = \frac{K \exp[(r_i - qE - (1/2)\sigma_i^2)t + \sigma_i W_t]}{\left[(K/B_0) + r_i \int_0^t \exp[(r_i - qE - (1/2)\sigma_i^2)s + \sigma_i W_s] ds\right]} \quad i = 1, 2. \quad (12)$$

## 78 2.2 The Mean- reverting spot price

The version of the Ornstein-Uhlenbeck (OU) process we employ here is described by

$$dP_t = \theta(\bar{p}_0 - \bar{p}_1 h - P_t)dt + \sigma_P dW_P(t) \quad (13)$$

where the parameters are positive constants,  $\theta$  is the reversion speed,  $\bar{p}_0$  is a maximum price,  $\bar{p}_1$  is the slope of the inverse demand curve and  $\sigma_p$  is the volatility of the spot price. Note that the mean (or long-term) price  $\bar{p}_0 - \bar{p}_1 h$  may depend upon the harvest level.  $W_P(t)$  is standardized Brownian motion as before. Its solution for an initial condition  $P(0) = p$  is

$$P_t = p e^{-\theta t} + (\bar{p}_0 - \bar{p}_1 h)(1 - e^{-\theta t}) + \sigma \int_0^t e^{-\theta(t-s)} dW_s \quad (14)$$

One of the most convenient properties is that

$$P_t \sim N \left( \bar{p}_0 - \bar{p}_1 h + (p + \bar{p}_0 - \bar{p}_1 h)e^{-\theta t}, \frac{\sigma^2}{2\theta}(1 - e^{-2\theta t}) \right) \quad (15)$$

### 2.3 The optimization problem

The cost of harvest per unit time is assumed to depend on effort and to have a quadratic form given by

$$c(B_t, E_t) = (c_1 + c_2 E(t))E(t) \quad (16)$$

80 where  $c_1, c_2 > 0$  are constants. The quadratic cost structure incorporates the case where the fishermen need to use less efficient vessels and fishing technologies  
82 or pay higher overtime wages to implement an extraordinary high effort (see (20), (21)).

By substituting the values in equation 5, the profit function can be rearranged as:

$$\pi(B_t, P_t, E) = (qB_t P_t - c_1 - c_2 E(t))E(t) \quad (17)$$

84 where,  $c_1$  and  $c_2$  are positive parameters.

For a time  $t$  in the horizon  $[0, T]$ , we define a performance criterion for each  $i \in \mathcal{S}$  as:

$$V_i(t, b_t, p_t) = \mathbb{E}_{b_t, p_t, i} \left[ \int_t^T e^{-\beta(s-t)} \frac{\pi(B_s, P_s, E_s, i)^{1-\gamma}}{1-\gamma} ds + e^{-\beta(T-t)} V_i(B_T) \right] \quad (18)$$

$$= \mathbb{E}_{b_t, p_t, i} \left[ \int_t^T e^{-\beta(s-t)} l(s, B_s, P_s, E_s, i) ds + e^{-\beta(T-t)} m(T, B_T) \right] \quad (19)$$

We will start the optimization at time  $t = 0$ . Let  $b_0 = b, p_0 = p$  with  $b, p \in ]0; +\infty]$ , we have

$$V_i(0, b, p) = \mathbb{E}_{b, p, i} \left[ \int_0^T e^{-\beta s} \frac{\pi(B_s, P_s, E_s, i)^{1-\gamma}}{1-\gamma} ds + e^{-\beta T} V_i(B_T) \right] \quad (20)$$

$$= \mathbb{E}_{b, p, i} \left[ \int_0^T e^{-\beta s} l(s, B_s, P_s, E_s, i) ds + e^{-\beta T} m(T, B_T) \right] \quad (21)$$

86 Here  $\mathbb{E}_{b, p, i}$  is the conditional expectation given  $B(0) = b, P(0) = p$  and  $\alpha(0) = i$  under  $\mathbb{P}$ , where  $T$  is the finite time horizon  $\beta > 0$  is a discount factor.



88 We say that the control process  $E(t)$  is admissible if the following tree con-  
 89 ditions are satisfied:

- 90 **1.** the SDE (4) for the state process  $B(t)$  has a unique strong solution;
- 2.** the SDE (6) for the state process  $P(t)$  has a unique strong solution;
- 92 **3.**  $\mathbb{E}_{b,p,i} \left[ \int_0^T |e^{-\beta t} \frac{\pi(B_t, P_t, E_t, i)^{1-\gamma}}{1-\gamma}| dt + |e^{-\beta T} V(B_T)| \right] < \infty$ .

Write  $\mathcal{A}$  for the set of admissible controls. The number of tools, gears, hours,  
 94 vessels and manpower is finite and limited, so we require the set  $\mathcal{A}$  in which the  
 controls take values to be bounded. The stochastic control problem is to find  
 96 an optimal control  $E^* \in \mathcal{A}_i$  such that:

$$v_i(b, p) = \sup_{E \in \mathcal{A}_i} V_i(b, p) \quad (22)$$

### 3 Main results

The Hamilton-Jacobi-Bellman equations associated with this problem is a  
 variational inequality involving, at least heuristically, a nonlinear second order  
 parabolic differential equations :

$$\frac{\partial v_i}{\partial t}(t, b_t, p_t) + \sup_{E \in \mathcal{A}_i} \left\{ -\beta v_i(t, b_t, p_t) + \frac{\pi(B_t, P_t, E_t)^{1-\gamma}}{1-\gamma} + \mathcal{L}v_i(t, b_t, p_t) \right\} = 0, \quad (23)$$

$$v_i(T, b_t, p_t) = \kappa^\gamma \frac{B_T^{1-\gamma}}{1-\gamma} \quad \text{for } i \in \{0; 1\}; \quad \kappa > 0 \quad (24)$$

where  $\mathcal{L}$  is an operator defined by:

$$\begin{aligned} \mathcal{L}v_i(t, B, P) &= \theta(\bar{p}_0 - \bar{p}_1 q E B - P) \frac{\partial v_i}{\partial p}(t, B, P) + f(B, E_i) \frac{\partial v_i}{\partial b}(t, B, P) \\ &+ \frac{1}{2} \sigma_P^2 \frac{\partial^2 v_i}{\partial p^2}(t, B, P) + \frac{1}{2} g^2(B, E) \frac{\partial^2 v_i}{\partial b^2}(t, B, P) + q_{ij}(v_j(t, B, P) - v_i(t, B, P)) \end{aligned} \quad (25)$$

As it is well-known, there is not in general a smooth solution of the equation  
 (23) hence we find the solution in the viscosity sense, as introduced by (22), in

subsection 3.2. Recall that  $E_{free}^*$  is the optimal solution of equation 23. The optimal harvest rule  $E^*(t)$  can be described as follows

$$E^*(t) = \begin{cases} 0 & \text{if } E_{free}^*(t) < 0 \\ E_{free}^*(t) & \text{if } 0 \leq E_{free}^*(t) \leq E_{max} \\ E_{max} & \text{if } E_{free}^*(t) > E_{max} \end{cases} \quad (26)$$

98 In addition to these, we know that the fishery is valueless if the stock goes  
extinct and therefore add the condition  $V_i(0, P_t) = 0$ , which must hold for all  
100  $P_t$  and  $i$ .

### 3.1 On the regularity of value functions

In this section, we study the growth and continuity properties of the value functions.

We shall make the following assumptions: there exist  $\rho > 0$  such that for all  $s, t \in [0; T], b, b' \in \mathbb{R}_+, p, p' \in \mathbb{R}_+$  and  $E \in \mathcal{A}$

$$|l(t, b, p, E) - l(s, b', p', E)| + |m(b, p) - m(b', p')| \leq \rho[|t - s| + |b - b'| + |p - p'|] \quad (27)$$

and the global linear growth conditions:

$$|l(t, b, p, E)| + |m(b, p)| \leq \rho[1 + |b| + |p|] \quad (28)$$

**Lemma 3.1.** *Let (27) and (28) hold. For any  $k \in [0; 2]$  there exists  $C = C(k; K; T) > 0$  such that for all  $h, t \in [0; T], b, p, b_t, p_t \in \mathbb{R}_+$ :*

$$\mathbb{E}|B_h^{t, b_t}|^k \leq C(1 + |b_t|^k); \quad \mathbb{E}|P_h^{t, p_t}|^k \leq C(1 + |p_t|^k) \quad (29)$$

$$\mathbb{E}|B_h^{t, b_t} - b_t|^k \leq C(1 + |b_t|^k)h^{\frac{k}{2}}; \quad \mathbb{E}|P_h^{t, p_t} - p_t|^k \leq C(1 + |p_t|^k)h^{\frac{k}{2}} \quad (30)$$

$$\mathbb{E}|B_h^{t, b_t} - B_h^{t, b'_t}|^k \leq C|b_t - b'_t|^2; \quad \mathbb{E}|P_h^{t, p_t} - P_h^{t, p'_t}|^k \leq C|p_t - p'_t|^2 \quad (31)$$

$$\mathbb{E}\left[\sup_{0 \leq s \leq h} |B_h^{t, b_t}|\right]^k \leq C(1 + |b_t|^k)h^{\frac{k}{2}}; \quad \mathbb{E}\left[\sup_{0 \leq s \leq h} |P_h^{t, p_t}|\right]^k \leq C(1 + |p_t|^k)h^{\frac{k}{2}} \quad (32)$$

102 **Proof 3.1.** See Appendix B

**Proposition 3.1.** For any  $i \in \mathcal{S}$ , the value function denoted by  $v_i(s, b, p)$  satisfies a linear growth condition and is also Lipschitz in  $(b, p)$  uniformly in  $t$ . There exists a constant  $C > 0$ , such that

$$0 \leq v_i(s, b_s, p_s) \leq C(1 + |b_s| + |p_s|),$$

$$\forall (s, b_s, p_s) \in [0; T] \times \mathbb{R}_+ \times \mathbb{R}_+ \quad (33)$$

$$|v_i(s, b_s, p_s) - v_i(s, b'_s, p'_s)| \leq C(|b_s - b'_s| + |p_s - p'_s|),$$

$$\forall s \in [0; T], \quad b_s, b'_s \in \mathbb{R}_+, \quad p_s, p'_s \in \mathbb{R}_+ \quad (34)$$

**Proof 3.2.** We first show that  $v$  is Lipschitz in  $(b, p)$ , uniformly in  $t$  and its linear growth condition.

$$v_i(s, b_s, p_s) = \sup_{E \in \mathcal{A}} \mathbb{E} \left[ \int_s^T e^{-\beta(u-s)} l(i, u, B_u^{s, b_s}, P_u^{s, p_s}, E_u) du + e^{-\beta(T-s)} m(B_T^{s, b_s}, P_T^{s, p_s}) \right] \quad (35)$$

1. Using elementary inequality  $|\sup A - \sup B| \leq \sup |A - B|$ , Lipschitz con-

dition (27) on  $l$  ;  $m$  and from estimate (3.1), with  $k=1$ ,

$$\begin{aligned}
& |v_i(s, b_s, p_s) - v_i(s, b'_s, p'_s)| \\
& \leq \sup_{E \in \mathcal{A}_i} \left| \mathbb{E} \left[ \int_s^T e^{-\beta(u-s)} \left( l(i, u, B_u^{s, b_s}, P_u^{s, p_s}, E_u) - l(i, u, B_u^{s, b'_s}, P_u^{s, p'_s}, E_u) \right) du \right. \right. \\
& \quad \left. \left. + e^{-\beta(T-s)} \left( m(B_T^{s, b_s}, P_T^{s, p_s}) - m(B_T^{s, b'_s}, P_T^{s, p'_s}) \right) \right] \right| \\
& \leq \sup_{E \in \mathcal{A}_i} \mathbb{E} \left[ \int_s^T \left| \left( l(i, u, B_u^{s, b_s}, P_u^{s, p_s}, E_u) - l(i, u, B_u^{s, b'_s}, P_u^{s, p'_s}, E_u) \right) \right| du \right. \\
& \quad \left. + \left| \left( m(B_T^{s, b_s}, P_T^{s, p_s}) - m(B_T^{s, b'_s}, P_T^{s, p'_s}) \right) \right| \right] \\
& \leq \sup_{E \in \mathcal{A}_i} \mathbb{E} \left[ \int_s^T \left( |B_u^{s, b_s} - B_u^{s, b'_s}| + |P_u^{s, p_s} - P_u^{s, p'_s}| \right) du + \left( |B_T^{s, b_s} - B_T^{s, b'_s}| + |P_T^{s, p_s} - P_T^{s, p'_s}| \right) \right] \\
& \leq \sup_{E \in \mathcal{A}_i} \left[ \int_s^T \mathbb{E} \left( |B_u^{s, b_s} - B_u^{s, b'_s}| + \mathbb{E} |P_u^{s, p_s} - P_u^{s, p'_s}| \right) du + \left( \mathbb{E} |B_T^{s, b_s} - B_T^{s, b'_s}| + \mathbb{E} |P_T^{s, p_s} - P_T^{s, p'_s}| \right) \right] \\
& \quad |v_i(s, b_s, p_s) - v_i(s, b'_s, p'_s)| \leq C \left( |b_s - b'_s| + |p_s - p'_s| \right) \quad (36)
\end{aligned}$$

2. from linear growth condition (28) on  $l$  ;  $m$  and from estimate (3.1), with  $k=1$ ,

$$|v_i(s, b_s, p_s)| \leq \sup_{E \in \mathcal{A}_i} \mathbb{E} \left[ \int_s^T \left| l(i, u, B_u^{s, b_s}, P_u^{s, p_s}, E_u) \right| du + \left| m(B_T^{s, b_s}, P_T^{s, p_s}) \right| \right] \quad (37)$$

$$|v_i(s, b_s, p_s)| \leq \rho \sup_{E \in \mathcal{A}_i} \mathbb{E} \left[ \int_s^T \left( 1 + |B_u^{s, b_s}| + |P_u^{s, p_s}| \right) du + \left( 1 + |B_T^{s, b_s}| + |P_T| \right) \right] \quad (38)$$

$$|v_i(s, b_s, p_s)| \leq \rho \sup_{E \in \mathcal{A}_i} \left[ \int_s^T \left( 1 + \mathbb{E} |B_u^{s, b_s}| + \mathbb{E} |P_u^{s, p_s}| \right) du + \left( 1 + \mathbb{E} |B_T^{s, b_s}| + \mathbb{E} |P_T^{s, p_s}| \right) \right] \quad (39)$$

$$|v_i(s, b_s, p_s)| \leq C (1 + |b_s| + |p_s|) \quad (40)$$

**Proposition 3.2.** Under assumptions (49) and (28) the value function  $v \in \mathcal{C}^0([0; T] \times \mathbb{R}_+ \times \mathbb{R}_+)$ . More precisely, there exists a constant  $C > 0$  such that

for all  $t, s \in [0; T]$ ,  $b_t, b_s \in \mathbb{R}_+$ ,  $p_t, p_s \in \mathbb{R}_+$ ,

$$|v_i(t, b_t, p_t) - v_i(s, b_s, p_s)| \leq C \left[ (1 + |b_t| + |p_t|) |s - t|^{\frac{1}{2}} + |b_t - b_s| + |p_t - p_s| \right] \quad (41)$$

**Proof 3.3.** Let  $0 \leq t < s \leq T$ . To prove continuity property in time  $t$ , we use the dynamic programming principle.

$$v_i(t, b, p) = \sup_{E \in \mathcal{A}_i} \mathbb{E} \left[ \int_t^T e^{-\beta(u-t)} l(i, u, B_u^{t, b_t}, P_u^{t, p_t}, E_u) du + e^{-\beta(T-t)} m(B_T^{t, b_t}, P_T^{t, p_t}) \right] \quad (42)$$

$$= \sup_{E \in \mathcal{A}_i} \mathbb{E} \left[ \int_t^s e^{-\beta(u-t)} l(i, u, B_u^{t, b_t}, P_u^{t, p_t}, E_u) du + e^{-\beta(s-t)} v(s, B_s^{t, b_t}, P_s^{t, p_t}, i) \right] \quad (43)$$

$$= \sup_{E \in \mathcal{A}_i} \mathbb{E} \left[ \int_0^{s-t} e^{-\beta u} l(i, t+u, B_{t+u}^{t, b_t}, P_{t+u}^{t, p_t}, E_{t+u}) du + e^{-\beta(s-t)} v(s, B_{s-t}^{t, b_t}, P_{s-t}^{t, p_t}, i) \right] \quad (44)$$

$$\begin{aligned} 0 \leq v_i(t, b_t, p_t) - v_i(s, b_s, p_s) &= \sup_{E \in \mathcal{A}_i} \mathbb{E} \left[ \int_0^{s-t} e^{-\beta(u)} l(i, u, B_u^{t, b_t}, P_u^{t, p_t}, E_u) du \right. \\ &\quad \left. + e^{-\beta(s-t)} \left( v(s, B_{s-t}^{t, b_t}, P_{s-t}^{t, p_t}, i) - v(s, b_s, p_s, i) \right) \right. \\ &\quad \left. + \left( e^{-\beta(s-t)} - 1 \right) v(s, b_s, p_s, i) \right] \quad (45) \end{aligned}$$

Applying linear growth condition (27) on  $l$ , noting that  $0 \leq 1 - e^{-\beta(s-t)} \leq \beta(s-t)$

and  $v$  satisfies (3.1), we deduce that:

$$\begin{aligned}
& |v_i(t, b_t, p_t) - v_i(s, b_s, p_s)| \\
& \leq \sup_{E \in \mathcal{A}_i} \mathbb{E} \left[ \int_0^{s-t} |l(i, u, B_u^{t, b_t}, P_u^{t, p_t}, E_u)| du + \left| e^{-\beta(s-t)} (v(s, B_{s-t}^{t, b_t}, P_{s-t}^{t, p_t}, i) - v(s, b_s, p_s, i)) \right| \right. \\
& \quad \left. + \left| (e^{-\beta(s-t)} - 1) v(s, b_s, p_s, i) \right| \right] \\
& \leq (s-t)^{\frac{1}{2}} \left( \int_0^{s-t} \sup_{E \in \mathcal{A}_i} \mathbb{E} |l(i, u, B_u^{t, b_t}, P_u^{t, p_t}, E_u)|^2 du \right)^{\frac{1}{2}} + \sup_{E \in \mathcal{A}_i} \mathbb{E} \left| e^{-\beta(s-t)} (v(s, B_{s-t}^{t, b_t}, P_{s-t}^{t, p_t}, i) - v(s, b_s, p_s, i)) \right| \\
& \quad + \sup_{E \in \mathcal{A}_i} \mathbb{E} \left| (e^{-\beta(s-t)} - 1) v(s, b_s, p_s, i) \right| \\
& \leq (s-t)^{\frac{1}{2}} \left( \int_0^{s-t} \rho^2 \sup_{E \in \mathcal{A}_i} (1 + \mathbb{E}|B_u^{t, b_t}| + \mathbb{E}|P_u^{t, p_t}|)^2 du \right)^{\frac{1}{2}} + \sup_{E \in \mathcal{A}_i} \mathbb{E} \left| (v(s, B_{s-t}^{t, b_t}, P_{s-t}^{t, p_t}, i) - v(s, b_s, p_s, i)) \right| \\
& \quad + \beta |s-t| \sup_{E \in \mathcal{A}} \mathbb{E} |v(s, b_s, p_s, i)| \\
& \leq |s-t|^{\frac{1}{2}} \left( \int_0^{s-t} \rho \sup_{E \in \mathcal{A}_i} (1 + \mathbb{E}|B_u^{t, b_t}| + \mathbb{E}|P_u^{t, p_t}|) du \right) + \sup_{E \in \mathcal{A}_i} \mathbb{E} \left| (v(s, B_{s-t}^{t, b_t}, P_{s-t}^{t, p_t}, i) - v(s, b_s, p_s, i)) \right| \\
& \quad + \beta |s-t| \sup_{E \in \mathcal{A}_i} \mathbb{E} |v(s, b_s, p_s, i)| \\
& \leq C' \left( |s-t|^{\frac{1}{2}} \int_0^{s-t} (1 + \mathbb{E}|B_u^{t, b_t}| + \mathbb{E}|P_u^{t, p_t}|) du + (|b_t - b_s| + |p_t - p_s|) + \beta(1 + |b_s| + |p_s|)|s-t| \right) \\
& \leq C \left[ (1 + |b_t| + |p_t|)|s-t|^{\frac{1}{2}} + |b_t - b_s| + |p_t - p_s| \right] \quad (46)
\end{aligned}$$

### 3.2 Existence of viscosity solution

104 In this section we will first define what we mean by viscosity solutions. Then we will prove that the value function is a viscosity solution.

From the optimization problem (23), we derive the Bellman equations as follows:

$$\begin{aligned}
& \frac{\partial v_i}{\partial t}(t, B, P) + \sup_{E \in \mathcal{A}_i} \left\{ -\beta v_i(t, B, P) + \frac{\pi^{1-\gamma}}{1-\gamma} + \theta(\bar{p}_0 - \bar{p}_1 q E B - P) \frac{\partial v_i}{\partial p}(t, B, P) \right. \\
& + \left[ r_i B \left(1 - \frac{B}{K}\right) - q E B \right] \frac{\partial v_i}{\partial b}(t, B, P) + \frac{1}{2} \sigma_P^2 \frac{\partial^2 v_i}{\partial p^2}(t, B, P) + \frac{1}{2} \sigma^2 B^2 \frac{\partial^2 v_i}{\partial b^2}(t, B, P) \\
& \quad \left. + q_{ij}(v_j(t, B, P) - v_i(t, B, P)) \right\} = 0 \quad (47)
\end{aligned}$$

The corresponding Hamiltonian has the following form:

$$\begin{aligned} & \mathcal{H} \left( i, s, B, P, u_i, \frac{\partial u_i}{\partial s}, \frac{\partial u_i}{\partial b}, \frac{\partial u_i}{\partial p}, \frac{\partial^2 u_i}{\partial b^2}, \frac{\partial^2 u_i}{\partial p^2} \right) \\ &= \frac{\partial u_i}{\partial s}(s, B, P) + \sup_{E \in \mathcal{A}_i} \left\{ -\beta u_i(s, B, P) + \frac{\pi(B_s, P_s, E_s)^{1-\gamma}}{1-\gamma} + \mathcal{L}u_i(s, B, P) \right\} = 0 \end{aligned} \quad (48)$$

We have the following systems:

$$\begin{cases} \mathcal{H} \left( i, s, B, P, u_i, \frac{\partial u_i}{\partial s}, \frac{\partial u_i}{\partial b}, \frac{\partial u_i}{\partial p}, \frac{\partial^2 u_i}{\partial b^2}, \frac{\partial^2 u_i}{\partial p^2} \right) = 0 & \text{for } (i, s, B, P) \in \mathcal{S} \times [0; T_i] \times \mathbb{R}_+ \times \mathbb{R}_+ \\ u_i(T, B, P) = \kappa^\gamma \frac{B_T^{1-\gamma}}{1-\gamma} & \text{for } i, j \in \{0; 1\} \quad \kappa > 0. \end{cases} \quad (49)$$

we recall that

$$\pi(B_s, P_s, E_s) = (qB_s P_s - c_1 - c_2 E(s))E(s) \quad (50)$$

106 In order to study the possibility of existence and uniqueness of a solution of  
(49), we use a notion of viscosity solution introduced by (22).

108

Let denote the set of measurable functions on  $[0; T] \times \mathbb{R}_+ \times \mathbb{R}_+$  with polynomial growth of degree  $q \geq 0$  as,

$$\begin{aligned} & \mathcal{C}_q([0; T] \times \mathbb{R}_+ \times \mathbb{R}_+) \\ &= \{ \phi : [0; T] \times \mathbb{R}_+ \times \mathbb{R}_+, \text{ measurable} \mid \exists C > 0, |\phi(t, b, p)| \leq C(1 + |b|^q + |p|^q) \}. \end{aligned} \quad (51)$$

**Definition 3.1.** We say that  $u_i \in \mathcal{C}^0([0; T] \times \mathbb{R}_+ \times \mathbb{R}_+)$  is called

*i. a viscosity subsolution of (49) if for any  $i \in \mathcal{S}$ ,  $u_i(T, b, p) \leq \kappa^\gamma \frac{b_T^{1-\gamma}}{1-\gamma}$ , for all  $b \in \mathbb{R}_+$ ,  $p \in \mathbb{R}_+$  and for all functions  $\phi \in \mathcal{C}^{1,2,2}([0; T] \times \mathbb{R}_+ \times \mathbb{R}_+) \cap \mathcal{C}_2([0; T] \times \mathbb{R}_+ \times \mathbb{R}_+)$  and  $(\bar{t}, \bar{b}, \bar{p})$  such that  $u_i - \phi$  attains its local maximum at  $(\bar{t}, \bar{b}, \bar{p})$ ,*

$$\mathcal{H} \left( i, \bar{t}, \bar{b}, \bar{p}, \phi(\bar{t}, \bar{b}, \bar{p}), \frac{\partial \phi(\bar{t}, \bar{b}, \bar{p})}{\partial s}, \frac{\partial \phi(\bar{t}, \bar{b}, \bar{p})}{\partial b}, \frac{\partial \phi(\bar{t}, \bar{b}, \bar{p})}{\partial p}, \frac{\partial^2 \phi(\bar{t}, \bar{b}, \bar{p})}{\partial b^2}, \frac{\partial^2 \phi(\bar{t}, \bar{b}, \bar{p})}{\partial p^2} \right) \geq 0 \quad (52)$$

ii. a viscosity supersolution of (49) if for any  $i \in \mathcal{S}$ ,  $u_i(T, b, p) \geq \kappa^\gamma \frac{b_T^{1-\gamma}}{1-\gamma}$ , for all  $b \in \mathbb{R}_+$ ,  $p \in \mathbb{R}_+$  and if for all functions  $\phi \in \mathcal{C}^{1,2,2}([0; T] \times \mathbb{R}_+ \times \mathbb{R}_+) \cap \mathcal{C}_2([0; T] \times \mathbb{R}_+ \times \mathbb{R}_+)$  and  $(\underline{t}, \underline{b}, \underline{p})$  such that  $u_i - \phi$  attains its local minimum at  $(\underline{t}, \underline{b}, \underline{p})$ ,

$$\mathcal{H} \left( i, \underline{t}, \underline{b}, \underline{p}, \phi(\underline{t}, \underline{b}, \underline{p}), \frac{\partial \phi(\underline{t}, \underline{b}, \underline{p})}{\partial s}, \frac{\partial \phi(\underline{t}, \underline{b}, \underline{p})}{\partial b}, \frac{\partial \phi(\underline{t}, \underline{b}, \underline{p})}{\partial p}, \frac{\partial^2 \phi(\underline{t}, \underline{b}, \underline{p})}{\partial b^2}, \frac{\partial^2 \phi(\underline{t}, \underline{b}, \underline{p})}{\partial p^2} \right) \leq 0 \quad (53)$$

110 iii. a viscosity solution of (49) if it is both a viscosity sub- and a supersolution of equation (49)

112 **Theorem 3.1.** Under assumptions (27), the value function  $v$  is a viscosity solution of (47).

114 **Proof 3.4.** We establish the viscosity super- and sub-solution properties, respectively in the following two steps.

Step 1.  $v_i(t, b_t, p_t)$ ,  $i = 1; 2$  is a viscosity super-solution of (47).

We already know that  $v \in \mathcal{C}^0([0; T] \times \mathbb{R}_+ \times \mathbb{R}_+)$ . We first note that  $v_i(T, b, p) = \kappa^\gamma \frac{B_T^{1-\gamma}}{1-\gamma}$  so, the boundary condition at time  $t = T$  is clearly satisfied. Let  $(s, b_s, p_s) \in [0; T] \times \mathbb{R}_+ \times \mathbb{R}_+$ ,  $i \in \mathcal{S}$  and  $\phi \in \mathcal{C}^{1,2,2}([0; T] \times \mathbb{R}_+ \times \mathbb{R}_+) \cap \mathcal{C}_2([0; T] \times \mathbb{R}_+ \times \mathbb{R}_+)$  such that  $v_i(\cdot, \cdot, \cdot) - \phi(\cdot, \cdot, \cdot)$  has a local minimum at  $(s, b_s, p_s)$ . Let  $\mathbf{N}(b_s, p_s)$  a neighborhood of  $(s, b_s, p_s)$  where  $v_i(\cdot, \cdot, \cdot) - \phi(\cdot, \cdot, \cdot)$  take its minimum, we introduce a new test-function  $\psi$  as follows:

$$\psi(\cdot, \cdot, \cdot, j) = \begin{cases} \phi(\cdot, \cdot, \cdot) + [v_i(s, b_s, p_s) - \phi(s, b_s, p_s)], & \text{if } j = i \\ v_i(\cdot, \cdot, \cdot), & \text{if } j \neq i. \end{cases} \quad (54)$$

116 This helps us to suppose without any loss of generality that this minimum is equal to 0.

Let  $\tau_\alpha$  be the first jump time of  $\alpha(t)$  ( $= \alpha(t)^{b_s, p_s, i}$ ), i.e.  $\tau_\alpha = \min\{t \geq s : \alpha(t) \neq i\}$ . Then  $\tau_\alpha > s$ , a.s. Let  $\theta_s \in (s, \tau_\alpha)$  be such that the state



$(B_t^{b_s, i}, P_t^{p_s, i})$  starts at  $(b_s, p_s)$  and stays in  $\mathbb{N}(b_s, p_s)$  for  $s \leq t \leq \theta_s$ . Applying the generalized Itô's formula to the switching process  $e^{-\beta t} \psi(t, B_t, P_t, \alpha(t))$ , taking integral from  $t = s$  to  $t = \theta_s \wedge h$ , where  $h > 0$  is a positive constant, and then taking expectation we have

$$\begin{aligned}
& \mathbb{E}_{b_s, p_s, i} \left[ e^{-\beta \theta_s \wedge h} \psi(\theta_s \wedge h, B_{\theta_s \wedge h}, P_{\theta_s \wedge h}, \alpha(\theta_s \wedge h)) \right] \\
&= \psi(s, B_s, P_s, i) + \mathbb{E}_{b_s, p_s, i} \left[ \int_s^{\theta_s \wedge h} e^{-\beta t} \left\{ -\beta \psi(t, B_t, P_t, \alpha(t)) + \frac{\partial \psi(t, B_t, P_t, \alpha(t))}{\partial t} \right. \right. \\
&+ \left[ r_i B_t \left( 1 - \frac{B_t}{K} \right) - q E B_t \right] \frac{\partial \psi(t, B_t, P_t, \alpha(t))}{\partial b} + \theta (\bar{p}_0 - \bar{p}_1 q E B_t - P_t) \frac{\partial \psi(t, B_t, P_t, \alpha(t))}{\partial p} \\
&\quad + \frac{1}{2} \sigma^2 B_t^2 \frac{\partial^2 \psi(t, B_t, P_t, \alpha(t))}{\partial b^2} + \frac{1}{2} \sigma_P^2 \frac{\partial^2 \psi(t, B_t, P_t, \alpha(t))}{\partial p^2} \\
&\quad \left. \left. + q_{\alpha(t)j} (\psi(t, B_t, P_t, j) - \psi(t, B_t, P_t, \alpha(t))) \right\} dt \right]; \quad \alpha(t) \neq j \quad (55)
\end{aligned}$$

From hypothesis, for any  $t \in [s, \theta_s \wedge h]$

$$v_i(t, B_t^{b_s}, P_t^{p_s}) \geq \phi(t, B_t^{b_s}, P_t^{p_s}) + v_i(s, b_s, p_s) - \phi(s, b_s, p_s) \geq \psi(t, B_t^{b_s}, P_t^{p_s}, i) \quad (56)$$

Recalling that  $(B_s^{b_s}, P_s^{p_s}) = (b_s, p_s)$  and using Equations (54) and (56), we have

$$\begin{aligned}
& \mathbb{E}_{b_s, p_s, i} \left[ e^{-\beta \theta_s \wedge h} \psi(\theta_s \wedge h, B_{\theta_s \wedge h}, P_{\theta_s \wedge h}, \alpha(\theta_s \wedge h)) \right] \geq \\
&+ v_i(s, b_s, p_s) + \mathbb{E}_{b_s, p_s, i} \left[ \int_s^{\theta_s \wedge h} e^{-\beta t} \left\{ -\beta v_i(t, B_t, P_t) + \frac{\partial \psi(t, B_t, P_t, \alpha(t))}{\partial t} \right. \right. \\
&+ \left[ r_i B_t \left( 1 - \frac{B_t}{K} \right) - q E B_t \right] \frac{\partial \psi(t, B_t, P_t, \alpha(t))}{\partial b} + \theta (\bar{p}_0 - \bar{p}_1 q E B_t - P_t) \frac{\partial \psi(t, B_t, P_t, \alpha(t))}{\partial p} \\
&\quad + \frac{1}{2} \sigma^2 B_t^2 \frac{\partial^2 \psi(t, B_t, P_t, \alpha(t))}{\partial b^2} + \frac{1}{2} \sigma_P^2 \frac{\partial^2 \psi(t, B_t, P_t, \alpha(t))}{\partial p^2} \\
&\quad \left. \left. + q_{ij} (v_j(t, B_t, P_t) - v_i(t, B_t, P_t)) \right\} dt \right] \quad (57)
\end{aligned}$$

By Bellman's principle

$$\begin{aligned}
\psi(s, b_s, p_s, i) &= v_i(s, b_s, p_s) = \sup_{E \in \mathcal{A}_i} \mathbb{E}_{b_s, p_s, i} \left[ \int_s^{\theta_s \wedge h} e^{-\beta t} l(i, t, B_t^{s, b_s}, P_t^{s, p_s}, E_t) dt \right. \\
&\quad \left. + e^{-\beta(\theta_s \wedge h)} v_i(\theta_s \wedge h, B_{\theta_s \wedge h}^{s, b_s}, P_{\theta_s \wedge h}^{s, p_s}) \right] \\
&\geq \sup_{E \in \mathcal{A}_i} \mathbb{E}_{b_s, p_s, i} \left[ \int_s^{\theta_s \wedge h} e^{-\beta t} l(i, t, B_t^{s, b_s}, P_t^{s, p_s}, E_t) dt \right. \\
&\quad \left. + e^{-\beta(\theta_s \wedge h)} \psi(\theta_s \wedge h, B_{\theta_s \wedge h}^{s, b_s}, P_{\theta_s \wedge h}^{s, p_s}, i) \right] \quad (58)
\end{aligned}$$

Setting  $\tau = \mathbb{E}(\theta_s \wedge h)$  combining (57) and (58) and multiplying both sides by  $1/(\tau - s) > 0$ , we obtain

$$\begin{aligned}
&\sup_{E \in \mathcal{A}_i} \mathbb{E}_{b_s, p_s, i} \left[ \frac{1}{\tau - s} \int_s^{\theta_s \wedge h} e^{-\beta t} \left\{ \beta v_i(t, B_t, P_t) - \frac{\partial \psi(t, B_t, P_t, \alpha(t))}{\partial t} \right. \right. \\
&- \left[ r_i B_t \left( 1 - \frac{B_t}{K} \right) - q E B_t \right] \frac{\partial \psi(t, B_t, P_t, \alpha(t))}{\partial b} - \theta(\bar{p}_0 - \bar{p}_1 q E B_t - P_t) \frac{\partial \psi(t, B_t, P_t, \alpha(t))}{\partial p} \\
&- \frac{1}{2} \sigma^2 B_t^2 \frac{\partial^2 \psi(t, B_t, P_t, \alpha(t))}{\partial b^2} - \frac{1}{2} \sigma_P^2 \frac{\partial^2 \psi(t, B_t, P_t, \alpha(t))}{\partial p^2} \\
&\left. \left. - q_{ij} [v_j(t, B_t, P_t) - v_i(t, B_t, P_t)] - l(i, t, B_t^{s, b_s}, P_t^{s, p_s}, E_t) \right\} dt \right] \geq 0 \quad (59)
\end{aligned}$$

letting  $\tau \downarrow s$  and using the dominated convergence theorem, it turns out that

$$\begin{aligned}
&e^{-\beta s} \left[ - \frac{\partial \psi(s, b_s, p_s, i)}{\partial t} + \inf_{E \in \mathcal{A}_i} \left\{ \beta v_i(s, b_s, p_s) - \right. \right. \\
&\left[ r_i b_s \left( 1 - \frac{b_s}{K} \right) - q E b_s \right] \frac{\partial \psi(s, b_s, p_s, i)}{\partial b} - \theta(\bar{p}_0 - \bar{p}_1 q E b_s - p_s) \frac{\partial \psi(s, b_s, p_s, i)}{\partial p} \\
&- \frac{1}{2} \sigma^2 b_s^2 \frac{\partial^2 \psi(s, b_s, p_s, i)}{\partial b^2} - \frac{1}{2} \sigma_P^2 \frac{\partial^2 \psi(s, b_s, p_s, i)}{\partial p^2} \\
&\left. \left. - q_{ij} [v_j(s, b_s, p_s) - v_i(s, b_s, p_s)] - l(i, s, b_s, p_s, E_s) \right\} \right] \geq 0 \quad (60)
\end{aligned}$$

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This shows that the value function  $v_i(t, b_t, p_t)$ ,  $i = 1, 2$  satisfies the viscosity super-solution property (53).

Step 2.  $v_i(t, b_t, p_t)$ ,  $i = 1, 2$  is a viscosity sub-solution of (47).

We argue by contradiction. Assume that there exist an  $i_0 \in \mathcal{S}$ , a point

$(s, b_s, p_s) \in [0; T] \times \mathbb{R}_+^* \times \mathbb{R}_+^*$  and a testing function  $\phi_{i_0} \in \mathcal{C}^{1,2,2}([0; T] \times \mathbb{R}_+^* \times \mathbb{R}_+^*) \cap \mathcal{C}_2([0; T] \times \mathbb{R}_+^* \times \mathbb{R}_+^*)$  such that  $v_{i_0}(\cdot, \cdot, \cdot) - \phi_{i_0}(\cdot, \cdot, \cdot)$  has a local maximum at  $(s, b_s, p_s)$  in a bounded neighborhood  $\mathbb{N}(b_s, p_s)$ ,  $v_{i_0}(s, b_s, p_s) = \phi_{i_0}(s, b_s, p_s)$ , and

$$\begin{aligned} & \min \left[ -\frac{\partial \phi_{i_0}(s, b_s, p_s)}{\partial t} + \inf_{E \in \mathcal{A}_{i_0}} \left\{ \beta v_{i_0}(s, b_s, p_s) - \right. \right. \\ & \left. \left[ r_{i_0} b_s \left( 1 - \frac{b_s}{K} \right) - q E b_s \right] \frac{\partial \phi_{i_0}(s, b_s, p_s)}{\partial b} - \theta (\bar{p}_0 - \bar{p}_1 q E b_s - p_s) \frac{\partial \phi_{i_0}(s, b_s, p_s)}{\partial p} \right. \\ & \left. - \frac{1}{2} \sigma^2 b_s^2 \frac{\partial^2 \phi_{i_0}(s, b_s, p_s)}{\partial b^2} - \frac{1}{2} \sigma_P^2 \frac{\partial^2 \phi_{i_0}(s, b_s, p_s)}{\partial p^2} - q_{i_0 j} [v_j(s, b_s, p_s) - v_{i_0}(s, b_s, p_s)] \right. \\ & \left. - l_{(i_0, s, b_s, p_s, E_s)} \right\}; \quad v_{i_0}(T, b_s, p_s) - \kappa^\gamma \frac{B_T^{1-\gamma}}{1-\gamma} \Big] > 0 \quad i_0 \neq j \quad (61) \end{aligned}$$

120

By the continuity of all functions involved in (61) ( $v_{i_0}, \phi'_{i_0}, \phi''_{i_0}, q_{ij}, l, \dots$ ), there exist a  $\delta > 0$  and an open ball  $\mathbb{B}_\delta(b_s, p_s) \subset \mathbb{N}(b_s, p_s)$  such that

$$\begin{aligned} & -\frac{\partial \phi_{i_0}(t, b_t, p_t)}{\partial t} + \inf_{E \in \mathcal{A}_{i_0}} \left\{ \beta v_{i_0}(t, b_t, p_t) - \right. \\ & \left[ r_{i_0} b_t \left( 1 - \frac{b_t}{K} \right) - q E b_t \right] \frac{\partial \phi_{i_0}(t, b_t, p_t)}{\partial b} - \theta (\bar{p}_0 - \bar{p}_1 q E b_t - p_t) \frac{\partial \phi_{i_0}(t, b_t, p_t)}{\partial p} \\ & \left. - \frac{1}{2} \sigma^2 b_t^2 \frac{\partial^2 \phi_{i_0}(t, b_t, p_t)}{\partial b^2} - \frac{1}{2} \sigma_P^2 \frac{\partial^2 \phi_{i_0}(t, b_t, p_t)}{\partial p^2} - q_{i_0 j} [v_j(t, b_t, p_t) - v_{i_0}(t, b_t, p_t)] \right. \\ & \left. - l_{(i_0, t, b_t, p_t, E_t)} \right\} > \delta \quad i_0 \neq j; \quad (t, b_t, p_t) \in \mathbb{B}_\delta(b_s, p_s) \quad (62) \end{aligned}$$

and

$$v_{i_0}(T, b_t, p_t) - \kappa^\gamma \frac{B_T^{1-\gamma}}{1-\gamma} > \delta \quad (t, b_t, p_t) \in \mathbb{B}_\delta(b_s, p_s) \quad (63)$$

Let  $\theta_\delta = \min\{t \geq s : (t, B_t, P_t) \notin \mathbb{B}_\delta(b_s, p_s)\}$  be the first exit time of  $(t, B_t, P_t) (= (t, B_t^{s, b_s}, P_t^{s, p_s}))$  from  $\mathbb{B}_\delta(b_s, p_s)$ . Let  $\theta = \theta_\delta \wedge \tau_\alpha$  where  $\tau_\alpha$  is the first jump time of  $\alpha(t)^{b_s, p_s, i_0}$ . Then  $\theta > 0$ , a.s.. For  $0 \leq t \leq \theta$ , we

have

$$\begin{aligned}
& \beta v_{i_0}(t, B_t, P_t) - \frac{\partial \phi_{i_0}(t, B_t, P_t)}{\partial t} \\
& - \left[ r_{i_0} b_t \left( 1 - \frac{B_t}{K} \right) - q E_{i_0} B_t \right] \frac{\partial \phi_{i_0}(t, B_t, P_t)}{\partial b} - \theta (\bar{p}_0 - \bar{p}_1 q E B_t - P_t) \frac{\partial \phi_{i_0}(t, B_t, P_t)}{\partial p} \\
& - \frac{1}{2} \sigma^2 b_s^2 \frac{\partial^2 \phi_{i_0}(t, B_t, P_t)}{\partial b^2} - \frac{1}{2} \sigma_P^2 \frac{\partial^2 \phi_{i_0}(t, B_t, P_t)}{\partial p^2} - q_{i_0 j} [v_j(t, B_t, P_t) - v_{i_0}(t, B_t, P_t)] \\
& - l_{(i_0, t, B_t, P_t, E_t)} > \delta \quad i_0 \neq j; \quad (t, B_t, P_t) \in \mathcal{B}_\delta(b_s, p_s) \quad (64)
\end{aligned}$$

and

$$v_{i_0}(T, b_t, p_t) - \kappa^\gamma \frac{B_T^{1-\gamma}}{1-\gamma} > \delta \quad (t, b_t, p_t) \in \mathcal{B}_\delta(b_s, p_s) \quad (65)$$

As previously, we can replace  $\phi_{i_0}$  by a new test-function  $\psi$  defined as follows:

$$\psi(\cdot, \cdot, \cdot, j) = \begin{cases} \phi_{i_0}(\cdot, \cdot, \cdot), & \text{if } j = i_0 \\ v_{i_0}(\cdot, \cdot, \cdot), & \text{if } j \neq i_0. \end{cases} \quad (66)$$

For any stopping time  $\tau \in [s; T]$ . Applying Itô's formula to the switching process  $e^{-\beta t} \psi(t, B_t, P_t, \alpha(t))$ , taking integral from  $t = s$  to  $t = (\theta_s \wedge \tau)$  – and then taking expectation yield

$$\begin{aligned}
& \mathbb{E}_{b_s, p_s, i} \left[ e^{-\beta \theta \wedge \tau} \psi(\theta \wedge \tau, B_{\theta \wedge \tau}, P_{\theta \wedge \tau}, \alpha(\theta \wedge \tau)) \right] \\
& = v_{i_0}(s, b_s, p_s) + \mathbb{E}_{b_s, p_s, i} \left[ \int_s^{(\theta \wedge \tau)-} e^{-\beta t} \left\{ -\beta \psi(t, B_t, P_t, \alpha(t)) + \frac{\partial \psi(t, B_t, P_t, \alpha(t))}{\partial t} \right. \right. \\
& + \left[ r_i B_t \left( 1 - \frac{B_t}{K} \right) - q E_i B_t \right] \frac{\partial \psi(t, B_t, P_t, \alpha(t))}{\partial b} + \theta (\bar{p}_0 - \bar{p}_1 q E B_t - P_t) \frac{\partial \psi(t, B_t, P_t, \alpha(t))}{\partial p} \\
& + \frac{1}{2} \sigma^2 B_t^2 \frac{\partial^2 \psi(t, B_t, P_t, \alpha(t))}{\partial b^2} + \frac{1}{2} \sigma_P^2 \frac{\partial^2 \psi(t, B_t, P_t, \alpha(t))}{\partial p^2} \\
& \left. \left. + q_{\alpha(t)j} [v_j(t, B_t, P_t) - \psi(t, B_t, P_t, \alpha(t))] \right\} dt \right]; \quad \alpha(t) \neq j \quad (67)
\end{aligned}$$

in which we used  $\mathbb{E}_{b_s, p_s, i} \left[ e^{-\beta \theta \wedge \tau} \psi(\theta \wedge \tau, B_{\theta \wedge \tau}, P_{\theta \wedge \tau}, \alpha(\theta \wedge \tau)) \right] = \mathbb{E}_{b_s, p_s, i} \left[ e^{-\beta \theta_s \wedge \tau} \psi(\theta \wedge \tau, B_{\theta \wedge \tau}, P_{\theta \wedge \tau}, \alpha(\theta \wedge \tau)) \right]$  due to continuity. Noting that the integrand in the RHS of (67) is continuous in  $t$ . Using (64), (65) and that  $v_{i_0}(t, B_t, P_t) \leq$

$\phi_{i_0}(t, B_t, P_t)$  in (67). Also noting that  $\alpha(t) = i_0$  for  $0 \leq t \leq \theta$ , it follows

$$\begin{aligned}
& v_{i_0}(s, b_s, p_s) \\
& \geq \mathbb{E}_{b_s, p_s, i_0} \left[ e^{-\beta\theta \wedge \tau} \phi_{i_0}(\theta \wedge \tau, B_{\theta \wedge \tau}, P_{\theta \wedge \tau}, \alpha(\theta \wedge \tau)) \right. \\
& \quad \left. + \int_s^{(\theta \wedge \tau)} e^{-\beta t} \left\{ \beta v_{i_0}(t, B_t, P_t) - \frac{\partial \phi_{i_0}(t, B_t, P_t)}{\partial t} \right. \right. \\
& \quad - \left[ r_i B_t \left( 1 - \frac{B_t}{K} \right) - q E_i B_t \right] \frac{\partial \phi_{i_0}(t, B_t, P_t)}{\partial b} - \theta (\bar{p}_0 - \bar{p}_1 q E B_t - P_t) \frac{\partial \phi_{i_0}(t, B_t, P_t)}{\partial p} \\
& \quad - \frac{1}{2} \sigma^2 B_t^2 \frac{\partial^2 \phi_{i_0}(t, B_t, P_t)}{\partial b^2} - \frac{1}{2} \sigma_P^2 \frac{\partial^2 \phi_{i_0}(t, B_t, P_t)}{\partial p^2} \\
& \quad \left. \left. - q_{i_0 j} [v_j(t, B_t, P_t) - v_{i_0}(t, B_t, P_t)] \right\} dt \right]; \quad i_0 \neq j \quad (68)
\end{aligned}$$

122

i.e

$$\begin{aligned}
& v_{i_0}(s, b_s, p_s) \\
& \geq \mathbb{E}_{b_s, p_s, i_0} \left[ e^{-\beta\tau} v_{i_0}(\tau, B_\tau, P_\tau, \alpha(\tau)) \mathbf{1}_{\{\tau < \theta\}} + e^{-\beta\theta} v_{i_0}(\theta, B_\theta, P_\theta, \alpha(\theta)) \mathbf{1}_{\{\tau \geq \theta\}} \right. \\
& \quad \left. + \int_s^{(\theta \wedge \tau)} e^{-\beta t} \{ l(i_0, t, B_t, P_t, E_t) + \delta \} dt \right] \\
& \geq \mathbb{E}_{b_s, p_s, i_0} \left[ e^{-\beta\tau} \left[ \kappa^\gamma \frac{B_T^{1-\gamma}}{1-\gamma} + \delta \right] \mathbf{1}_{\{\tau < \theta\}} + e^{-\beta\theta} v_{i_0}(\theta, B_\theta, P_\theta, \alpha(\theta)) \mathbf{1}_{\{\tau \geq \theta\}} \right. \\
& \quad \left. + \int_s^{(\theta \wedge \tau)} e^{-\beta t} \{ l(i_0, t, B_t, P_t, E_t) + \delta \} dt \right] \\
& \geq \mathbb{E}_{b_s, p_s, i_0} \left[ + \int_s^{(\theta \wedge \tau)} e^{-\beta t} \{ l(i_0, t, B_t, P_t, E_t) \} dt + e^{-\beta\theta} v_{i_0}(\theta, B_\theta, P_\theta, \alpha(\theta)) \mathbf{1}_{\{\tau \geq \theta\}} \right. \\
& \quad \left. + e^{-\beta\tau} \left[ \kappa^\gamma \frac{B_T^{1-\gamma}}{1-\gamma} \right] \mathbf{1}_{\{\tau < \theta\}} \right] + \delta \mathbb{E}_{b_s, p_s, i_0} \left[ \int_s^{(\theta \wedge \tau)} e^{-\beta t} dt + e^{-\beta\tau} \mathbf{1}_{\{\tau < \theta\}} \right] \\
& \hspace{15em} (69)
\end{aligned}$$

Now the estimate of the term  $\mathbb{E}_{b_s, p_s, i_0} \left[ \int_s^{(\theta \wedge \tau)} e^{-\beta t} dt + e^{-\beta\tau} \mathbf{1}_{\{\tau < \theta\}} \right]$ . There exist a positive constant  $C_0$  such that

$$\mathbb{E}_{b_s, p_s, i_0} \left[ \int_s^{(\theta \wedge \tau)} e^{-\beta t} dt + e^{-\beta\tau} \mathbf{1}_{\{\tau < \theta\}} \right] \geq C_0 (1 - \mathbb{E}_{b_s, p_s, i_0} [e^{-\beta\tau\alpha}]) \quad (70)$$

For details see (23). It follows that

$$\begin{aligned}
v_{i_0}(s, b_s, p_s) &\geq \sup_{\tau \in [s; T]; E \in \mathcal{A}} \mathbb{E}_{b_s, p_s, i_0} \left[ + \int_s^{(\theta \wedge \tau)} e^{-\beta t} \{l(i_0, t, B_t, P_t, E_t)\} dt \right. \\
&\quad \left. + e^{-\beta \theta} v_{i_0}(\theta, B_\theta, P_\theta, \alpha(\theta)) \mathbf{1}_{\{\tau \geq \theta\}} + e^{-\beta \tau} \left[ \kappa^\gamma \frac{B_T^{1-\gamma}}{1-\gamma} \right] \mathbf{1}_{\{\tau < \theta\}} \right] \\
&\quad + C_0 \delta (1 - \mathbb{E}_{b_s, p_s, i_0} [e^{-\beta \tau \alpha}]) \quad (71)
\end{aligned}$$

which is a contradiction to the DP principle since  $\mathbb{E}_{b_s, p_s, i_0} [e^{-\beta \tau \alpha}] < 1$ .

124 Therefore the value function  $v_i(t, b_t, p_t)$ ,  $i = 1; 2$  is a viscosity sub-solution of the system (2.8).

126 This completes the proof of Theorem 3.1

### 3.3 Comparison principle: uniqueness of the viscosity solution

128 In this section, we prove a comparison result from which we obtain the uniqueness of the solution of the coupled system of partial differential equations.  
130 In proving comparison results for viscosity solutions, the notion of parabolic superjet and subjet defined by Crandall, Ishii and Lions [19] is particularly  
132 useful. Thus, we begin by

**Definition 3.2.** Given  $v \in \mathcal{C}^o([0; T] \times \mathbb{R} \times \mathbb{R} \times \mathcal{S})$  and  $(t; b; p; i) \in [0; T] \times \mathbb{R} \times \mathbb{R} \times \mathcal{S}$ , we define the parabolic superjet:

$$\begin{aligned}
\mathcal{P}^{2,+} v(t, b, p, i) &= \left\{ (c, q, M) \in \mathbb{R} \times \mathbb{R}^2 \times \mathbb{S}^2 : v(s, b', p', i) \leq v(t, b, p, i) \right. \\
&\quad + c(s - t) + q \cdot ((b' - b), (p' - p)) + \frac{1}{2} ((b' - b), (p' - p)) \cdot M((b' - b), (p' - p)) \\
&\quad \left. + o(|((b' - b), (p' - p))|^2) \text{ as } (s; b', p') \rightarrow (t; b, p) \right\} \quad (72)
\end{aligned}$$

and its closure:

$$\begin{aligned}
\bar{\mathcal{P}}^{2,+} v(t, b, p, i) &= \left\{ (c, q, M) = \lim_{n \rightarrow \infty} (c_n, q_n, M_n) \right. \\
&\quad \text{with } (c_n, q_n, M_n) \in \mathcal{P}^{2,+} v(t_n, b_n, p_n, i) \text{ and} \\
&\quad \left. \lim_{n \rightarrow \infty} (t_n, b_n, p_n, v(t_n, b_n, p_n, i)) = (t, b, p, v(t, b, p, i)) \right\} \quad (73)
\end{aligned}$$

Similarly, we define the parabolic subset  $\bar{\mathcal{P}}^{2,-}v(t, b, p, i) = -\bar{\mathcal{P}}^{2,+}(-v)(t, b, p, i)$   
 134 and its closure  $\bar{\mathcal{P}}^{2,-}v(t, b, p, i) = -\bar{\mathcal{P}}^{2,+}(-v)(t, b, p, i)$

It is proved in (24) that

$$\mathcal{P}^{2,+(-)}v(t, b, p, i) = \left\{ \left( \frac{\phi}{\partial t}(t, b, p, i), D_{(b,p)}\phi(t, b, p, i), D_{(b,p)}^2\phi(t, b, p, i) \right. \right. \\ \left. \left. \text{and } v - \phi \text{ has a global maximum (minimum) at } (t, b, p, i) \right) \right\} \quad (74)$$

The previous notions lead to new definition of viscosity solutions.

**Definition 3.3.**  $u_i \in C^0([0; T] \times \mathbb{R}_+^* \times \mathbb{R}_+^*)$  satisfying the polynomial growth  
 136 condition is a viscosity solution of (49)

138 if

(1) for any test-function  $\phi \in C^{1,2,2}([0; T] \times \mathbb{R}_+^* \times \mathbb{R}_+^*)$  if  $(t, b, p)$  is a local  
 maximum point of  $u_i(\cdot, \cdot, \cdot) - \phi(\cdot, \cdot, \cdot)$  and if  $(c, q, L_1) \in \bar{\mathcal{P}}^{2,+}u(t, b, p, i)$   
 with  $c = \partial\phi(t, b, p)/\partial t$ ;  $q = D_{(b,p)}\phi(t, b, p)$  and  $L_1 \leq D_{(b,p)}^2\phi(t, b, p)$ , then

$$\mathcal{H} \left( i, s, b, p, u_i, \frac{\partial u_i}{\partial s}, \frac{\partial u_i}{\partial b}, \frac{\partial u_i}{\partial p}, \frac{\partial^2 u_i}{\partial b^2}, \frac{\partial^2 u_i}{\partial p^2} \right) \leq 0 \quad (75)$$

in this case  $u$  is a viscosity subsolution,

(2) for any test-function  $\phi \in C^{1,2,2}([0; T] \times \mathbb{R}_+^* \times \mathbb{R}_+^*)$  if  $(t, b, p)$  is a local  
 minimum point of  $u_i(\cdot, \cdot, \cdot) - \phi(\cdot, \cdot, \cdot)$  and if  $(c, q, L_2) \in \bar{\mathcal{P}}^{2,-}u(t, b, p, i)$   
 with  $c = \partial\phi(t, b, p)/\partial t$ ;  $q = D_{(b,p)}\phi(t, b, p)$  and  $L_2 \geq D_{(b,p)}^2\phi(t, b, p)$ , then

$$\mathcal{H} \left( i, s, b, p, u_i, \frac{\partial u_i}{\partial s}, \frac{\partial u_i}{\partial b}, \frac{\partial u_i}{\partial p}, \frac{\partial^2 u_i}{\partial b^2}, \frac{\partial^2 u_i}{\partial p^2} \right) \geq 0 \quad (76)$$

140 in this case  $u$  is a viscosity supersolution,

It is proved in (25) that this new definition and the previous one are equiva-  
 142 lent. We refer the reader to the mentioned paper for a proof. The last definition  
 is particular suitable for the discussion of a maximum principle which is the  
 144 backbone of the uniqueness problem for the viscosity solutions theory.

146 Before state next lemma, we first introduce the inf and sup-convolution  
 operations we are going to use.

**Definition 3.4.** For any usc (upper semi-continuous) function  $U : \mathbb{R}^m \rightarrow \mathbb{R}$  and any lsc (lower semi-continuous) function  $V : \mathbb{R}^m \rightarrow \mathbb{R}$ , we set

$$R^\alpha[U](z, r) = \sup_{|Z-z| \leq 1} \left\{ U(Z) - r \cdot (Z - z) - \frac{|Z - z|}{2\alpha} \right\} \quad (77)$$

$$R_\alpha[V](z, r) = \inf_{|Z-z| \leq 1} \left\{ V(Z) + r \cdot (Z - z) + \frac{|Z - z|}{2\alpha} \right\} \quad (78)$$

148  $R^\alpha[U](z, r)$  is called the modified sup-convolution and  $R_\alpha[V](z, r)$  the modified  
inf-convolution. Notice that  $R_\alpha[V](z, r) = -R^\alpha[-U](z, r)$

150 **Lemma 3.2.** (nonlocal Jensen-Ishii's lemma (25))

For any  $i \in \mathcal{S}$ , let  $u_i(\cdot, \cdot, \cdot)$  and  $v_i(\cdot, \cdot, \cdot)$  be, respectively, a usc and lsc function  
152 defined on  $[0; T] \times \mathbb{R}_+ \times \mathbb{R}_+$  and  $\phi \in \mathcal{C}^{1,2,2}([0; T] \times \mathbb{R}_+^2 \times \mathbb{R}_+^2) \cap \mathcal{C}_2([0; T] \times \mathbb{R}_+^2 \times \mathbb{R}_+^2)$   
if  $(\hat{t}, (\hat{b}_1, \hat{p}_1), (\hat{b}_2, \hat{p}_2)) \in [0; T] \times \mathbb{R}_+^2 \times \mathbb{R}_+^2$  is a zero global maximum point of  
154  $u_i(t, b, p) - v_i(t, b', p') - \phi(t, (b, p), (b', p'))$  and if  $c - d := D_t \phi(\hat{t}, (\hat{b}_1, \hat{p}_1), (\hat{b}_2, \hat{p}_2))$ ,  
 $q := D_{(b,p)} \phi(\hat{t}, (\hat{b}_1, \hat{p}_1), (\hat{b}_2, \hat{p}_2))$ ,  $r := -D_{(b',p')} \phi(\hat{t}, (\hat{b}_1, \hat{p}_1), (\hat{b}_2, \hat{p}_2))$ , then for  
156 any  $K > 0$ , there exists  $\alpha(K) > 0$  such that, for any  $0 < \alpha < \alpha(K)$ , we  
have: there exist sequences  $t_k \rightarrow \hat{t}$ ,  $(b_k, p_k) \rightarrow (\hat{b}_1, \hat{p}_1)$ ,  $(b'_k, p'_k) \rightarrow (\hat{b}_2, \hat{p}_2)$ ,  
158  $q_k \rightarrow q$ ,  $r_k \rightarrow r$ , matrices  $M_k, N_k$  and a sequence of functions  $\phi_k$ , converging  
to the function  $\phi_\alpha := R^\alpha[\phi]((b, p), (b', p'), (q, r))$  uniformly in  $\mathbb{R}_+^2 \times \mathbb{R}_+^2$  and in  
160  $C^2(B((\hat{t}, (\hat{b}_1, \hat{p}_1), (\hat{b}_2, \hat{p}_2)), K))$ , such that

$$u_i(t_k, (b_k, p_k)) \rightarrow u_i(\hat{t}, (\hat{b}_1, \hat{p}_1)), \quad v_i(t_k, (b'_k, p'_k)) \rightarrow v_i(\hat{t}, (\hat{b}_2, \hat{p}_2)) \quad (79)$$

$(t_k, (b_k, p_k), (b'_k, p'_k))$  is a global maximum of  $u_i(\cdot, (\cdot, \cdot)) - v_i(\cdot, (\cdot, \cdot)) - \phi(\cdot, (\cdot, \cdot), (\cdot, \cdot))$

$$(c_k, q_k, M_k) \in \bar{\mathcal{P}}^{2,+} u_i(t_k, (b_k, p_k)) \quad (80)$$

$$(-d_k, r_k, N_k) \in \bar{\mathcal{P}}^{2,-} v_i(t_k, (b'_k, p'_k)) \quad (81)$$

$$-\frac{1}{\alpha} \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} M_k & 0 \\ 0 & -N_k \end{pmatrix} \leq D_{(b,p),(b',p')}^2 \phi(t_k, (b_k, p_k), (b'_k, p'_k)) \quad (82)$$



Here  $c_k - d_k = \nabla_t \phi(t_k, (b_k, p_k), (b'_k, p'_k))$ ,  $q_k = \nabla_{(b,p)} \phi(t_k, (b_k, p_k), (b'_k, p'_k))$ ,  $r_k =$   
 $\nabla_{(b',p')} \phi(t_k, (b_k, p_k), (b'_k, p'_k))$  and  $\phi_\alpha(\hat{t}, (\hat{b}_1, \hat{p}_1), (\hat{b}_2, \hat{p}_2)) = \phi(\hat{t}, (\hat{b}_1, \hat{p}_1), (\hat{b}_2, \hat{p}_2))$ ,  
 $\nabla \phi_\alpha(\hat{t}, (\hat{b}_1, \hat{p}_1), (\hat{b}_2, \hat{p}_2)) = \nabla \phi(\hat{t}, (\hat{b}_1, \hat{p}_1), (\hat{b}_2, \hat{p}_2))$

Now we can state our comparison result.

**Theorem 3.2.** (*comparison principle*):

If  $u_i(t, b, p)$  and  $v_i(t, b, p)$  are continuous in  $(t, b, p)$  and are, respectively, viscosity subsolution and supersolution of the HJB system (47) with at most linear growth then

$$u_i(t, b, p) \leq v_i(t, b, p) \text{ for all } (t, b, p, i) \in [0; T] \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathcal{S} \quad (83)$$

**Proof 3.5.** For  $\varrho, \epsilon, \delta, \lambda > 0$ , we define the auxiliary functions  $\phi : (0; T] \times \mathbb{R}_+^2 \times \mathbb{R}_+^2 \rightarrow \mathbb{R}$  and  $\Xi : [0; T] \times \mathbb{R}_+^2 \times \mathbb{R}_+^2 \times \mathcal{S}$  by

$$\phi(t, (b, p), (b', p')) = \frac{\varrho}{t} + \frac{1}{2\epsilon} |(b, p) - (b', p')|^2 + \delta e^{\lambda(T-t)} (|(b, p)|^2 + |(b', p')|^2) \quad (84)$$

and

$$\Xi(t, (b, p), (b', p'), i) = v_i(t, b, p) - u_i(t, b', p') - \phi(t, (b, p), (b', p')) \quad (85)$$

By using the linear growth of  $v_i$  and  $u_i$ , we have for each  $i \in \mathcal{S}$

$$\lim_{|(b,p)|+|(b',p')| \rightarrow \infty} \Xi(t, (b, p), (b', p'), i) = -\infty \quad (86)$$

Then, since  $v_i$  and  $u_i$  are uniformly continuous with respect to  $(t, b, p)$  on each compact subset of  $[0; T] \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathcal{S}$  and that  $\mathcal{S}$  is a finite set,  $\Xi$  attains its global maximum at some finite point belonging to a compact  $K \subset [0; T] \times \mathbb{R}_+^2 \times \mathbb{R}_+^2 \times \mathcal{S}$  say,  $(t_{\delta\epsilon}, (b_{1\delta\epsilon}, p_{1\delta\epsilon}), (b_{2\delta\epsilon}, p_{2\delta\epsilon}), \alpha_{\delta\epsilon})$ . Observing that  $2\Xi(t_{\delta\epsilon}, (b_{1\delta\epsilon}, p_{1\delta\epsilon}), (b_{2\delta\epsilon}, p_{2\delta\epsilon}), \alpha_{\delta\epsilon}) \geq \Xi(t_{\delta\epsilon}, (b_{1\delta\epsilon}, p_{1\delta\epsilon}), (b_{2\delta\epsilon}, p_{2\delta\epsilon}), \alpha_{\delta\epsilon}) + \Xi(t_{\delta\epsilon}, (b_{1\delta\epsilon}, p_{1\delta\epsilon}), (b_{2\delta\epsilon}, p_{2\delta\epsilon}), \alpha_{\delta\epsilon})$  and using the uniform continuity of  $v_i$  and  $u_i$  on  $K$  we have

$$\begin{aligned} & \frac{1}{\epsilon} |(b_{1\delta\epsilon}, p_{1\delta\epsilon}) - (b_{2\delta\epsilon}, p_{2\delta\epsilon})|^2 \\ & \leq v_i(t_{\delta\epsilon}, (b_{1\delta\epsilon}, p_{1\delta\epsilon})) - v_i(t_{\delta\epsilon}, (b_{2\delta\epsilon}, p_{2\delta\epsilon})) + u_i(t_{\delta\epsilon}, (b_{1\delta\epsilon}, p_{1\delta\epsilon})) - u_i(t_{\delta\epsilon}, (b_{2\delta\epsilon}, p_{2\delta\epsilon})) \\ & \leq 2C |(b_{1\delta\epsilon}, p_{1\delta\epsilon}) - (b_{2\delta\epsilon}, p_{2\delta\epsilon})| \quad (87) \end{aligned}$$

Thus,

$$|(b_{1\delta\epsilon}, p_{1\delta\epsilon}) - (b_{2\delta\epsilon}, p_{2\delta\epsilon})| \leq 2C\epsilon \quad (88)$$

166 where  $C$  is a positive constant independent of  $\rho, \epsilon, \delta, \lambda$ . From the inequality,

$$2\Xi(T, (0, 0), (0, 0), \alpha_{\delta\epsilon}) \leq 2\Xi(t_{\delta\epsilon}, (b_{1\delta\epsilon}, p_{1\delta\epsilon}), (b_{2\delta\epsilon}, p_{2\delta\epsilon}), \alpha_{\delta\epsilon}) \quad (89)$$

and the linear growth for  $v_i$  and  $u_i$ , we have:

$$\begin{aligned} \delta(|(b_{1\delta\epsilon}, p_{1\delta\epsilon})|^2 + |(b_{2\delta\epsilon}, p_{2\delta\epsilon})|^2) &\leq e^{-\lambda(T-t_{\delta\epsilon})} \left[ v_i(t_{\delta\epsilon}, b_{1\delta\epsilon}, p_{1\delta\epsilon}) - v_i(T, 0, 0) \right. \\ &\quad \left. + u_i(T, 0, 0) - u_i(t_{\delta\epsilon}, b_{2\delta\epsilon}, p_{2\delta\epsilon}) \right] \\ &\leq e^{-\lambda(T-t_{\delta\epsilon})} C_2 (1 + |(b_{1\delta\epsilon}, p_{1\delta\epsilon})| + |(b_{2\delta\epsilon}, p_{2\delta\epsilon})|) \quad (90) \end{aligned}$$

It follows that

$$\frac{\delta(|(b_{1\delta\epsilon}, p_{1\delta\epsilon})|^2 + |(b_{2\delta\epsilon}, p_{2\delta\epsilon})|^2)}{(1 + |(b_{1\delta\epsilon}, p_{1\delta\epsilon})| + |(b_{2\delta\epsilon}, p_{2\delta\epsilon})|)} \leq C_2 \quad (91)$$

Consequently, there exists  $C_\delta > 0$  such that

$$|(b_{1\delta\epsilon}, p_{1\delta\epsilon})| + |(b_{2\delta\epsilon}, p_{2\delta\epsilon})| \leq C_\delta \quad (92)$$

This inequality implies that for any fixed  $\delta \in (0, 1)$ , the sets  $\{(b_{1\delta\epsilon}, p_{1\delta\epsilon}), \epsilon > 0\}$  and  $\{(b_{2\delta\epsilon}, p_{2\delta\epsilon}), \epsilon > 0\}$  are bounded by  $C_\delta$  independent of  $\epsilon$ . It follows from inequalities (90) and (92) that, possibly if necessary along a subsequence, named again  $(t_{\delta\epsilon}, (b_{1\delta\epsilon}, p_{1\delta\epsilon}), (b_{2\delta\epsilon}, p_{2\delta\epsilon}), \alpha_{\delta\epsilon})$  that there exist  $(b_{1\delta 0}, p_{1\delta 0}) \in \mathbb{R}_+^2$ ,  $t_{\delta 0} \in (0, T]$  and  $\alpha_{\delta 0} \in \mathcal{S}$  such that:  $\lim_{\epsilon \downarrow 0} (b_{1\delta\epsilon}, p_{1\delta\epsilon}) = (b_{1\delta 0}, p_{1\delta 0}) = \lim_{\epsilon \downarrow 0} (b_{1\delta\epsilon}, p_{1\delta\epsilon})$ ,

$$\lim_{\epsilon \downarrow 0} t_{\delta\epsilon} = t_{\delta 0}, \quad \lim_{\epsilon \downarrow 0} \alpha_{\delta\epsilon} = \alpha_{\delta 0}.$$

If  $t_{\delta\epsilon} = T$  then writing that  $\Xi(t, (b, p), (b, p), \alpha_{\delta\epsilon}) \leq \Xi(T, (b_{1\delta\epsilon}, p_{1\delta\epsilon}), (b_{2\delta\epsilon}, p_{2\delta\epsilon}), \alpha_{\delta\epsilon})$ ,

we have

$$\begin{aligned}
& u_i(t, b, p) - v_i(t, b, p) - \frac{\varrho}{t} - 2\delta e^{\lambda(T-t)}(|(b, p)|^2) \\
& \leq u_i(T, (b_{1\delta\epsilon}, p_{1\delta\epsilon})) - v_i(T, (b_{2\delta\epsilon}, p_{2\delta\epsilon})) - \frac{\varrho}{T} \\
& - \frac{1}{2\epsilon} |(b_{1\delta\epsilon}, p_{1\delta\epsilon}) - (b_{2\delta\epsilon}, p_{2\delta\epsilon})|^2 - \delta(|(b_{1\delta\epsilon}, p_{1\delta\epsilon})|^2 + |(b_{2\delta\epsilon}, p_{2\delta\epsilon})|^2) \\
& \leq u_i(T, (b_{1\delta\epsilon}, p_{1\delta\epsilon})) - v_i(T, (b_{2\delta\epsilon}, p_{2\delta\epsilon})) \\
& = [u_i(T, (b_{1\delta\epsilon}, p_{1\delta\epsilon})) - v_i(T, (b_{1\delta\epsilon}, p_{1\delta\epsilon}))] \\
& + [v_i(T, (b_{1\delta\epsilon}, p_{1\delta\epsilon})) - v_i(T, (b_{2\delta\epsilon}, p_{2\delta\epsilon}))] \\
& \leq C_1 |(b_{1\delta\epsilon}, p_{1\delta\epsilon}) - (b_{2\delta\epsilon}, p_{2\delta\epsilon})| \quad (93)
\end{aligned}$$

where the last inequality follows from the uniform continuity of  $v_i$  and by assumption that  $u_i(T, (b_{1\delta\epsilon}, p_{1\delta\epsilon})) = \kappa^\gamma \frac{b_T^{1-\gamma}}{1-\gamma} = v_i(T, (b_{1\delta\epsilon}, p_{1\delta\epsilon}))$ . Sending  $\varrho, \epsilon, \delta \downarrow 0$  and using estimate (88), we have:  $u_i(t, b, p) \leq v_i(t, b, p)$ . Assume now that  $t_{\delta\epsilon} < T$ .

Applying Lemma 3.2 with  $u_i, v_i$  and  $\phi(t, (b, p), (b', p'))$  at the point  $(t_{\delta\epsilon}, (b_{1\delta\epsilon}, p_{1\delta\epsilon}), (b_{2\delta\epsilon}, p_{2\delta\epsilon}), \alpha_{\delta\epsilon}) \in (0; T) \times \mathbb{R}_+^2 \times \mathbb{R}_+^2 \times \mathbb{S}$ , for any  $\zeta \in (0, 1)$  there are  $d \in \mathbb{R}, M_{\delta\epsilon}, N_{\delta\epsilon} \in \mathbb{S}^2$  such that:

$$\begin{aligned}
& \left( d - \frac{\varrho}{t^2} - \lambda \delta e^{\lambda(T-t)}(|(b_{\delta\epsilon}, p_{\delta\epsilon})|^2 + |(b'_{\delta\epsilon}, p'_{\delta\epsilon})|^2), \frac{1}{\epsilon}((b_{\delta\epsilon}, p_{\delta\epsilon}) - (b'_{\delta\epsilon}, p'_{\delta\epsilon})) \right. \\
& \quad \left. + 2\delta e^{\lambda(T-t)}(b_{\delta\epsilon}, p_{\delta\epsilon}), M_{\delta\epsilon} + 2\delta e^{\lambda(T-t)}I \right) \in \bar{\mathcal{P}}^{2,+}v(t, b, p, i) \\
& \left( d, \frac{1}{\epsilon}((b_{\delta\epsilon}, p_{\delta\epsilon}) - (b'_{\delta\epsilon}, p'_{\delta\epsilon})) - 2\delta e^{\lambda(T-t)}(b'_{\delta\epsilon}, p'_{\delta\epsilon}), N_{\delta\epsilon} - 2\delta e^{\lambda(T-t)}I \right) \in \bar{\mathcal{P}}^{2,-}v(t, b, p, i)
\end{aligned} \quad (94)$$

and

$$\begin{aligned}
& -\frac{1}{\zeta} \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} M_{\delta\epsilon} & 0 \\ 0 & -N_{\delta\epsilon} \end{pmatrix} \leq D_{(b,p),(b',p')}^2 \phi(t_{\delta\epsilon}, (b_{\delta\epsilon}, p_{\delta\epsilon}), (b'_{\delta\epsilon}, p'_{\delta\epsilon})) \\
& \quad + \zeta \left( D_{(b,p),(b',p')}^2 \phi(t_{\delta\epsilon}, (b_{\delta\epsilon}, p_{\delta\epsilon}), (b'_{\delta\epsilon}, p'_{\delta\epsilon})) \right)^2 \\
& \leq \frac{\epsilon + \zeta(2 + 4\delta\epsilon e^{\lambda(T-t)})}{\epsilon^2} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + (2\delta + 4\zeta\delta^2\epsilon e^{\lambda(T-t)}) e^{\lambda(T-t)} \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}
\end{aligned} \tag{95}$$

Letting  $\delta \downarrow 0$  and taking  $\zeta = \frac{\epsilon}{2}$ , we obtain

$$-\frac{1}{\epsilon} \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} M_{\delta\epsilon} & 0 \\ 0 & -N_{\delta\epsilon} \end{pmatrix} \leq \frac{2}{\epsilon} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} \tag{96}$$

It follows that

$$\begin{aligned}
& (b_{\delta\epsilon}, p_{\delta\epsilon}) M_{\delta\epsilon} \begin{pmatrix} b_{\delta\epsilon} \\ p_{\delta\epsilon} \end{pmatrix} - (b'_{\delta\epsilon}, p'_{\delta\epsilon}) N_{\delta\epsilon} \begin{pmatrix} b'_{\delta\epsilon} \\ p'_{\delta\epsilon} \end{pmatrix} \\
& = ((b_{\delta\epsilon}, p_{\delta\epsilon}), (b'_{\delta\epsilon}, p'_{\delta\epsilon})) \begin{pmatrix} M_{\delta\epsilon} & 0 \\ 0 & -N_{\delta\epsilon} \end{pmatrix} \begin{pmatrix} \begin{pmatrix} b_{\delta\epsilon} \\ p_{\delta\epsilon} \end{pmatrix} \\ \begin{pmatrix} b'_{\delta\epsilon} \\ p'_{\delta\epsilon} \end{pmatrix} \end{pmatrix} \\
& \leq ((b_{\delta\epsilon}, p_{\delta\epsilon}), (b'_{\delta\epsilon}, p'_{\delta\epsilon})) \left[ \frac{2}{\epsilon} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} \right] \begin{pmatrix} \begin{pmatrix} b_{\delta\epsilon} \\ p_{\delta\epsilon} \end{pmatrix} \\ \begin{pmatrix} b'_{\delta\epsilon} \\ p'_{\delta\epsilon} \end{pmatrix} \end{pmatrix} \\
& \leq \frac{2}{\epsilon} |(b_{\delta\epsilon}, p_{\delta\epsilon}) - (b'_{\delta\epsilon}, p'_{\delta\epsilon})|^2 \tag{97}
\end{aligned}$$

Furthermore, the definition of the viscosity subsolution  $u_i$  and supersolution  $v_i$

implies that

$$\begin{aligned}
& \min \left[ \beta u_{i_0}(t_{\delta\epsilon}, b_{\delta\epsilon}, p_{\delta\epsilon}) - \left( d - \frac{\rho}{t_{\delta\epsilon}^2} - \lambda \delta e^{\lambda(T-t_{\delta\epsilon})} (|(b_{\delta\epsilon}, p_{\delta\epsilon})|^2 + |(b'_{\delta\epsilon}, p'_{\delta\epsilon})|^2) \right) \right. \\
& \quad + \inf_{E \in \mathcal{A}_{i_0}} \left\{ - \left[ r_{i_0} b_{\delta\epsilon} \left( 1 - \frac{b_{\delta\epsilon}}{K} \right) - q E b_{\delta\epsilon} \right] \left( \frac{1}{\epsilon} (b_{\delta\epsilon} - b'_{\delta\epsilon}) + 2 \delta e^{\lambda(T-t)} b_{\delta\epsilon} \right) \right. \\
& \quad - \theta (\bar{p}_0 - \bar{p}_1 q E b_{\delta\epsilon} - p_s) \left( \frac{1}{\epsilon} (p_{\delta\epsilon} - p'_{\delta\epsilon}) + 2 \delta e^{\lambda(T-t)} p_{\delta\epsilon} \right) - \frac{1}{2} (\sigma b_{\delta\epsilon}; \sigma_P) (M_{\delta\epsilon} + 2 \delta e^{\lambda(T-t)} I) \begin{pmatrix} \sigma b_{\delta\epsilon} \\ \sigma_P \end{pmatrix} \\
& \quad \left. - q_{i_0 j} [u_j(t_{\delta\epsilon}, b_{\delta\epsilon}, p_{\delta\epsilon}) - u_{i_0}(t_{\delta\epsilon}, b_{\delta\epsilon}, p_{\delta\epsilon})] \right. \\
& \quad \left. - l(i_0, t_{\delta\epsilon}, b_{\delta\epsilon}, p_{\delta\epsilon}, E t_{\delta\epsilon}) \right\}; \quad u_{i_0}(T, \cdot, b_{\delta\epsilon}, p_{\delta\epsilon}) - \kappa \gamma \frac{B_T^{1-\gamma}}{1-\gamma} \Big] \leq 0 \quad i_0 \neq j \quad (98)
\end{aligned}$$

168 and

$$\begin{aligned}
& \min \left[ \beta v_{i_0}(t_{\delta\epsilon}, b_{\delta\epsilon}, p_{\delta\epsilon}) - d + \inf_{E \in \mathcal{A}_{i_0}} \left\{ - \left[ r_{i_0} b_{\delta\epsilon} \left( 1 - \frac{b_{\delta\epsilon}}{K} \right) - q E b_{\delta\epsilon} \right] \left( \frac{1}{\epsilon} (b_{\delta\epsilon} - b'_{\delta\epsilon}) + 2 \delta e^{\lambda(T-t)} b'_{\delta\epsilon} \right) \right. \right. \\
& \quad - \theta (\bar{p}_0 - \bar{p}_1 q E b_{\delta\epsilon} - p_s) \left( \frac{1}{\epsilon} (p_{\delta\epsilon} - p'_{\delta\epsilon}) + 2 \delta e^{\lambda(T-t)} p'_{\delta\epsilon} \right) - \frac{1}{2} (\sigma b'_{\delta\epsilon}; \sigma_P) (N_{\delta\epsilon} - 2 \delta e^{\lambda(T-t)} I) \begin{pmatrix} \sigma b'_{\delta\epsilon} \\ \sigma_P \end{pmatrix} \\
& \quad \left. - q_{i_0 j} [v_j(t_{\delta\epsilon}, b'_{\delta\epsilon}, p'_{\delta\epsilon}) - v_{i_0}(t_{\delta\epsilon}, b'_{\delta\epsilon}, p'_{\delta\epsilon})] \right. \\
& \quad \left. - l(i_0, t_{\delta\epsilon}, b'_{\delta\epsilon}, p'_{\delta\epsilon}, E t_{\delta\epsilon}) \right\}; \quad v_{i_0}(T, \cdot, b_{\delta\epsilon}, p_{\delta\epsilon}) - \kappa \gamma \frac{B_T^{1-\gamma}}{1-\gamma} \Big] \geq 0 \quad i_0 \neq j \quad (99)
\end{aligned}$$

Let us define operators  $A^E(x, v, \phi, X, Z)$  and  $B^E(x, v)$ .

$$A^E(t, b, p, w, X, YZ) = \left[ r_{i_0} b \left( 1 - \frac{b}{K} \right) - q E b \right] X + \theta (\bar{p}_0 - \bar{p}_1 q E b - p_s) Y + \frac{1}{2} w Z w' \quad (100)$$

$$B^E(t, b, p, v) = q_{i_0 j} [v_j(t, b, p) - v_{i_0}(t, b, p)] \quad (101)$$

170 by Subtracting these last two inequalities and remarking that  $\min(x; y) - \min(z; t) \leq 0$  implies either  $x - z \leq 0$  or  $y - t \leq 0$ , we divide our consideration into two cases:

Case 1

$$\begin{aligned}
& \beta [u_{i_0}(t_{\delta\epsilon}, b_{\delta\epsilon}, p_{\delta\epsilon}) - v_{i_0}(t_{\delta\epsilon}, b_{\delta\epsilon}, p_{\delta\epsilon})] + \frac{\rho}{t_{\delta\epsilon}^2} + \lambda \delta e^{\lambda(T-t_{\delta\epsilon})} (|(b_{\delta\epsilon}, p_{\delta\epsilon})|^2 + |(b'_{\delta\epsilon}, p'_{\delta\epsilon})|^2) \\
& \leq \sup_{E \in \mathcal{A}_{i_0}} \{l(i_0, t_{\delta\epsilon}, b_{\delta\epsilon}, p_{\delta\epsilon}, E_{t_{\delta\epsilon}}) - l(i_0, t_{\delta\epsilon}, b'_{\delta\epsilon}, p'_{\delta\epsilon}, E_{t_{\delta\epsilon}})\} \\
& + \sup_{E \in \mathcal{A}_{i_0}} \left\{ A^E \left( t_{\delta\epsilon}, b_{\delta\epsilon}, p_{\delta\epsilon}, (\sigma b_{\delta\epsilon}; \sigma_P), \frac{1}{\epsilon} (b_{\delta\epsilon} - b'_{\delta\epsilon}) + 2\delta e^{\lambda(T-t_{\delta\epsilon})} b_{\delta\epsilon}, \frac{1}{\epsilon} (p_{\delta\epsilon} - p'_{\delta\epsilon}) + 2\delta e^{\lambda(T-t_{\delta\epsilon})} p_{\delta\epsilon}, \right. \right. \\
& \quad \left. \left. M_{\delta\epsilon} + 2\delta e^{\lambda(T-t_{\delta\epsilon})} I \right) \right. \\
& - A^E \left( t_{\delta\epsilon}, b'_{\delta\epsilon}, p'_{\delta\epsilon}, (\sigma b'_{\delta\epsilon}; \sigma_P), \frac{1}{\epsilon} (b_{\delta\epsilon} - b'_{\delta\epsilon}) + 2\delta e^{\lambda(T-t_{\delta\epsilon})} b'_{\delta\epsilon}, \frac{1}{\epsilon} (p_{\delta\epsilon} - p'_{\delta\epsilon}) + 2\delta e^{\lambda(T-t_{\delta\epsilon})} p'_{\delta\epsilon}, \right. \\
& \quad \left. \left. N_{\delta\epsilon} - 2\delta e^{\lambda(T-t_{\delta\epsilon})} I \right) \right\} \\
& + \sup_{E \in \mathcal{A}_{i_0}} \left\{ B^E(t_{\delta\epsilon}, b_{\delta\epsilon}, p_{\delta\epsilon} u) - B^E(t_{\delta\epsilon}, b'_{\delta\epsilon}, p'_{\delta\epsilon} v) \right\} \equiv \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 \quad (102)
\end{aligned}$$

In view of condition (27) on  $l$  and from estimate (3.1), we have the classical estimates of  $\mathcal{I}_1$  and  $\mathcal{I}_2$ :

$$\mathcal{I}_1 \leq C |(b_{\delta\epsilon}, p_{\delta\epsilon}) - (b'_{\delta\epsilon}, p'_{\delta\epsilon})| \quad (103)$$

$$\mathcal{I}_2 \leq C \left( \frac{1}{\epsilon} |(b_{\delta\epsilon}, p_{\delta\epsilon}) - (b'_{\delta\epsilon}, p'_{\delta\epsilon})|^2 + 2\delta e^{\lambda(T-t_{\delta\epsilon})} (1 + |(b_{\delta\epsilon}, p_{\delta\epsilon})|^2 + |(b'_{\delta\epsilon}, p'_{\delta\epsilon})|^2) \right) \quad (104)$$

Using the Lipschitz condition for  $u$  and  $v$ , we have

$$\mathcal{I}_3 \leq 2C |(b_{\delta\epsilon}, p_{\delta\epsilon}) - (b'_{\delta\epsilon}, p'_{\delta\epsilon})| \quad (105)$$

Writing that  $\Xi(t, (b, p), (b, p), i) \leq \Xi(t_{\delta\epsilon}, (b_{\delta\epsilon}, p_{\delta\epsilon}), (b_{\delta\epsilon}, p_{\delta\epsilon}), i)$  for  $i \in \mathcal{S}$  and using the inequality (102),

$$\begin{aligned}
& u_i(t, b, p) - v_i(t, b, p) - \frac{\rho}{t} - 2\delta e^{\lambda(T-t)} |(b, p)|^2 \leq \\
& v_i(t_{\delta\epsilon}, b_{\delta\epsilon}, p_{\delta\epsilon}) - u_i(t_{\delta\epsilon}, b_{\delta\epsilon}, p_{\delta\epsilon}) - \frac{\rho}{t_{\delta\epsilon}} - 2\delta e^{\lambda(T-t)} |(b_{\delta\epsilon}, p_{\delta\epsilon})|^2 \leq \\
& \frac{1}{\beta} [\mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3] - \frac{\rho}{\beta t_{\delta\epsilon}^2} - \frac{\lambda}{\beta} \delta e^{\lambda(T-t_{\delta\epsilon})} (|(b_{\delta\epsilon}, p_{\delta\epsilon})|^2 + |(b'_{\delta\epsilon}, p'_{\delta\epsilon})|^2) \quad (106)
\end{aligned}$$

this implies

$$\begin{aligned}
& u_i(t, b, p) - v_i(t, b, p) - \frac{\rho}{t} - 2\delta e^{\lambda(T-t)} |(b, p)|^2 \leq \\
& \frac{1}{\beta} [\mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3] - \frac{\lambda}{\beta} \delta e^{\lambda(T-t_{\delta\epsilon})} (|(b_{\delta\epsilon}, p_{\delta\epsilon})|^2 + |(b'_{\delta\epsilon}, p'_{\delta\epsilon})|^2) \quad (107)
\end{aligned}$$

Sending  $\epsilon \downarrow 0$ , with the above estimates of  $(\mathcal{I}_1) - (\mathcal{I}_2) - (\mathcal{I}_3)$ , we obtain:

$$u_i(t, b, p) - v_i(t, b, p) - \frac{\varrho}{t} - 2\delta e^{\lambda(T-t)} |(b, p)|^2 \leq \frac{2\delta}{\beta} e^{\lambda(T-t_0)} \left[ C(1+2|(b_0, p_0)|^2) - \lambda|(b_0, p_0)|^2 \right] \quad (108)$$

172 Choose  $\lambda$  sufficiently large positive ( $\lambda \geq 2C$ ) and send  $\varrho, \delta \rightarrow 0^+$  to conclude that  $u_i(t, b, p) \leq v_i(t, b, p)$

Case 2 the second case occurs if

$$u_{i_0}(T, , b_{\delta\epsilon}, p_{\delta\epsilon}) - v_{i_0}(T, , b_{\delta\epsilon}, p_{\delta\epsilon}) \leq 0 \quad (109)$$

174 and

finally that  $u_i(t, b, p) \leq v_i(t, b, p)$

176 This completes the proof.

The following corollary follows from Theorems 3.1 and 3.2.

178 **Corollary 3.1.** *The value function  $v$  is a unique viscosity solution of (47) that has at most a linear growth.*

## 180 4 Monotone Finite Difference and Simulation

The determination of the effort value requires numerical computations. Thus, 182 instead of arbitrary parameters values, we have decided to use realistic values. We found a quite complete set of parameter values in (2) and (11). The time 184 horizon was set at  $T_i = 5$  years. The complete set of parameter values is listed in Table 1.

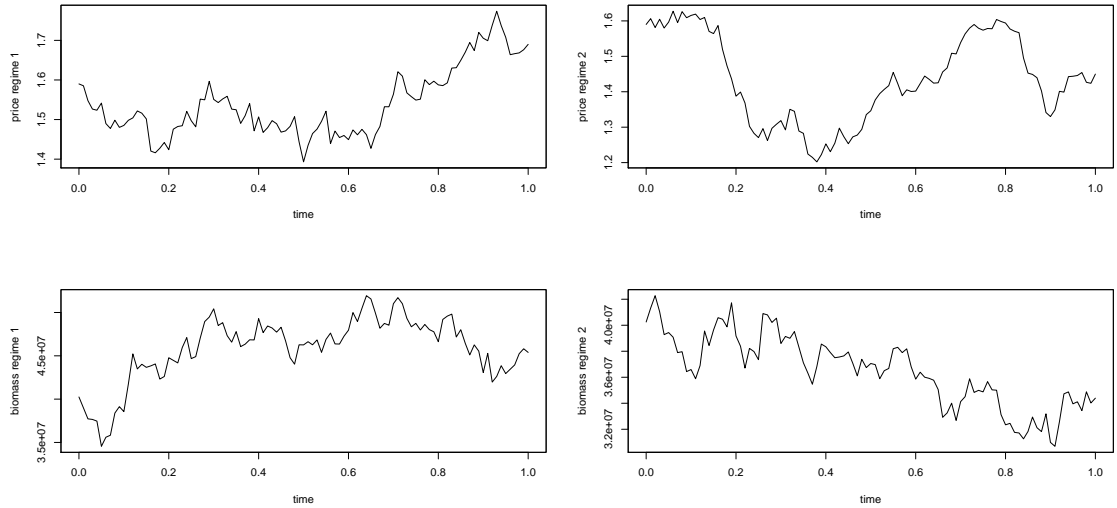
### 186 4.1 Sample Realisations of Price and Stock

We chose the maximum fishing effort value for these sample realisations.

Parameters	Description	Values	Units
$r_1; r_2$	Intrinsic growth rate	0.71; 0.68	$year^{-1}$
$K$	Carrying capacity	$80.5 \times 10^6$	kg
$q$	Catchability coefficient	$3.30 \times 10^{-6}$	$SFU^{-1}year^{-1}$
$E_{max}$	Maximum fishing effort	$0.7r/q$	$SFU$
$B_o$	Initial population size	$0.5K$	kg
$\beta$	Discount factor	0.05	$year^{-1}$
$p_0$	Price per unit yield	1.59	$kg^{-1}$
$c_1$	Linear cost parameter	$96 \times 10^{-6}$	$SFU^{-1}year^{-1}$
$c_2$	Quadratic cost parameter	$0.10 \times 10^{-6}$	$SFU^{-2}year^{-1}$
$\theta$	Mean-reversion speed	0.59	
$\bar{p}_0$	Price of the stock	1.211	
$\bar{p}_1$	Strength of demand	0.0001	
$\sigma$	Volatility of the stock	0.3	
$\sigma_p$	Volatility of the price	0.3	
$T$	Time horizon	5	years
$\gamma$	risk aversion coefficient	0.3	

Table 1: Numerical parameters

188





190 4.2 The Numerical Approximation

In this section, we present a numerical solution. We consider the switching process  $\alpha(t)$  where  $\alpha(t) \in \mathcal{S} = \{1, 2\}$  represents the season. In particular,  $\alpha(t) = 1$  stands for the flood period with reduced fishing and intensive reproduction. and  $\alpha(t) = 2$  the dry season with intensive fishing and reduced reproduction. The generator of  $\alpha(t)$  is given by

$$\begin{pmatrix} 2.0367 & -2.0367 \\ -1.9821 & 1.9821 \end{pmatrix} \quad (110)$$

192 For our problems we need to ensure that our discretization methods converge to the viscosity solution and determine the optimal effort. Using the basic results of (26) and (27), this ensures that our numerical solutions convergence  
194 to the viscosity solution. For this purpose, we use the fully implicit upwind scheme which is unconditionally monotone.

196

To approximate the solution to (23) we discretize variables  $t$ ,  $p$  and  $B$  with stepsizes  $\Delta t$ ,  $\Delta p$  and  $\Delta B$  respectively. The value of  $v_i$  at a grid point  $(t_n; p_k; B_l)$  in the regime  $i$  is denoted by  $v_{k,l}^n(i)$ . The derivatives of  $v_i$  are approximated by

$$\begin{aligned} \frac{\partial v_i}{\partial t} &\approx \frac{v_{k,l}^{n+1}(i) - v_{k,l}^n(i)}{\Delta t}, \\ \frac{\partial^2 v_i}{\partial b^2} &\approx \frac{v_{k,l+1}^{n+1}(i) + v_{k,l-1}^{n+1}(i) - 2v_{k,l}^{n+1}(i)}{(\Delta b)^2}, \\ \frac{\partial^2 v_i}{\partial p^2} &\approx \frac{v_{k+1,l}^{n+1}(i) + v_{k-1,l}^{n+1}(i) - 2v_{k,l}^{n+1}(i)}{(\Delta p)^2}, \\ \Theta_i \frac{\partial v_i}{\partial p} &\approx \begin{cases} \Theta_i \frac{v_{k+1,l}^{n+1}(i) - v_{k,l}^{n+1}(i)}{2\Delta p} & \text{if } \Theta_i > 0 \\ \Theta_i \frac{v_{k,l}^{n+1}(i) - v_{k-1,l}^{n+1}(i)}{2\Delta p} & \text{if } \Theta_i < 0 \end{cases} \quad \text{and} \\ \Phi_i \frac{\partial v_i}{\partial b} &\approx \begin{cases} \Phi_i \frac{v_{k,l+1}^{n+1}(i) - v_{k,l}^{n+1}(i)}{2\Delta b} & \text{if } \Phi_i > 0 \\ \Phi_i \frac{v_{k,l}^{n+1}(i) - v_{k,l-1}^{n+1}(i)}{2\Delta b} & \text{if } \Phi_i < 0 \end{cases} . \end{aligned}$$

Discretizing equation (23)

$$\begin{aligned}
& \frac{v_{k,l}^{n+1} - v_{k,l}^n}{\Delta t} + \sup_{E \in \mathcal{A}_i} \left\{ \left( -\beta + q_{ij} \right) v_{k,l}^{n+1} \right. \\
& + \max \left( \theta(\bar{p}_0 - \bar{p}_1 q E B_l - P_k); 0 \right) \frac{v_{k+1,l}^{n+1} - v_{k,l}^{n+1}}{\Delta p} - \min \left( \theta(\bar{p}_0 - \bar{p}_1 q E B_l - P_k); 0 \right) \frac{v_{k,l}^{n+1} - v_{k-1,l}^{n+1}}{\Delta p} \\
& + \max \left( r_i B_l \left( 1 - \frac{B_l}{K} \right) - q E B_l; 0 \right) \frac{v_{k,l+1}^{n+1} - v_{k,l}^{n+1}}{\Delta b} - \min \left( r_i B_l \left( 1 - \frac{B_l}{K} \right) - q E B_l; 0 \right) \frac{v_{k,l}^{n+1} - v_{k,l-1}^{n+1}}{\Delta b} \\
& + \frac{1}{2} \sigma_p^2 \frac{v_{k+1,l}^{n+1} + v_{k-1,l}^{n+1} - 2v_{k,l}^{n+1}}{(\Delta p)^2} + \frac{1}{2} \sigma_B^2 B_l^2 \frac{v_{k,l+1}^{n+1} + v_{k,l-1}^{n+1} - 2v_{k,l}^{n+1}}{(\Delta b)^2} \\
& \left. + \frac{(q E B_l P_k - c_1 E - c_2 E^2)^{1-\gamma}}{1-\gamma} - q_{ij} v_{k,l}^{n+1}(j) \right\} = 0 \quad (111)
\end{aligned}$$

and rearranging the terms, we obtain

$$\begin{aligned}
& \left( -\beta + \frac{1}{\Delta t} - \frac{\sigma_p^2}{(\Delta p)^2} - \frac{\sigma_B^2 B_l^2}{(\Delta b)^2} + q_{ij} - \frac{1}{\Delta p} \theta(\bar{p}_0 - \bar{p}_1 q E B_l - P_k) - \frac{1}{\Delta b} \left( r_i B_l \left( 1 - \frac{B_l}{K} \right) - q E B_l \right) \right) v_{k,l}^{n+1} \\
& + \sup_{E \in \mathcal{A}_i} \left\{ \left( \frac{\sigma_p^2}{2(\Delta p)^2} + \frac{\max \left( \theta(\bar{p}_0 - \bar{p}_1 q E B_l - P_k); 0 \right)}{\Delta p} \right) v_{k+1,l}^{n+1} \right. \\
& + \left( \frac{\sigma_p^2}{2(\Delta p)^2} + \frac{\min \left( \theta(\bar{p}_0 - \bar{p}_1 q E B_l - P_k); 0 \right)}{\Delta p} \right) v_{k-1,l}^{n+1} \\
& + \left( \frac{\sigma_B^2 B_l^2}{2(\Delta b)^2} + \frac{\max \left( r_i B_l \left( 1 - \frac{B_l}{K} \right) - q E B_l; 0 \right)}{\Delta b} \right) v_{k,l+1}^{n+1} \\
& + \left( \frac{\sigma_B^2 B_l^2}{2(\Delta b)^2} + \frac{\min \left( r_i B_l \left( 1 - \frac{B_l}{K} \right) - q E B_l; 0 \right)}{\Delta b} \right) v_{k,l-1}^{n+1} \\
& \left. - q_{ij} v_{k,l}^{n+1}(j) + \frac{(q E B_l P_k - c_1 E - c_2 E^2)^{1-\gamma}}{1-\gamma} \right\} = \frac{v_{k,l}^n}{\Delta t} \quad (112)
\end{aligned}$$

198 In addition, we consider that:

- the optimization starts at time  $t = 0$  and ends at time  $t = T < +\infty$

200 • the time interval is uniformly partitioned as  $0 = t_0 < t_1 < \dots < t_N = T$   
with

202  $t_{n+1} - t_n = \Delta t = \frac{T}{N}, n = 0, 1, \dots, N - 1;$

- the state variable of biomass takes values within the interval  $[0, 2K]$ , which

204 is uniformly partitioned as  $0 = B_0 < B_1 < \dots < B_m = 2K$  with  $B_{l+1} -$   
 $B_l = \Delta B = 2K/m$ ,  $l = 0, 1, \dots, m - 1$ ;

206 • the state variable of prices takes values within the interval  $[0, p_{max}]$ , which  
is uniformly partitioned as  $0 = p_0 < p_1 < \dots < p_m = p_{max}$  with

208  $p_{k+1} - p_k = \Delta p = \frac{p_{max}}{m}$ ,  $k = 0, 1, \dots, m - 1$ ;

210 • we have boundary conditions, a terminal conditions  $v_i(T, b_t, p_t) = \kappa^\gamma \frac{B_T^{1-\gamma}}{1-\gamma}$ ,  
and a initial condition  $v_i(0, b, p) = 0$ .

If we define the constants

$$a_i = 1 - \beta \Delta t - \frac{\sigma_p^2 \Delta t}{(\Delta p)^2} - \frac{\sigma_B^2 B_l^2 \Delta t}{(\Delta B)^2} + q_{ij} \Delta t - \frac{\Delta t}{\Delta p} \theta (\bar{p}_0 - \bar{p}_1 q E B_l - P_k) - \frac{\Delta t}{\Delta b} \left( r_i B_l \left( 1 - \frac{B_l}{K} \right) - q E B_l \right) \quad (113)$$

$$b_i = \frac{\sigma_p^2 \Delta t}{2(\Delta p)^2} + \frac{\max(\theta(\bar{p}_0 - \bar{p}_1 q E B_l - P_k); 0)}{\Delta p} \Delta t \quad (114)$$

$$c_i = \frac{\sigma_p^2 \Delta t}{2(\Delta p)^2} + \frac{\min(\theta(\bar{p}_0 - \bar{p}_1 q E B_l - P_k); 0)}{\Delta p} \Delta t \quad (115)$$

$$d_i = \frac{\sigma_B^2 B_l^2 \Delta t}{2(\Delta b)^2} + \frac{\max(r_i B_l (1 - \frac{B_l}{K}) - q E B_l; 0)}{\Delta B} \Delta t \quad (116)$$

$$e_i = \frac{\sigma_B^2 B_l^2 \Delta t}{2(\Delta b)^2} + \frac{\min(r_i B_l (1 - \frac{B_l}{K}) - q E B_l; 0)}{\Delta B} \Delta t \quad (117)$$

$$f_i = \frac{(q E B_l P_k - c_1 E - c_2 E^2)^{1-\gamma}}{1-\gamma} \Delta t \quad (118)$$

We can rewrite this difference equation in a more manageable form:

$$\sup_{E \in \mathcal{A}_i} \left\{ a_i v_{k,l}^{n+1} + b_i v_{k+1,l}^{n+1} + c_i v_{k-1,l}^{n+1} + d_i v_{k,l+1}^{n+1} + e_i v_{k,l-1}^{n+1} - q_{ij} \Delta t v_{k,l}^{n+1}(j) + f_i \right\} = v_{k,l}^n \quad (119)$$

Writing (119) in a appropriate matrix form,

$$\sup_{E \in \mathcal{A}_i} \left\{ \mathbf{A}_i^E \mathbf{v}_i^{n+1} - \mathbf{\Lambda}_{ji} \mathbf{v}_j^{n+1} + \mathbf{F}_i^{n+1} - \mathbf{v}_i^n \right\} = 0 \quad (120)$$

### 4.3 Howard's algorithm

212 We denote by  $\mathbf{v}_i^n$  and  $\mathbf{v}_i^{n+1}$  the approximations at time  $n$  and  $n + 1$ .

**Step 0:** start with an initial value for the control  $E^0$ . Compute the solution  $v_i^0$   
 214 of  $\mathbf{A}_i^{E^0} w - \mathbf{\Lambda}_{ji} \mathbf{v}_j^{n+1} + \mathbf{F}_i^{n+1} - \mathbf{v}_i^n = 0$ .

**Step  $j \rightarrow j+1$ :** given  $v_h^j$ , find  $E^{j+1} \in \mathcal{A}_i$  maximizing  $\mathbf{A}_i^{E^{j+1}} w - \mathbf{\Lambda}_{ji} \mathbf{v}_j^{n+1} + \mathbf{F}_i^{n+1} -$   
 216  $\mathbf{v}_i^n = 0$ . Then compute the solution  $v_h^{j+1}$  of  $\mathbf{A}_i^{E^{j+1}} w - \mathbf{\Lambda}_{ji} \mathbf{v}_j^{n+1} + \mathbf{F}_i^{n+1} -$   
 $\mathbf{v}_i^n = 0$ .

**Final step** :if  $|\mathbf{v}_i^{j+1} - \mathbf{v}_i^j| < \epsilon$ , then set  $\mathbf{v}_i^{n+1} = \mathbf{v}_i^{j+1}$

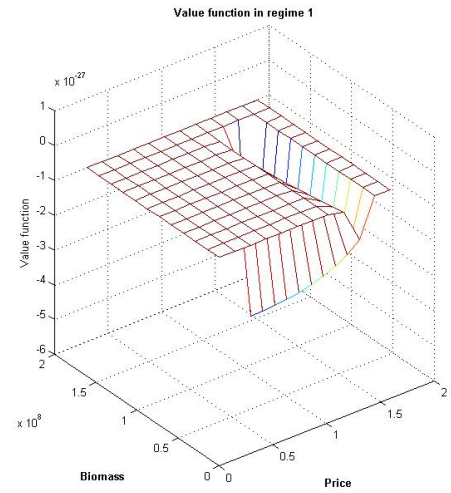
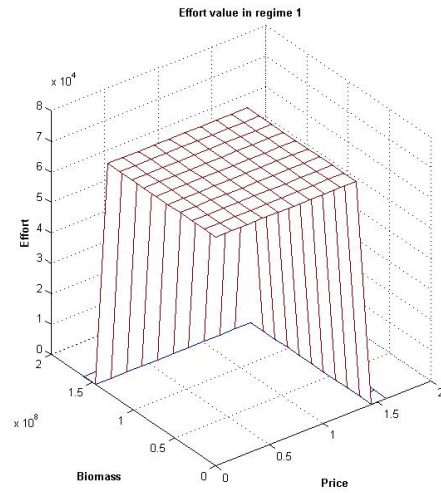
### 4.4 Optimal effort

220 We applied the Howard's algorithm: we compute the optimal effort in both regimes. As result:

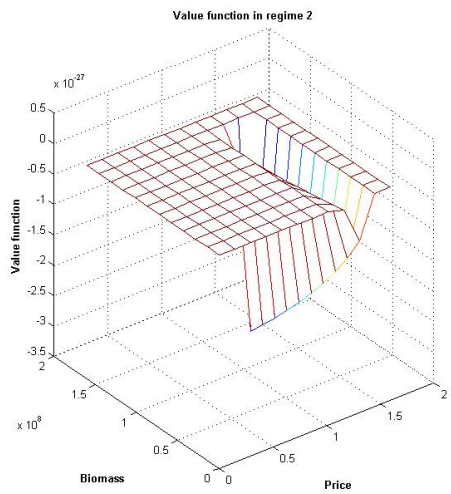
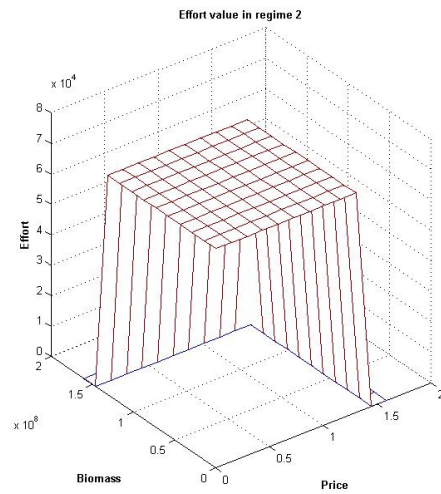
Regimes	Optimal effort
1	$7.5303 \times 1.0e + 04$
2	$7.2121 \times 1.0e + 04$

Table 2: Optimal Effort

222



224



226

## 5 Conclusions

228 We treat an finite-horizon optimal fishery problem in switching diffusion models. Using the viscosity solution approach, we prove that the value function

230 is the unique viscosity solution of the associated system of HJB equations. As  
an application, the optimal effort is deduced by using Howard's algorithm.

232 These methodologies can be applied to similar comparison studies and other  
fishery models. This will be the subject of a further paper.

234 **Appendix A Stochastic logistic growth with harvesting**

$$dB_t = rB_t \left(1 - \frac{B_t}{K}\right) dt - qEB_t dt + \sigma B_t dW(t) \quad (\text{A.1})$$

$$= rB_t \left(1 - \frac{qE}{r} - \frac{B_t}{K}\right) dt + \sigma B_t dW(t) \quad (\text{A.2})$$

Recalling the Itô's chain rules for solving the SDE  $dX_t = f(X, t)dt + g(X, t)dW_t$   
 $t \geq 0$  or

$$dV(X(t), t) = \left( V_t(X(t), t) + f(X(t), t)V_X + \frac{1}{2}g^2(X(t), t)V_{XX} \right) dt + g(X(t), t)V_X dW(t)$$

Let  $V(B_t, t) = B_t^{-1}$

$$\frac{\partial V}{\partial t} = 0 \quad \frac{\partial V}{\partial B_t} = -B_t^{-2} \quad \frac{\partial^2 V}{\partial B_t^2} = 2B_t^{-3}$$

$$f(B_t, t) = rB_t \left(1 - \frac{qE}{r} - \frac{B_t}{K}\right) \quad g(B_t, t) = \sigma B_t$$

$$dV_t = \left[ 0 + rB_t \left(1 - \frac{qE}{r} - \frac{B_t}{K}\right) (-B_t^{-2}) + \frac{1}{2}\sigma^2 B_t^2 (2B_t^{-3}) \right] dt + \sigma B_t (-B_t^{-2}) dW_t \quad (\text{A.3})$$

$$d(B_t^{-1}) = \left[ -r \left( B_t^{-1} \left(1 - \frac{qE}{r}\right) - \frac{1}{K} \right) + \sigma^2 B_t^{-1} \right] dt - \sigma B_t^{-1} dW_t \quad (\text{A.4})$$

To linearize (A.3), set  $B_t^{-1} = y_t$  so that

$$dy_t = \left[ -r \left( y_t \left(1 - \frac{qE}{r}\right) - \frac{1}{K} \right) + \sigma^2 y_t \right] dt - \sigma y_t dW_t \quad (\text{A.5})$$

$$= \left[ \frac{r}{K} + (-r + \sigma^2 + qE) y_t \right] dt - \sigma y_t dW_t \quad (\text{A.6})$$

We are looking for a solution to (A.5) of the form  $y(t) = y_1(t) \cdot y_2(t)$  where

$$dy_1(t) = (-r + \sigma^2 + qE)y_1 dt - \sigma y_1 dW_t, \quad y_1(0) = 1$$

$$dy_2(t) = a_t dt + b_t dW_t, \quad y_2(0) = y(0) = y_0$$

and the process coefficients  $a_t, b_t$  are, at this point, unknown.

$$\begin{aligned}
dy_t &= d(y_1 \cdot y_2) \\
&= y_1 dy_2 + y_2 dy_1 + dy_1 dy_2 \\
&= y_1 dy_2 + y_2 dy_1 + [(-r + \sigma^2 + qE)y_1 dt - \sigma y_1 dW_t][a_t dt + b_t dW_t] \\
&= y_1 dy_2 + y_2 dy_1 - \sigma b_t y_1 dt \\
&= y_1(a_t dt + b_t dW_t) + y_2[(-r + \sigma^2 + qE)y_1 dt - \sigma y_1 dW_t] - \sigma b_t y_1 dt \\
&= y_1(a_t dt + b_t dW_t) + (-r + \sigma^2 + qE)y_2 dt - \sigma y_2 dW_t - \sigma b_t y_1 dt
\end{aligned}$$

Now, select  $a_t, b_t$  so that

$$y_1(a_t dt + b_t dW_t) - \sigma b_t y_1 dt = \frac{r}{K} dt$$

Thus

$$b_t = 0 \quad a_t = \frac{r}{K} y_1^{-1} \quad (\text{A.7})$$

We integrate  $dy_1(t)$  and  $dy_2(t)$  to

$$\begin{aligned}
y_1(t) &= \exp \left[ \int_0^t -\sigma dW_s + \int_0^t \left( (-r + \sigma^2 + qE) - \frac{1}{2}\sigma^2 \right) ds \right] \\
&= \exp \left[ -\sigma W_t + \left( -r + qE + \frac{1}{2}\sigma^2 \right) t \right] \\
y_2(t) &= y_0 + \int_0^t \frac{r}{K} y_1^{-1}(s) ds
\end{aligned}$$

Thus

$$\begin{aligned}
y(t) &= \exp \left[ -\sigma W_t + \left( -r + qE + \frac{1}{2}\sigma^2 \right) t \right] \times \\
&\quad \left[ y_0 + \frac{r}{K} \int_0^t \exp \left[ \sigma W_s + \left( r - qE - \frac{1}{2}\sigma^2 \right) s \right] ds \right]
\end{aligned}$$

Hence

$$B_t = \frac{\exp \left[ (r - qE - (1/2)\sigma^2) t + \sigma W_t \right]}{\left[ B_0^{-1} + (r/K) \int_0^t \exp \left[ (r - qE - (1/2)\sigma^2) s + \sigma W_s \right] ds \right]} \quad (\text{A.8})$$

$$B_t = \frac{K \exp \left[ (r - qE - (1/2)\sigma^2) t + \sigma W_t \right]}{\left[ K/B_0 + r \int_0^t \exp \left[ (r - qE - (1/2)\sigma^2) s + \sigma W_s \right] ds \right]} \quad (\text{A.9})$$

## Appendix B Proof of lemma 3.1

1. Let  $k \in [0; 2]$  and  $h = s - t$ .

According to Hölder inequality  $E\eta^k \leq [E\eta^2]^{k/2}$  for  $\forall k \geq 0$ ,

$$E|P_h^{t,p_t}|^k \leq [E|P_h^{t,p_t}|^2]^{k/2} \quad (\text{B.1})$$

Given that

$$dP(t) = f'(t, P(t))dt + g'(t, P(t))dw(t) \quad (\text{B.2})$$

According to the elementary inequality  $|\sum_{i=1}^M a_i|^2 \leq M \sum_{i=1}^M |a_i|^2$ ,

$$|P_h^{t,p_t}|^2 \leq 3 \left\{ |p_t|^2 + \left| \int_0^h f'(u+t, P_u^{t,p_t}, i) du \right|^2 + \left| \int_0^h g'(u+t, P_u^{t,p_t}, i) dw_u \right|^2 \right\} \quad (\text{B.3})$$

Thus, using Ito-isometry and Fubini's theorem

$$E|P_h^{t,p_t}|^2 \leq 3 \left\{ |p_t|^2 + \int_0^h E \left| f'(u+t, P_u^{t,p_t}, i) \right|^2 du + \int_0^h E \left| g'(u+t, P_u^{t,p_t}, i) \right|^2 du \right\} \quad (\text{B.4})$$

$f'$  and  $g'$  satisfying the growth condition by definition, thus there exists

$C_1 \in \mathbb{R}$  such that

$$E|P_h^{t,p_t}|^2 \leq C_1 \left\{ 1 + |p_t|^2 + \int_0^h E |P_u^{t,p_t}|^2 du \right\} \quad (\text{B.5})$$

Applying Gronwall's inequality we obtain

$$E|P_h^{t,p_t}|^2 \leq C_1 e^{C_1 h} [1 + |p_t|^2] \quad (\text{B.6})$$

i.e

$$E|P_h^{t,p_t}|^2 \leq C [1 + |p_t|^2] \quad (\text{B.7})$$

Using (B.1) and elementary inequalities  $(a_1 + a_2)^k \leq 2^{k-1}(|a_1|^k + |a_2|^k)$

and  $(\sqrt{|a_1 + a_2|} \leq \sqrt{|a_1|} + \sqrt{|a_2|})$  we deduce

$$E|P_h^{t,p_t}|^k \leq C [1 + |p_t|^k] \quad (\text{B.8})$$

The same reasoning gives....

$$E|B_h^{t,b_t}|^k \leq C [1 + |b_t|^k] \quad (\text{B.9})$$



2. We have

$$|P_h^{t,p_t} - p_t|^2 \leq 2 \left\{ \left| \int_0^h f'(u+t, P_u^{t,p_t}, i) du \right|^2 + \left| \int_0^h g'(u+t, P_u^{t,p_t}, i) dw_u \right|^2 \right\} \quad (\text{B.10})$$

Similar arguments as above we deduce

$$\mathbb{E}|P_h^{t,p_t} - p_t|^2 \leq C_1 \int_0^h [1 + \mathbb{E}|P_u^{t,p_t}|^2] du \quad (\text{B.11})$$

using (B.7) we deduce

$$\mathbb{E}|P_h^{t,p_t} - p_t|^2 \leq C(1 + |p_t|^2)h \quad (\text{B.12})$$

Hence

$$\mathbb{E}|P_h^{t,p_t} - p_t|^k \leq C(1 + |p_t|^k)h^{k/2} \quad (\text{B.13})$$

3. Let us define the process  $P_s^{t,p_t} - P_s^{t,p'_t}$ . Put  $\bar{f}(u+t, P_u^{t,p_t}, P_u^{t,p'_t}, i) = f'(u+t, P_u^{t,p_t}, i) - f'(u+t, P_u^{t,p'_t}, i)$  and  $\bar{g}(u+t, P_u^{t,p_t}, P_u^{t,p'_t}, i) = g'(u+t, P_u^{t,p_t}, i) - g'(u+t, P_u^{t,p'_t}, i)$ . Then

$$\begin{aligned} \mathbb{E}|B_h^{t,p_t} - B_h^{t,p'_t}|^2 &\leq 3 \left( |p_t - p'_t|^2 + \mathbb{E} \left| \int_0^h \bar{f}(u+t, P_u^{t,p_t}, P_u^{t,p'_t}, i) du \right|^2 + \mathbb{E} \left| \int_0^h \bar{g}(u+t, P_u^{t,p_t}, P_u^{t,p'_t}, i) dw_u \right|^2 \right) \\ \mathbb{E}|P_h^{t,p_t} - P_h^{t,p'_t}|^2 &\leq 3 \left( |p_t - p'_t|^2 + \mathbb{E} \int_0^h |\bar{f}(u+t, P_u^{t,p_t}, P_u^{t,p'_t}, i)|^2 du + \mathbb{E} \int_0^h |\bar{g}(u+t, P_u^{t,p_t}, P_u^{t,p'_t}, i)|^2 du \right) \\ \mathbb{E}|P_h^{t,p_t} - P_h^{t,p'_t}|^2 &\leq C \left( |p_t - p'_t|^2 + \int_0^h \mathbb{E} |P_u^{t,p_t} - P_u^{t,p'_t}|^2 du \right) \end{aligned} \quad (\text{B.14})$$

Hence

$$\mathbb{E}|P_h^{t,p_t} - P_h^{t,p'_t}|^2 \leq C|p_t - p'_t|^2 \quad (\text{B.15})$$

Similar arguments as above we deduce

$$\mathbb{E}|B_h^{t,b_t} - B_h^{t,b'_t}|^k \leq C|b_t - b'_t|^2; \quad \mathbb{E}|P_h^{t,p_t} - P_h^{t,p'_t}|^k \leq C|p_t - p'_t|^2 \quad (\text{B.16})$$

4. Using Doob's inequality for submartingale. We get

$$\mathbb{E} \left[ \sup_{0 \leq s \leq h} |B_h^{t,b_t}|^k \right] \leq C(1 + |b_t|^k)h^{\frac{k}{2}}; \quad \mathbb{E} \left[ \sup_{0 \leq s \leq h} |P_h^{t,p_t}|^k \right] \leq C(1 + |p_t|^k)h^{\frac{k}{2}} \quad (\text{B.17})$$

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