



**HAL**  
open science

# Robustness of Student link function in multinomial choice models

Jean Peyhardi

► **To cite this version:**

Jean Peyhardi. Robustness of Student link function in multinomial choice models. 2019. hal-02385029

**HAL Id: hal-02385029**

**<https://hal.umontpellier.fr/hal-02385029>**

Preprint submitted on 28 Nov 2019

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Robustness of Student link function in multinomial choice models

Jean Peyhardi

*Institut Montpellierain Alexander Grothendieck, 34000 Montpellier*

---

## Abstract

The Student distribution has already been used to obtain robust maximum likelihood estimator (MLE) in the framework of binary choice models. But, until recently, only the logit and probit binary models were extended to the case of multinomial choices, resulting in the multinomial logit (MNL) and the multinomial probit (MNP). The recently introduced family of reference models, well defines a multivariate extension of any binary choice model, i.e. for any link function. This paper highlights the robustness of reference models with Student link function, by showing that the influence function is bounded. Inference of the MLE is detailed through the Fisher's scoring algorithm, which is appropriated since reference models belong to the family of generalized linear model (GLM). These models are compared to the MNL on the benchmark dataset of travel mode choice between Sydney and Melbourne. The results obtained on this dataset with reference models are completely different compared with those usually obtained with MNL, nested logit (NL) or MNP that failed to select relevant attributes. It will be showed that the travel mode choice is totally deterministic according to the terminal waiting time. In fact, the use of Student link function allow us to detect the total artificial aspect of this famous dataset.

## Keywords:

Discrete choice model, Generalized linear model, Link function, Influence function

---

## 1. Introduction

The use of Student distribution to obtain robust estimations has been introduced by Lange et al. (1989) in the framework of linear regression. For a binary response variable, the use of Student distribution was suggested by Albert and Chib (1993) as an alternative to logit or probit regression. Liu (2004) called this model the robit regression model and demonstrated the robustness of the MLE. Koenker and Yoon (2009) studied the importance of the link function in binary choice models with a focus on Student link function. But the variety of link functions defined in the literature, is considerably decreasing when the number of alternative choice  $J$  is more than two. Only the logit and probit binary models were extended to the case of multinomial choices, resulting in the MNL and the MNP. Contrarily to the MNL, the probabilities of alternative choices have no analytic forms with the MNP. Their computation has to be made through approximations and this complexity often lead the practitioner to neglect the MNP when  $J > 3$ . This explains why the extension of the MNP with multivariate Student error distribution in the context of random utility maximisation (RUM), has not been yet investigated. An alternative way to extend link functions when  $J > 2$ , has been recently introduced by Peyhardi et al. (2015) in the context of GLM. The family of reference models was thereby introduced, for which all alternatives are compared to a reference alternative. Since each of these  $J - 1$  comparisons is binary, the link function through the linear predictor can be made with a cumulative distribution function (cdf). All usual econometric outputs (willingness-to-pay, elasticities, ...) of reference models have been determined by Bouscasse et al. (2019). The goal of the present paper is to demonstrate the robustness of reference models defined with Student link function. An usual way to study the robustness is to study the influence function (Hampel, 1974). This approach is well established in the GLM framework for the class of  $M$  estimators (Künsch et al., 1989). We propose to study the influence function for a reference model based on the score  $\psi = \partial l / \partial \beta$ . We extend the results on robustness of Student link function, showed by Liu (2004), to the case of  $J > 2$  alternatives. More precisely the influence function for a reference model is showed to be unbounded for the logistic and normal link functions and bounded for the Student link function.

Student and logistic link function are then compared on the well known dataset of travel mode choice between Sydney and Melbourne. This is certainly the most used benchmark dataset to compare different families of discrete choice models (Greene, 2003; Hensher and Greene, 2002). The MNL, the NL and the MNP models have been extensively studied among this dataset, principally to highlight the limitation of the independence of irrelevant alternatives (IIA) property. This paper presents the IIA property for reference models and relies it to invariance property under permutations of alternatives. It will be shown that, even if the NL or the MNP model do not share this property, they fail, as the MNL, to select relevant attributes. The model selection is clearly in favour of the Student link function against the MNL, NL and MNP. It reveals that the travel mode choice is driven only by the terminal time. Plotting the observed choices according to the terminal time, revealed that all individuals made the same choice for each given terminal time value. Moreover, using a three dimensional plot, it is noticed that the design experiment takes a geometric form of a cross. In fact, the use of Student link function allow us to highlight the total artificial aspect of this famous dataset and thus reconsider classical results obtained in the usual literature (Greene, 2003; Hensher and Greene, 2002).

The present paper is organized as follows. The second section presents the family of reference models as an extension of the MNL. It describes the invariance property of such models under permutations of the alternatives and relates it to the IIA property. It is highlighted that a reference model is depending on the reference alternative. Otherwise, the reference model is also depending on the degree of freedom parameter of the Student distribution. The Fisher's scoring algorithm is therefore firstly detailed for a given reference alternative and a given degree of freedom. Then the inference procedure is described when these two parameters are unknown. The influence function of a reference model is described according to the chosen link function, i.e. the chosen cdf. It is thus easily seen that the influence function is bounded with Student cdf and unbounded with logistic or normal cdf. The third section briefly presents the dataset. The model selection is then detailed, leading to a simple model with only the terminal time as attribute. This section then backs to the dataset and highlights its artificial aspect, using a well chosen plot of the dataset. It makes this dataset the perfect candidate to study the sensitivity of a model to different kind of noise. Indeed the general cost, for instance, can be considered as a noise in attribute (i.e., the corresponding parameter is null) and some observations can be considered as outliers (i.e., prediction and observation are the opposite).

## 2. Reference models

We first recall the notations used all along the paper. The individual subscript will be omitted for convenience, without loss of generality. Let  $Y$  denote the response variable corresponding to the choice with  $J$  the number of alternatives. Let  $\mathbf{x} \in \mathbb{R}^p$  denote the vector of  $p$  individual attributes and  $\boldsymbol{\omega} = \{\omega_j\}_{j=1,\dots,J} \in \mathbb{R}^{qJ}$  the vectors of  $q$  alternative specific attributes. Let  $\pi_j = P(Y = j)$  denote the probability of choosing the alternative  $j$  given the attributes  $\mathbf{x}$  and  $\boldsymbol{\omega}$ .

### 2.1. Multinomial logit models

Reference models can be presented as an extension of the MNL. The MNL is classically presented by the  $J$  equations

$$P(Y = j) = \frac{\exp(\eta_j^*)}{\sum_{k=1}^J \exp(\eta_k^*)}, \quad j = 1, \dots, J, \quad (1)$$

where  $\eta_j^*$  is the predictor associated with alternative  $j$  generally assumed to be linear in attributes. Different logit models are obtained according to the form of the linear predictors. They can be defined using individual attributes  $\mathbf{x}$  and/or alternative specific attributes  $\boldsymbol{\omega}$ . In the following we will use the more general parametrization (containing both individual and alternative specific attributes), i.e.,

$$\eta_j^* = \alpha_j^* + \boldsymbol{\omega}_j^t \boldsymbol{\gamma}^* + \mathbf{x}^t \boldsymbol{\delta}_j^*$$

for  $j = 1, \dots, J$ . But these predictors  $\eta_j^*$  are not identifiable and thus the intercept and slope parameters no more. By convention, the last alternative  $J$  is considered as the reference alternative. The up and down parts of the fraction in

equations (1) are divided by  $\exp(\eta_j^*)$ . Thanks to the exponentiation identity  $\exp(a+b) = \exp(a)\exp(b)$ , the  $J$  equations (1) are equivalent to the  $J - 1$  equations

$$P(Y = j) = \frac{\exp(\eta_j)}{1 + \sum_{k=1}^{J-1} \exp(\eta_k)}, \quad j = 1, \dots, J - 1, \quad (2)$$

where  $\eta_j = \eta_j^* - \eta_J^*$  for all  $j = 1, \dots, J - 1$ , and consequently the probability of the reference alternative is given by

$$P(Y = J) = \frac{1}{1 + \sum_{k=1}^{J-1} \exp(\eta_k)}.$$

This entails a translation of with respect to the reference alternative:

- alternative constants:  $\alpha_j = \alpha_j^* - \alpha_J^*$ ,
- individual attributes parameter:  $\delta_j = \delta_j^* - \delta_J^*$ ,
- alternative specific attributes:  $\omega_j - \omega_J$ ,

Remark that the alternative specific parameter  $\gamma = \gamma^*$  stays unchanged. The translated predictors are given by

$$\eta_j = \alpha_j + (\omega_j - \omega_J)^t \gamma + \mathbf{x}^t \delta_j, \quad j = 1, \dots, J - 1,$$

where  $\alpha_1, \dots, \alpha_{J-1}$ ,  $\delta_1, \dots, \delta_{J-1}$  and  $\gamma$  are identifiable parameters.

## 2.2. Extension of logit models

The equations (2) can be rewritten as

$$\frac{\pi_j}{\pi_j + \pi_J} = \frac{\exp(\eta_j)}{1 + \exp(\eta_j)}, \quad j = 1, \dots, J - 1. \quad (3)$$

The right part of these equations corresponds to the logistic cdf  $F(x) = e^x/(1 + e^x)$ . It is proposed to use another cdf. In this paper, focus is made on the use of Student cdf denoted by  $F_\nu$ , where  $\nu$  is the degree of freedom. The model is described by the  $J - 1$  equations

$$\frac{\pi_j}{\pi_j + \pi_J} = F_\nu(\eta_j), \quad j = 1, \dots, J - 1, \quad (4)$$

with

$$F_\nu(x) = \frac{1}{2} + \frac{x \Gamma\left(\frac{\nu+1}{2}\right) {}_2F_1\left(\frac{1}{2}, \frac{\nu+1}{2}; \frac{3}{2}; -\frac{x^2}{\nu}\right)}{\sqrt{\nu\pi} \Gamma\left(\frac{\nu}{2}\right)},$$

where  ${}_2F_1$  denotes the hypergeometric function.

Otherwise, the vector of linear predictors  $\eta = (\eta_1, \dots, \eta_{J-1})$  can be written as the product of the design matrix  $Z$  and the vector of parameters  $\boldsymbol{\beta} = (\alpha_1, \dots, \alpha_{J-1}, \gamma^t, \delta_1^t, \dots, \delta_{J-1}^t)$  where

$$Z = \begin{pmatrix} 1 & (\omega_1 - \omega_J)^t & \mathbf{x}^t & \dots \\ \vdots & \vdots & \vdots & \ddots \\ 1 & (\omega_{J-1} - \omega_J)^t & \mathbf{x}^t & \dots \end{pmatrix}.$$

The parametrization of the linear predictors  $\eta_1, \dots, \eta_{J-1}$  is characterized by the design matrix  $Z$ . A reference model is therefore specified by the choice of the cdf  $F$  and the design matrix  $Z$ . More generally, Peyhardi et al. (2015) introduced an unifying specification of GLM for categorical responses, including reference models as a special case. This specification is based on three components  $(r, F, Z)$  that characterize the equations:

$$r_j(\boldsymbol{\pi}) = F_\nu(\eta_j), \quad j = 1, \dots, J - 1,$$

where  $\boldsymbol{\pi} = (\pi_1, \dots, \pi_{J-1})$ . Three other ratios (adjacent, cumulative and sequential) can also be used but only if a total ordering assumption is made among the  $J$  alternatives. In the following we will denote a model by its corresponding  $(r, F, Z)$  triplet. For instance the MNL is fully characterized by the triplet (reference, logistic,  $Z$ ) and can thus be viewed as a special case of reference models. It is considered as the canonical reference model since it corresponds to the canonical link function in the framework of GLM. All along this paper we will only use the reference ratio  $r_j(\boldsymbol{\pi}) = \pi_j/(\pi_j + \pi_{j_0})$  since it is appropriate for non-ordered alternatives. It can be ensured by study the invariance properties under permutations of the alternatives.

Otherwise, an interesting property of the family of Student distributions is that normal and logistic distributions can be viewed as special cases. It is known that  $F_\nu$  converges towards the normal cdf when  $\nu$  tends to infinity. In application we will see that the (reference,  $F_{20}$ ,  $Z$ ) model and the (reference, normal,  $Z$ ) model give similar results. Moreover, Albert and Chib (1993) note that "logistic quantiles are approximately a linear function of  $\mathcal{T}(8)$  quantiles." In application we will see that the (reference,  $F_8$ ,  $Z$ ) model and the MNL give similar results. It should be remarked, to avoid confusion, that the (reference, normal,  $Z$ ) model and the MNP are different models. They are equivalent only in the binary case  $J = 2$ .

### 3. Invariance property under permutations of alternatives

#### 3.1. Transposition of the reference alternative

It can be shown that the MNL, or equivalently the (reference, logistic,  $Z$ ) model is invariant under all permutations of alternatives. It means that a permutation of alternatives only implies a (linear) transformation of the parameter vector  $\boldsymbol{\beta}$  such that the fitted probabilities stay unchanged. In particular, the MNL is invariant under any transposition of the reference alternative. It means that the choice of the reference alternative has no impact on the fitted probabilities. This is quite different for (reference,  $F_\nu$ ,  $Z$ ) models. They are invariant under all permutation except under transposition of the reference alternative (Peyhardi et al., 2015). We will therefore denote by (reference,  $F_\nu$ ,  $Z$ ) $_{j_0}$  the model defined with  $j_0$  as reference alternative, i.e., such that

$$\frac{\pi_j}{\pi_j + \pi_{j_0}} = F_\nu(\eta_j), \quad \forall j \neq j_0.$$

Otherwise, the  $J - 1$  non-reference alternatives  $j \neq j_0$  can be permuted without modifying the model. We will see in the application that this invariance property of reference models will be crucial. The reference alternative has thus to be selected as a part of the link function.

#### 3.2. Independence of irrelevant alternatives

Let us remark that the total invariance property of the MNL is related to the IIA property. For non-canonical reference models the IIA property partially holds, i.e., we have

$$\frac{\pi_j}{\pi_{j_0}} = \frac{F_\nu}{1 - F_\nu} \left\{ \alpha_j + (\boldsymbol{\omega}_j - \boldsymbol{\omega}_{j_0})' \boldsymbol{\gamma} + \mathbf{x}' \boldsymbol{\delta}_j \right\}, \quad \forall j \neq j_0. \quad (5)$$

These  $J - 1$  ratio of probabilities are independent of other alternatives. On the contrary for two non-reference alternatives  $j \neq j_0$  and  $k \neq j_0$  we have

$$\begin{aligned} \frac{\pi_j}{\pi_k} &= \frac{\pi_j/\pi_{j_0}}{\pi_k/\pi_{j_0}}, \\ \frac{\pi_j}{\pi_k} &= \frac{\frac{F_\nu}{1-F_\nu} \left\{ \alpha_j + (\boldsymbol{\omega}_j - \boldsymbol{\omega}_{j_0})' \boldsymbol{\gamma} + \mathbf{x}' \boldsymbol{\delta}_j \right\}}{\frac{F_\nu}{1-F_\nu} \left\{ \alpha_k + (\boldsymbol{\omega}_k - \boldsymbol{\omega}_{j_0})' \boldsymbol{\gamma} + \mathbf{x}' \boldsymbol{\delta}_k \right\}}. \end{aligned}$$

Therefore these  $\binom{J-1}{2}$  ratios of probabilities are depending on the reference alternative specific attributes  $\boldsymbol{\omega}_{j_0}$ . In the case of canonical link (i.e., logistic cdf), the function  $F/(1 - F)$  turns out to be the exponential function and thus the

dependence disappears:

$$\frac{\pi_j}{\pi_k} = \frac{\exp\{\alpha_j + (\omega_j - \omega_{j_0})^t \boldsymbol{\gamma} + \mathbf{x}^t \boldsymbol{\delta}_j\}}{\exp\{\alpha_k + (\omega_k - \omega_{j_0})^t \boldsymbol{\gamma} + \mathbf{x}^t \boldsymbol{\delta}_k\}},$$

$$\frac{\pi_j}{\pi_k} = \exp\{\alpha_j + (\omega_j - \omega_k)^t \boldsymbol{\gamma} + \mathbf{x}^t (\boldsymbol{\delta}_j - \boldsymbol{\delta}_k)\}.$$

## 4. Inference procedure

### 4.1. Fixed alternative reference $j_0$ and degree of freedom $\nu$

#### 4.1.1. Fisher's scoring algorithm

Reference models belong to the set of GLMs and thus the Fisher's scoring algorithm is easily computed. For maximum likelihood estimation, the  $t + 1^{\text{th}}$  iteration of Fisher's scoring algorithm is given by

$$\boldsymbol{\beta}^{[t+1]} = \boldsymbol{\beta}^{[t]} - \left\{ \mathbb{E} \left( \frac{\partial^2 l}{\partial \boldsymbol{\beta}^t \partial \boldsymbol{\beta}} \right)_{\boldsymbol{\beta}=\boldsymbol{\beta}^{[t]}} \right\}^{-1} \left( \frac{\partial l}{\partial \boldsymbol{\beta}} \right)_{\boldsymbol{\beta}=\boldsymbol{\beta}^{[t]}} ,$$

where  $l$  denotes the log-likelihood among the dataset  $(\mathbf{y}_i, \boldsymbol{\omega}_i, \mathbf{x}_i)_{i=1, \dots, n}$ . Using the chain rule and properties of the exponential family of distributions, the score becomes the following product of matrices

$$\begin{aligned} \frac{\partial l}{\partial \boldsymbol{\beta}} &= \sum_{i=1}^n \frac{\partial l_i}{\partial \boldsymbol{\beta}} \\ &= \sum_{i=1}^n \frac{\partial \boldsymbol{\eta}_i}{\partial \boldsymbol{\beta}} \frac{\partial \boldsymbol{\theta}_i}{\partial \boldsymbol{\eta}_i} \frac{\partial l_i}{\partial \boldsymbol{\theta}_i} \\ \frac{\partial l}{\partial \boldsymbol{\beta}} &= \sum_{i=1}^n \mathbf{Z}_i^t \frac{\partial \boldsymbol{\theta}_i}{\partial \boldsymbol{\eta}_i} (\mathbf{y}_i - \boldsymbol{\pi}_i) \end{aligned}$$

where  $\boldsymbol{\theta}_i := (\theta_{i,j})_{j \neq j_0}$ ,  $\boldsymbol{\eta}_i := (\eta_{i,j})_{j \neq j_0}$ ,  $\boldsymbol{\pi}_i := (\pi_{i,j})_{j \neq j_0}$  and  $\mathbf{y}_i := (y_{i,j})_{j \neq j_0}$  such that  $y_{i,j} = 1$  if the alternative  $j$  is chosen by the individual  $i$  and  $y_{i,j} = 0$  otherwise. The parameter  $\boldsymbol{\theta}_i$  is the natural parameter of the multinomial distribution, viewed as member of the exponential family. In this framework, it can be shown that  $\theta_{i,j} = \ln(\pi_{i,j}/\pi_{i,j_0})$ . Now, using the equation (5) we obtain  $\theta_{i,j} = \ln F_\nu(\eta_{i,j}) - \ln\{1 - F_\nu(\eta_{i,j})\}$  and thus

$$\frac{\partial \boldsymbol{\theta}_i}{\partial \boldsymbol{\eta}_i} = \mathbf{D}_i = \text{diag}_{j \neq j_0} \left[ \frac{f_\nu(\eta_{i,j})}{F_\nu(\eta_{i,j})\{1 - F_\nu(\eta_{i,j})\}} \right],$$

where  $f_\nu$  denotes the probability density function (pdf) of the Student distribution with  $\nu$  degree of freedom, i.e.,

$$f_\nu(x) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu\pi} \Gamma\left(\frac{\nu}{2}\right) \left(1 + \frac{x^2}{\nu}\right)^{(\nu+1)/2}}.$$

Following the same way, the Fisher's information matrix is given by

$$\mathbb{E} \left( \frac{\partial^2 l}{\partial \boldsymbol{\beta}^t \partial \boldsymbol{\beta}} \right) = - \sum_{i=1}^n \mathbf{Z}_i^t \mathbf{D}_i \text{Cov}(Y_i) \mathbf{D}_i \mathbf{Z}_i,$$

with  $\text{Cov}(Y_i) = \text{diag}(\boldsymbol{\pi}_i) - \boldsymbol{\pi}_i \boldsymbol{\pi}_i^t$ . It could be remarked that this algorithm is simplified if the logistic cdf  $F$  is used (instead of the Student cdf  $F_\nu$ ) since  $f = F(1 - F)$  and therefore  $\mathbf{D}_i$  becomes the identity matrix of size  $J - 1$ .

#### 4.1.2. Robustness of estimator

An usual way to study the robustness of the estimator is to study the influence function (Hampel, 1974). It measures the influence of a new observation on a  $M$  estimator. According to Künsch et al. (1989), the influence function of a new observation  $(\mathbf{y}^*, \boldsymbol{\omega}^*, \mathbf{x}^*)$  on the MLE  $\hat{\boldsymbol{\beta}}$  of a GLM, is given by

$$\text{IF}\{(\mathbf{y}^*, \mathbf{x}^*), \hat{\boldsymbol{\beta}}\} = \left\{ \text{E} \left( \frac{\partial^2 l}{\partial \boldsymbol{\beta}' \partial \boldsymbol{\beta}} \right)_{\boldsymbol{\beta}=\hat{\boldsymbol{\beta}}} \right\}^{-1} \left( \frac{\partial l^*}{\partial \boldsymbol{\beta}} \right)_{\boldsymbol{\beta}=\hat{\boldsymbol{\beta}}},$$

where  $l$  (respectively  $l^*$ ) denotes the log-likelihood computed on the full dataset  $\{(\mathbf{y}_i, \boldsymbol{\omega}_i, \mathbf{x}_i)\}_{i=1, \dots, n}$  (respectively on the new observation  $(\mathbf{y}^*, \boldsymbol{\omega}^*, \mathbf{x}^*)$ ). Since the left part is not depending on the new observation, it is sufficient to show that the score vector  $\partial l / \partial \boldsymbol{\beta}$  is bounded according to  $(\mathbf{y}, \boldsymbol{\omega}, \mathbf{x}) \in \{0, 1\}^{J-1} \times \mathbb{R}^{qJ+p}$ . Without loss of generality, focus is made on one coordinate of this vector, i.e., for  $k \in \{1, \dots, p\}$  focus is made on quantity

$$\sup_{y_j \in \{0,1\}, \boldsymbol{\omega} \in \mathbb{R}^{qJ}, \mathbf{x} \in \mathbb{R}^p} x_k \frac{f_v(\eta_j)}{F_v(\eta_j)\{1 - F_v(\eta_j)\}} (y_j - \pi_j).$$

The right part  $(y_j - \pi_j)$  lies in  $(-1, 1)$ . The parameter  $\boldsymbol{\beta}$  being constant, we can consider that  $x_k$  and  $\eta_j$  are equivalent. Focus is thus made on the superior bound on  $\mathbb{R}$  of the function

$$\frac{\eta f_v(\eta)}{F_v(\eta)\{1 - F_v(\eta)\}}. \quad (6)$$

This function being continue on  $\mathbb{R}$ , we only study limit when  $\eta \rightarrow -\infty$  or  $\eta \rightarrow +\infty$ .

$$\lim_{\eta \rightarrow -\infty} \frac{\eta f_v(\eta)}{F_v(\eta)\{1 - F_v(\eta)\}} = \lim_{\eta \rightarrow -\infty} \frac{\eta f_v(\eta)}{F_v(\eta)}.$$

As noted by Liu (2004), we have

$$\lim_{\eta \rightarrow -\infty} \frac{\eta f_v(\eta)}{F_v(\eta)} = 1 - \lim_{\eta \rightarrow -\infty} \frac{(1 + \nu)\eta^2}{\nu + \eta^2} = -\nu.$$

The other limit is given by

$$\lim_{\eta \rightarrow +\infty} \frac{\eta f_v(\eta)}{F_v(\eta)\{1 - F_v(\eta)\}} = \lim_{\eta \rightarrow +\infty} \frac{\eta f_v(\eta)}{1 - F_v(\eta)}.$$

By symmetry of the Student distribution we have

$$\lim_{\eta \rightarrow +\infty} \frac{\eta f_v(\eta)}{1 - F_v(\eta)} = \lim_{\eta \rightarrow +\infty} \frac{\eta f_v(-\eta)}{F_v(-\eta)} = \nu.$$

The influence function of a (reference,  $F$ ,  $Z$ ) model is therefore bounded if  $F$  is the Student cdf. On the contrary it is unbounded when  $F$  is the logistic cdf since

$$\frac{\eta f(\eta)}{F(\eta)\{1 - F(\eta)\}} = \eta,$$

or if  $F$  is the normal cdf since

$$\lim_{\eta \rightarrow -\infty} \frac{\eta f(\eta)}{F(\eta)\{1 - F(\eta)\}} = \lim_{\eta \rightarrow -\infty} \eta^2.$$

These three different behaviours of function 6 according to the three cdfs (Student, logistic and normal) are well represented in Fig.1.

#### 4.2. Unknown alternative reference $j_0$ and degree of freedom $\nu$

The Fisher's scoring algorithm for a (reference,  $F_\nu$ ,  $Z$ ) $_{j_0}$  model has been detailed for a given link function, i.e., for a given reference alternative  $j_0$  and degree of freedom  $\nu$ . The link function is here selected, by simply comparing the log-likelihood of models for all values of  $j_0$  and  $\nu$ . First assume that the reference alternative is fixed. The aim is to select  $\nu$  among  $(0, +\infty)$ . Let us note that Student distributions with different degrees of freedom  $F_{\nu_1}$  and  $F_{\nu_2}$  are not

connected by a linear transformation and thus lead to different models (reference,  $F_{\nu_1}, Z$ ) and (reference,  $F_{\nu_2}, Z$ ), i.e., different likelihood maxima. That is why we have to estimate  $\nu$  contrarily to the location or scale parameter of the cdf. In practice a finite discretization of  $\nu$  values is proposed and all corresponding models are compared using the log-likelihood since the same parametrization is used (i.e., the same design matrix  $Z$ ). An interesting property, that ensure to find the MLE, is the concavity of the log-likelihood according to the degree of freedom. This concavity is empirically verified; see the Fig.2 representing the log-likelihood according to  $\nu$  in four situations corresponding to the four reference alternatives. You can also see the Appendix B of (Boucasse et al., 2019) for another example.

## 5. Application

### 5.1. The dataset

We use the benchmark dataset of travel modes between Sydney and Melbourn used by Louviere et al. (2000); Greene (2003). The dataset contains 210 observations of choice among  $J = 4$  travel modes: *air*, *bus*, *car* and *train*. The two considered alternative specific attributes were the general cost (GC) denoted by  $c = (c_{\text{air}}, c_{\text{bus}}, c_{\text{car}}, c_{\text{train}})$  and the terminal waiting time (TTime) denoted by  $t = (t_{\text{air}}, t_{\text{bus}}, t_{\text{car}}, t_{\text{train}})$ . The two considered individual attributes were the household income (Hinc) denoted by  $h$  and the number of people travelling (PSize) denoted by  $p$ .

### 5.2. Model selection

Let us first remark that the design matrix  $Z$  is depending on the reference alternative  $Z = Z_{j_0}$ . The linear predictors are given by

$$\eta_j = \alpha_j + (t_j - t_{j_0})\gamma^1 + (c_j - c_{j_0})\gamma^2 + (h\delta_{\text{air}}^1 + p\delta_{\text{air}}^2)\mathbf{1}_{(j=\text{air})} \quad (7)$$

for all  $j \neq j_0$ . This is the design proposed by Louviere et al. (2000) page 157. Using the bus as the reference alternative, for instance, the design matrix is given by

$$Z_{\text{bus}} = \begin{pmatrix} 1 & 0 & 0 & t_{\text{air}} - t_{\text{bus}} & c_{\text{air}} - c_{\text{bus}} & h & p \\ 0 & 1 & 0 & t_{\text{car}} - t_{\text{bus}} & c_{\text{car}} - c_{\text{bus}} & 0 & 0 \\ 0 & 0 & 1 & t_{\text{train}} - t_{\text{bus}} & c_{\text{train}} - c_{\text{bus}} & 0 & 0 \end{pmatrix}.$$

Note that, using the car as reference alternative, gives the design matrix

$$Z_{\text{car}} = \begin{pmatrix} 1 & 0 & 0 & t_{\text{air}} & c_{\text{air}} - c_{\text{car}} & h & p \\ 0 & 1 & 0 & t_{\text{bus}} & c_{\text{bus}} - c_{\text{car}} & 0 & 0 \\ 0 & 0 & 1 & t_{\text{train}} & c_{\text{train}} - c_{\text{car}} & 0 & 0 \end{pmatrix},$$

since the terminal time of the car is null  $t_{\text{car}} = 0$ . In the following we will denote (reference,  $F, Z$ ) $_{j_0}$  instead of (reference,  $F, Z_{j_0}$ ) $_{j_0}$  to simplify the notations.

*Selection of the link function.* For each reference alternative  $j_0 \in \{\text{air}, \text{bus}, \text{car}, \text{train}\}$ , the cdf  $F$  was selected among the family of Student cdfs  $(F_{\nu})_{\nu>0}$ . In practice, the values of  $\nu$  was discretized into a fine grid between 0 and 2 (scale of 0.05) and coarse grid between 2 and 20 (unit scale). The result was compared with the (reference, logistic,  $Z$ ) $_{j_0}$  model, i.e., the MNL, and the (reference, normal,  $Z$ ) $_{j_0}$  model. The results are represented in Fig.2, corresponding to the four reference alternatives. As expected we can check that:

- the log-likelihood of the (reference,  $F_{\nu}, Z$ ) $_{j_0}$  model converges towards those of the (reference, normal,  $Z$ ) $_{j_0}$  model when  $\nu \rightarrow +\infty$ , in the four situations  $j_0 \in \{\text{air}, \text{bus}, \text{car}, \text{train}\}$ ,
- the log-likelihood of the (reference,  $F_{\nu}, Z$ ) $_{j_0}$  model equals those of the MNL for values of  $\nu$  around 8, in the four situations  $j_0 \in \{\text{air}, \text{bus}, \text{car}, \text{train}\}$ ,
- the MNL gives the same result, in the four situations  $j_0 \in \{\text{air}, \text{bus}, \text{car}, \text{train}\}$  (invariance under permutation),
- the best (reference,  $F_{\nu^*}, Z$ ) $_{j_0}$  models outperforms the MNL in the four situations  $j_0 \in \{\text{air}, \text{bus}, \text{car}, \text{train}\}$ .



The log-likelihood obtained with the MNL is  $-185.91$  as obtained by Louviere et al. (2000) page 157. The log-likelihoods obtained with the (reference,  $F_{\nu^*}, Z$ ) $_{j_0}$  models were  $-185.65$ ,  $-183.79$ ,  $-142$ ,  $-183.49$  respectively with the four reference alternatives  $j_0 = \text{air}$ ,  $j_0 = \text{bus}$ ,  $j_0 = \text{car}$ ,  $j_0 = \text{train}$  and corresponding degree of freedom  $\nu^* = 3$ ,  $\nu^* = 30$ ,  $\nu^* = 0.2$ ,  $\nu^* = 1.35$ . The results are clearly in favour of the reference alternative  $j_0 = \text{car}$  since the gain in log-likelihood is  $43.92$  compared to MNL results, i.e.,  $24\%$  of the log-likelihood. It is a very huge difference compared to the results given in the literature (Louviere et al., 2000; Greene, 2003), obtained with MNL and also with NL.

*Selection of relevant attributes.* After the selection of the link function (reference alternative and degree of freedom), focus is made on parameter estimations and variables selection. Before to compare parameters of different models, a first step is to choose a normalized space  $\mathfrak{F}$  of cdfs, i.e., with fixed location and scale values. The normalized space  $\mathfrak{F}_{g_{0.95}}$  is proposed as the standard case; see (Bouscasse et al., 2019). The estimates obtained for the (reference, normal,  $Z$ ) $_{\text{car}}$  and (reference, logistic,  $Z$ ) $_{\text{car}}$  models have comparable scales compared to results obtained with the normalized space  $\mathfrak{F}_1$ ; see the Supplementary Material. This step is useful to obtain comparable scale values but not necessary since ratios of parameters stay constant for any normalized space.

Let us now focus on parameter estimations obtained with (reference,  $F_{\nu}, Z$ ) $_{\text{car}}$  models. The behaviour of the log-likelihood suggests that the MLE for  $\nu$  lies in  $(0, 1)$ ; see Fig.2.(C). Estimates of  $\nu$  are more chaotic near to 0 because of computation instabilities; see Fig.3. The log-likelihood is maximal for  $\nu = 0.2$  but parameter estimates are very high, especially for alternative constants; see Table .1. We thus chose the smallest  $\nu > 0.2$  such that at least the alternative constant estimates were significant (p-value less than 0.01). Results obtained for the (reference,  $F_{0.45}, Z$ ) $_{\text{car}}$  model are summarized in Table .2. Even if the parameter estimates obtained with  $\nu = 0.2$  have a different scale compared to those obtained with  $\nu = 0.45$  or the MNL, the ratio of parameters are comparable. For instance, the ratios of alternative constant are nearly identical between the two models:  $\alpha_{\text{air}}/\alpha_{\text{bus}}$  and  $\alpha_{\text{bus}}/\alpha_{\text{train}}$  are equal to  $1.9357$  and  $0.91843$  with  $\nu = 0.2$  compared to  $2.042$  and  $0.822$  with the MNL; see Table .3 for the MNL results. The two models specially differ by their slope parameters. The terminal time effect is favoured by the Student distribution. Recall that the willingness to pay for the terminal time is given by  $\text{WTP} = -\gamma'/\gamma^c$  (Bouscasse et al., 2019). The willingness to pay for the terminal time is  $-146.142$  for the (reference,  $F_{0.45}, Z$ ) $_{\text{car}}$  versus  $-4.263$  for the MNL. Moreover, regarding the p-values and their associated test, the three other variables have no significant effect (p-values more than 0.1).

We thus propose to use a more simple design with only the terminal time variable, i.e., we have  $\eta_j = \alpha_j + t_j\gamma$  for  $j \in \{\text{air}, \text{bus}, \text{train}\}$ . Equivalently the design matrix is given by

$$Z' = \begin{pmatrix} 1 & 0 & 0 & t_{\text{air}} \\ 0 & 1 & 0 & t_{\text{bus}} \\ 0 & 0 & 1 & t_{\text{train}} \end{pmatrix}.$$

Estimates of the (reference, logistic,  $Z'$ ) $_{\text{car}}$  and (reference,  $F_{0.45}, Z'$ ) $_{\text{car}}$  models are summarized in Tables .4 and .5. The (reference, logistic,  $Z'$ ) $_{\text{car}}$  model poorly fits the data, obtaining a log-likelihood of  $-206.82$  using only the TTime versus  $-185.91$  using all the variables. The (reference,  $F_{0.45}, Z'$ ) $_{\text{car}}$  model obtains log-likelihood of  $-146.68$  versus  $-145.89$  using all variables, confirming the essential role of the terminal time in travel mode choices for this dataset. To confirm these result, several (reference,  $F_{\nu}, Z'$ ) $_{\text{car}}$  models were estimated for  $\nu \in (0, 20]$ . The corresponding log-likelihoods are represented in Fig.4.(A). Looking at the result obtained near to 0 in Fig.4.(B), we see that more the degree of freedom is small better is the fit. But some problems occur in the standard error estimation; see the estimates of the (reference,  $F_{0.05}, Z'$ ) $_{\text{car}}$  model in Table .6. As noticed by Koenker and Yoon (2009), it may be due to poor evaluations of the pdf  $f_{\nu}$  and the cdf  $F_{\nu}$  when the degree of freedom  $\nu$  is small, impacting the Fisher's scoring algorithm computation. Nevertheless, the ratios of parameters stay consistent even if estimates are infinitesimal; see for instance  $\alpha_{\text{air}}/\alpha_{\text{bus}} = 1.943$  and  $\alpha_{\text{bus}}/\alpha_{\text{train}} = 0.979$  whereas estimates are less than  $10^{-4}$ .

Finally, using the Student link function, it is concluded that only the terminal time is discriminant in travel mode choice. This conclusion is totally different of those obtained with the MNL, which states that the general cost, the household income and the number of people travelling are discriminant attributes. It should be noted that the NL (see (Hensher and Greene, 2002) pages 169-170 or (Greene, 2003) pages 732-733) and the MNP (see (Greene, 2003) page 735) also selects these attributes as relevant. To go a step further, we will see that these attributes can be considered as noise regarding the travel mode choice.

### 5.3. Back on the dataset

The variables selection is totally different using the Student distribution instead of the logistic distribution. Using only the terminal time attribute, the model fitting is represented in Fig.5. Indeed, the observed and predicted probability  $P(Y = j|Y \in \{j, j_0\})$  are represented according to the terminal time of alternative  $j$ , where  $j_0$  is the reference alternative car and  $j$  is the air alternative in Fig.5.(A), the bus alternative in Fig.5.(B) and the train alternative in Fig.5.(C). Looking at these graphs, the dataset is very surprising since all observed proportions are either 0 or 1. It means that all individuals, sharing the same terminal time value, take the same decision. In other words, knowing the terminal time value, the travel mode choice is totally deterministic.

To check this surprising fact, the dataset has been reduced by conserving only the columns corresponding to the terminal time of air, bus and train. Then, all individuals being in the same situation, i.e., with the same terminal time values (for air, bus and train), have been aggregated and the weight (number of observations) has been added as a new column. Looking at the three dimensional representation of terminal time values  $(t_{\text{air}}^i, t_{\text{bus}}^i, t_{\text{train}}^i)_{i=1, \dots, n=210}$ , in Fig.6 (left), we clearly see two groups with the same geometric form of a cross. For a better view, these two groups have been represented separately. Let us focus on the first cross (similar comments can be made on the second cross). This cross has a middle point with coordinates  $t_0 = (69, 35, 34)$ . All other points are obtained by fixing two coordinates and letting free the last one. It corresponds to a theoretical design that aims to study the effect of terminal time modification for a given alternative when the two other are fixed. Now let see in Fig.6 (right) the travel mode chosen by individuals in each terminal time situation of the first cross. Recall that there is potentially several observations for the same situation (the size of points are represented proportionally to the number of individuals). For each terminal time situation, remark that all individual made the same travel mode choice. For instance, for the middle point  $t_0 = (69, 35, 34)$ , all the 22 individuals have chosen the car alternative.

Now imagine that this point corresponds to the starting point of stated preference study. Since all the 22 individual chose the car alternative in this situation, the terminal time values  $t_0 = (69, 35, 34)$  can be considered as a threshold value. Now let increase the terminal time value only for air alternative. The individual behaviour should take the car again. But the data show that all individuals in situation  $t = (t_{\text{air}}, 35, 34)$  with  $t_{\text{air}} > 69$  have chosen the air alternative. More generally we observe that all individuals in situation  $t = (t_{\text{air}}, 35, 34)$  have chosen the air alternative for any value of  $t_{\text{air}} \neq 69$ . The same observation is made when modifying the terminal time of bus or train and fixing the two others. This cross design is totally deterministic, each of the three axes corresponding to exactly one travel mode choice (air, bus or train) except the intersection that corresponds to the car choice. The same observation is made for the other cross; see Fig.6 (left) that represents all the dataset. We can conclude that neither the design experiment, or the travel mode choice corresponds to real observations in this dataset. In other words, this is totally improbable to observe these data.

### 5.4. Sensitivity of the link function to the noise

One can distinguish two kinds of noise: a noise concerning the attributes and another concerning the choice of alternative. The first kind corresponds to the addition of columns (variables) in the dataset and the second to the addition of rows (individuals). The influence function measure the sensitivity of the model to the addition of an outlier, i.e., an observation  $(\mathbf{y}, \mathbf{w}, \mathbf{x})$  for which the distance between  $\mathbf{y}$  and the prediction  $\boldsymbol{\pi}$  is high. Such an observation is also called bad leverage point or contamination. It can be considered as a noise in the choice of alternative (response variable in regression). But it is also interesting to study the impact on the model, of a noise in attributes (explanatory variables in regression). It corresponds to the addition of attributes that have no influence on alternative choice, i.e., whose the true parameter is null. One talk about over-fitting when the model is too sensitive to this kind of noise. As previously seen, the benchmark dataset of travel mode choice between Sydney and Melbourne is totally artificial. It is perfect to highlight the sensitivity of the model to these two kind of noise. We propose to compare the sensitivity of the logistic and Student link functions.

#### 5.4.1. Noise in attributes

As previously seen, the alternative choice is totally deterministic knowing the terminal time. It means that the general cost, the household income and the number of people traveling can be considered as noise in attributes. Fig.7 represents the probability of choosing alternative  $j$  versus  $j_0$  before and after the addition of these three non-informative attributes. The results are here presented for alternative  $j = \text{air}$  and  $j_0 = \text{car}$ , but similar results are holding

with alternatives bus and train. The first model is estimated, using only the terminal time; see Tables .4 and .5. It corresponds to the equations

$$\frac{\pi_j}{\pi_j + \pi_{j_0}} = F(\hat{\alpha}_j + \hat{\gamma}t_j). \quad (8)$$

This probability is represented for all values of terminal time  $t$  between 0 and 100. The second model is estimated, using the terminal time, the general cost, the household income and the number of people traveling; see Tables .2 and .3. It corresponds to the equations

$$\frac{\pi_j}{\pi_j + \pi_{j_0}} = F\left\{\hat{\alpha}_j + \hat{\gamma}^1 t_j + \hat{\gamma}^2 (c_j - c_{j_0}) + \hat{\delta}^1 h + \hat{\delta}^2 p\right\}.$$

This probability is represented for all values of terminal time  $t$  between 0 and 100 and for median (and also 25% and 75% percentiles) values of  $c$ ,  $h$  and  $p$ . It can be seen that amplitude in prediction error is small for Student link function and high for logistic link function. The Student link function is clearly less sensitive to noise in attributes.

#### 5.4.2. Noise in alternative choice

Now assume that only the terminal time is used as an attribute, i.e., the model is described by equation (8). Here we focus on the conditional choice between bus and car alternatives since it is more representative, but similar results are holding for the two other conditional probabilities. As previously seen, based on the Fig.5 (B), it can be seen that the individual behaviour is to choose the bus alternative while  $t < 35$  and the car alternative when  $t \geq 35$ . Note that the car alternative is chosen for two terminal time values:  $t = 35$  and  $t = 53$ .

The perfect situation is therefore to observe only bus travel choice when  $t < 35$  and only car travel choice when  $t \geq 35$ . These observations are represented by crosses in Fig.5 and are corresponding to a complete separation. According to Albert and Anderson (1984), for logistic link function, there is not a finite MLE but the maximum of likelihood is attained at infinity on the boundary of the parameter space. Therefore, if we stop the Fisher's scoring algorithm when the likelihood is closed to 1, we obtain a perfect classification of these completely separated data. The model estimated on this dataset (complete separation) is represented with a dotted line in Fig.5. The second situation corresponds to the addition of bus observations (resp. air and train) for terminal time values  $35 \leq t \leq 53$  (resp. for  $64 \leq t \leq 69$  and  $34 \leq t \leq 44$ ). These additional observations are represented by stars in Fig.5 and are corresponding to an overlap situation. According to Albert and Anderson (1984), for logistic link function, the MLE is finite and unique. The model estimated on this completed dataset (overlap) is represented with a dashed line in Fig.5. The third situation corresponds to the addition of bus observations (resp. air and train) for terminal time higher than  $t = 53$  (resp. 69 and 44). These additional observations are represented by points in Fig.5 and are corresponding to outliers. This is an overlap situation with outliers. The model estimated on the full dataset (overlap with outliers) is represented with a line in Fig.5. It can be seen that amplitude in prediction error is high for logistic link function (Fig.5 left) and small for Student link function (Fig.5 right). The Student link function is clearly less sensitive to outliers (noise in alternative choice).

## 6. Conclusion

After the introduction of reference models by Peyhardi et al. (2015) and the description of their economic outputs by Bouscasse et al. (2019), this paper has demonstrated the robustness of reference model defined with Student link function. More precisely, its influence function has shown to be bounded contrarily to the MNL, and it is thus less sensitive to outliers. It was also empirically showed that Student link function is less sensitive to over-fitting since non-informative attributes are not selected. The empirical comparison with classical models has been made on the benchmark dataset of travel mode choice between Sydney and Melbourne. It is thus easy to compare the present results with several other models, well studied in the literature. To our knowledge, the reference model with Student link function, is the first model that detects the total artificial aspect of this famous dataset, thanks to its robustness.

More generally, reference model takes several advantages of the GLM framework, such as the simple probability formulation, fast Fisher's scoring algorithm and the natural addition of random effects. Moreover, extension of reference models to the case of nested alternative has already been introduced by Peyhardi et al. (2016). All these properties make the reference model with Student link function, a good candidate among the huge family of discrete choice models.

## References

- Adelin Albert and John A Anderson. On the existence of maximum likelihood estimates in logistic regression models. *Biometrika*, 71(1):1–10, 1984.
- James H Albert and Siddhartha Chib. Bayesian analysis of binary and polychotomous response data. *Journal of the American statistical Association*, 88(422):669–679, 1993.
- Hélène Bouscasse, Iragaël Joly, and Jean Peyhardi. A new family of qualitative choice models: An application of reference models to travel mode choice. *Transportation Research Part B: Methodological*, 121:74–91, 2019.
- William H. Greene. *Econometric Analysis*. Pearson Education, New York, fifth edition, 2003.
- Frank R Hampel. The influence curve and its role in robust estimation. *Journal of the american statistical association*, 69(346):383–393, 1974.
- David A Hensher and William H Greene. Specification and estimation of the nested logit model: alternative normalisations. *Transportation Research Part B: Methodological*, 36(1):1–17, 2002.
- Roger Koenker and Jungmo Yoon. Parametric links for binary choice models: A fisherian–bayesian colloquy. *Journal of Econometrics*, 152(2): 120–130, 2009.
- Hans R Künsch, Leonard A Stefanski, and Raymond J Carroll. Conditionally unbiased bounded-influence estimation in general regression models, with applications to generalized linear models. *Journal of the American Statistical Association*, 84(406):460–466, 1989.
- Kenneth L Lange, Roderick JA Little, and Jeremy MG Taylor. Robust statistical modeling using the t distribution. *Journal of the American Statistical Association*, 84(408):881–896, 1989.
- Chuanhai Liu. Robit regression: a simple robust alternative to logistic and probit regression. *Applied Bayesian Modeling and Casual Inference from Incomplete-Data Perspectives*, pages 227–238, 2004.
- Jordan J Louviere, David A Hensher, and Joffre D Swait. *Stated choice methods: analysis and applications*. Cambridge University Press, 2000.
- Jean Peyhardi, Catherine Trottier, and Yann Guédon. A new specification of generalized linear models for categorical responses. *Biometrika*, 102(4):889–906, 2015.
- Jean Peyhardi, Catherine Trottier, and Yann Guédon. Partitioned conditional generalized linear models for categorical responses. *Statistical Modelling*, 2016.

Table .1: Summary results for the (reference,  $F_{0.2}, Z$ )<sub>car</sub> model using the normalized space  $\tilde{\mathcal{F}}_{q_{0.95}}$ .

Variables	Estimate	Std. Error	$p$ -value	Test
Air	338.29	233.69	0.015492	*
Bus	174.78	124.53	0.025326	*
Train	190.29	129.36	0.021186	*
GC	-0.41235	0.29453	0.031887	*
TTime	-5.524	3.7516	0.020684	*
Hinc(air)	1.3504	1.2499	0.26482	
PSize(air)	-38.208	27.798	0.055836	.
Loglikelihood	-141.998			
AIC	297.99			
BIC	321.43			
Pseudo R <sup>2</sup>	0.49958			

Table .2: Summary results for the (reference,  $F_{0.45}, Z$ )<sub>car</sub> model using the normalized space  $\tilde{\mathcal{F}}_{q_{0.95}}$ .

Variables	Estimate	Std. Error	$p$ -value	Test
Air	6.27604	2.21017	0.00384	**
Bus	3.00083	1.00849	0.00134	**
Train	3.04039	1.03155	0.00139	**
GC	-0.00066	0.00246	0.682	
TTime	-0.09698	0.03289	0.00251	**
Hinc(air)	0.00166	0.0081	0.689	
PSize(air)	-0.21431	0.26504	0.433	
Loglikelihood	-145.89			
AIC	305.79			
BIC	329.22			
Pseudo R <sup>2</sup>	0.48585			

Table .3: Summary results for the (reference, logistic,  $Z$ )<sub>car</sub> model.

Variables	Estimate	Std. Error	$p$ -value	Test
Air	7.3347943	0.945712	$8.8810^{-15}$	***
Bus	3.5916978	0.4754201	$4.19610^{-14}$	***
Train	4.3719054	0.477643	$< 10^{-16}$	***
GC	-0.0235074	0.005081	$3.71810^{-6}$	***
TTime	-0.1002126	0.010531	$< 10^{-16}$	***
Hinc(air)	0.0238154	0.011184	0.03322	*
PSize(air)	-1.1738153	0.2580201	$5.38210^{-6}$	***
Loglikelihood	-185.915			
AIC	385.83			
BIC	409.26			
Pseudo R <sup>2</sup>	0.34481			

Table .4: Summary results for the (reference, logistic,  $Z'$ )<sub>car</sub> model using the normalized space  $\mathfrak{F}_{q_{0.95}}$

Variables	Estimate	Std. Error	$p$ -value	Test
Air	5.9486392	0.6667107	$< 10^{-16}$	***
Bus	3.1236823	0.4499032	$3.84 \cdot 10^{-12}$	***
Train	3.5354086	0.4170212	$< 10^{-16}$	***
TTime	-0.101083	0.0104822	$< 10^{-16}$	***
Log-likelihood	-206.817			
AIC	421.634			
BIC	435.022			
Pseudo $R^2$	0.27115			

Table .5: Summary results for the (reference,  $F_{0.45}$ ,  $Z'$ )<sub>car</sub> model using the normalized space  $\mathfrak{F}_{q_{0.95}}$

Variables	Estimate	Std. Error	$p$ -value	Test
Air	6.59149	2.29819	0.00299	**
Bus	3.4303	1.23521	0.00425	**
Train	3.43999	1.23142	0.00393	**
TTime	-0.107856	0.037306	0.0026	**
Log-likelihood	-146.68			
AIC	301.37			
BIC	314.76			
Pseudo $R^2$	0.48307			

Table .6: Summary results for the (reference,  $F_{0.05}$ ,  $Z'$ ) model using the normalized space  $\mathfrak{F}_{q_{0.95}}$

Variables	Estimate	Std. Error	$p$ -value	Test
Air	$2.74 \cdot 10^{-5}$	$8.25 \cdot 10^{-5}$	0.59	
Bus	$1.41 \cdot 10^{-5}$	$4.32 \cdot 10^{-5}$	0.564	
Train	$1.44 \cdot 10^{-5}$	$4.49 \cdot 10^{-5}$	0.561	
TTime	$-5 \cdot 10^{-7}$	$1.4 \cdot 10^{-6}$	0.607	
Log-likelihood	-129.76			
AIC	267.52			
BIC	280.91			
Pseudo $R^2$	0.54271			

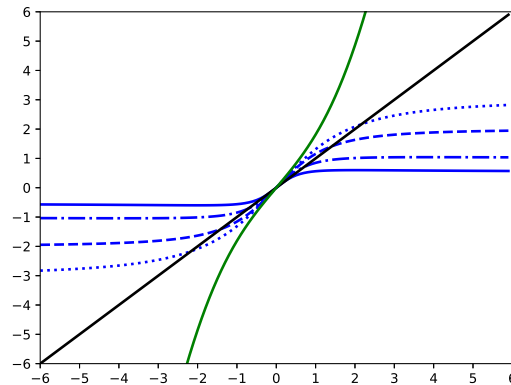


Figure .1: Function (6) with  $\nu = 0.5$  (blue line),  $\nu = 1$  (blue dashed-dotted line),  $\nu = 2$  (blue dashed line),  $\nu = 3$  (blue dotted line) and also with the logistic cdf (dark line) and the normal cdf (green line).

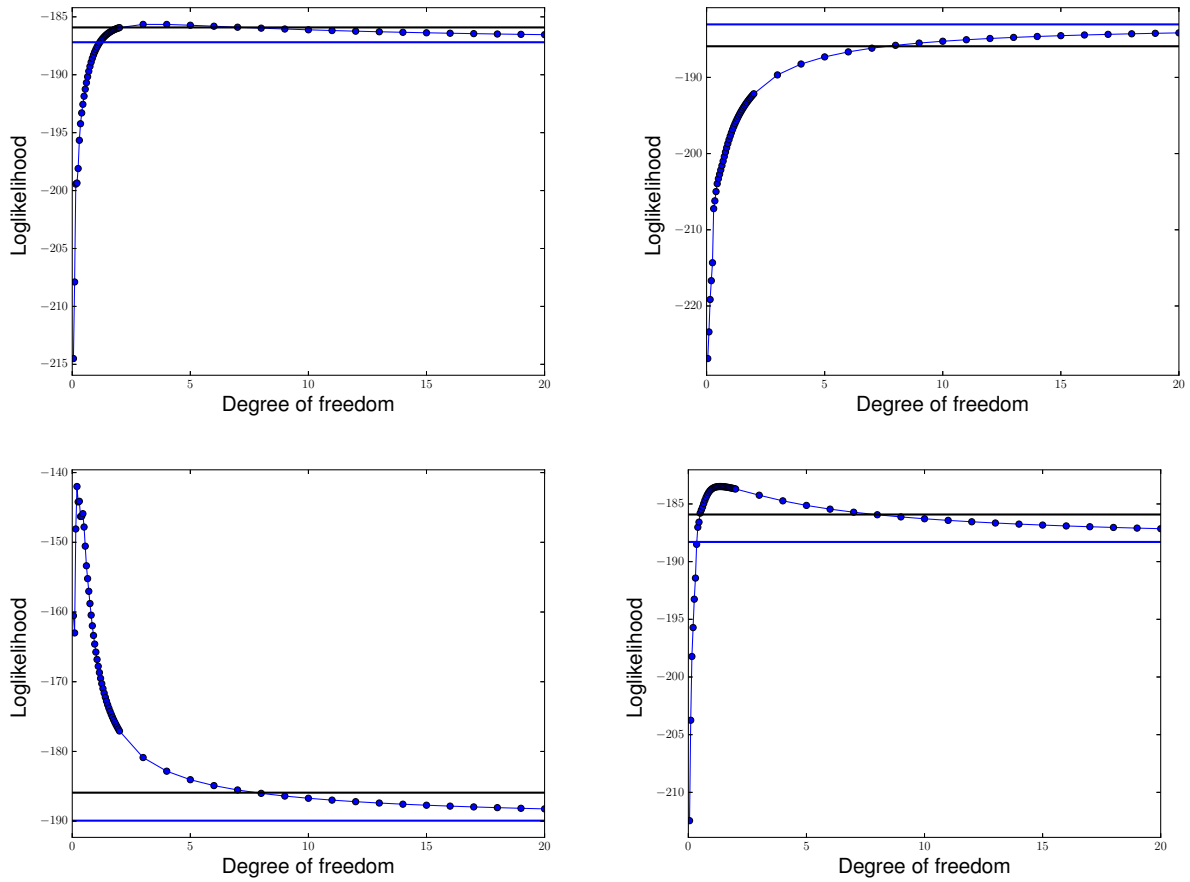


Figure .2: Log-likelihood of the (reference, normal,  $Z$ ) $_{j_0}$  model (blue line), the (reference, logistic,  $Z$ ) $_{j_0}$  model (dark line) and the (reference,  $F_\nu$ ,  $Z$ ) $_{j_0}$  models with  $\nu \in (0, 20]$  (blue points) in the four cases: (A)  $j_0 = \text{air}$ , (B)  $j_0 = \text{bus}$ , (C)  $j_0 = \text{car}$  and (D)  $j_0 = \text{train}$ .

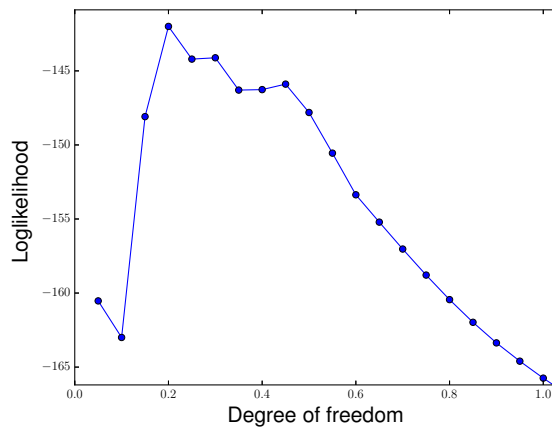


Figure .3: Log-likelihood of the (reference,  $\mathcal{T}(\nu)$ ,  $Z_0$ ) $_{\text{car}}$  models with  $\nu \in \{0.05, 0.1, \dots, 1\}$ .

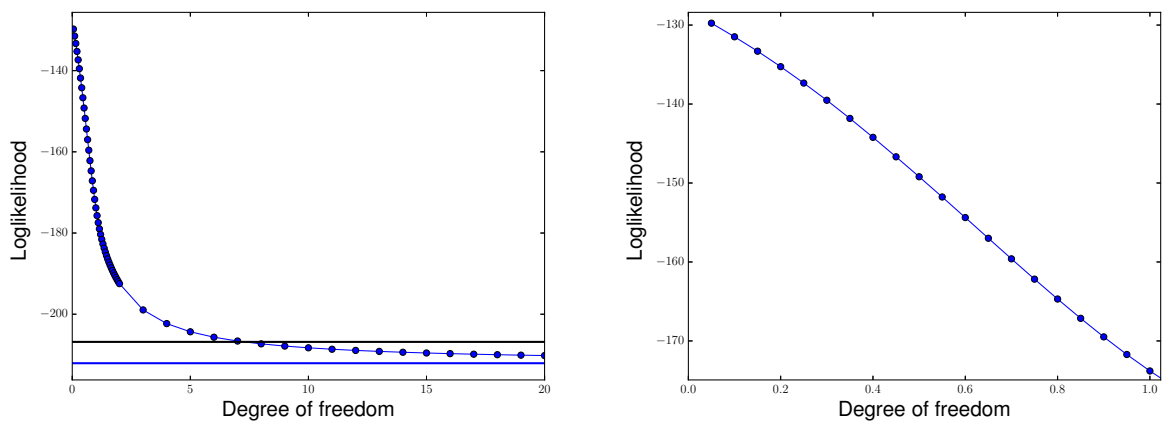
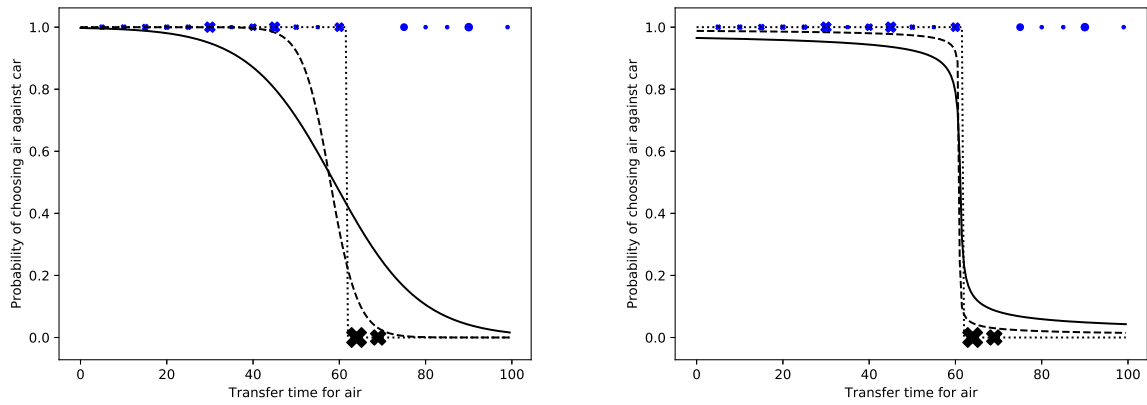


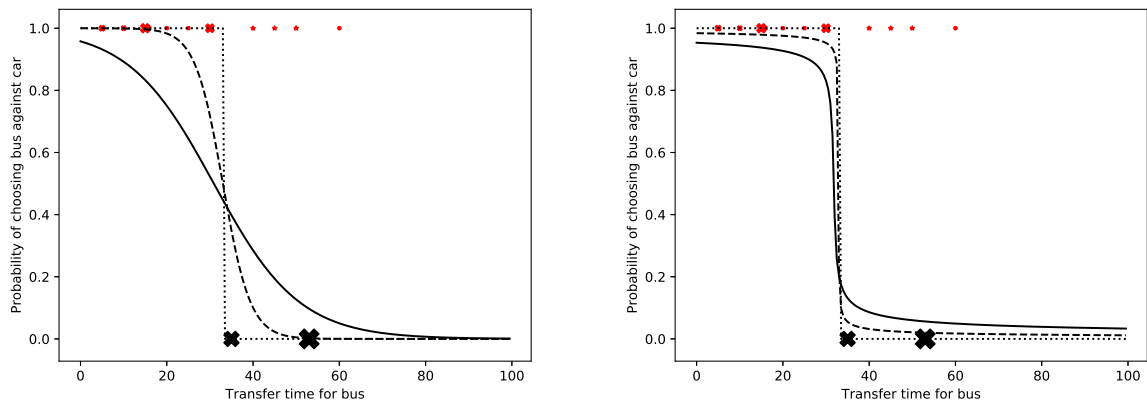
Figure 4: Log-likelihood of  $(\text{reference}, F_\nu, Z')_{\text{car}}$  models for (A)  $\nu \in (0, 20]$  (blue points),  $(\text{reference}, \text{normal}, Z')_{\text{car}}$  model (blue line) and  $(\text{reference}, \text{logistic}, Z')_{\text{car}}$  model (dark line) on the real dataset; (B) zooms on curve for  $\nu \in (0, 1)$ .



(A)



(B)



(C)

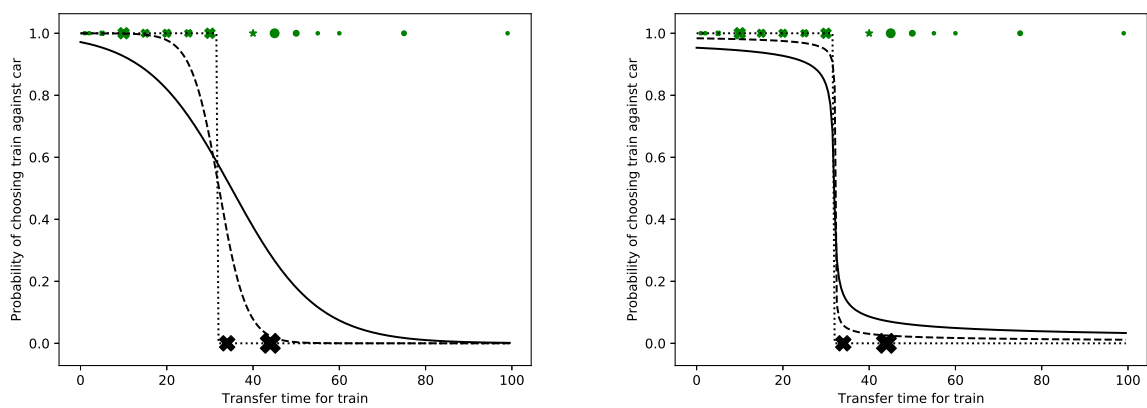


Figure .5: Conditional probabilities  $P(Y = j|Y \in \{j, j_0\}; t_j)$  with  $j_0 = car$  and (A)  $j = air$  (B)  $j = bus$  and (C)  $j = train$ , according to (reference,  $F$ ,  $Z'$ ) model where  $F$  is the logistic cdf (left) or Student  $F_{0.45}$  cdf (right). The corresponding observed proportions are represented by crosses (stars and points), whose the size depends on the number of observations. The color indicates the chosen alternative (blue for air, red for bus, green for train and dark for car).

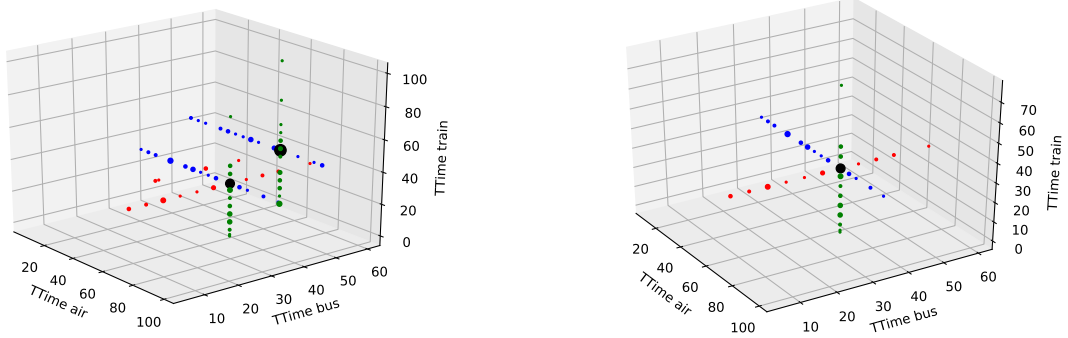


Figure .6: Three dimensional representation of observed terminal time values for A) the whole dataset B) the first group, and their associated travel mode choice (blue for air, red for bus, green for train and black for car). The size of each point is proportional to the number of individuals.

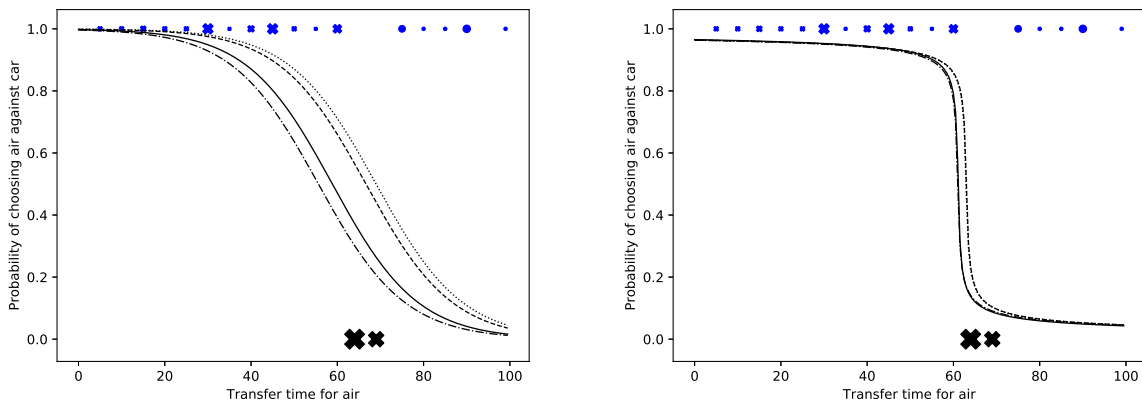


Figure .7: Conditional probabilities  $P(Y = j|Y \in \{j, j_0\}; t_j)$  with  $j_0 = car$  and  $j = air$ , according to (reference,  $F, Z'$ ) model where  $F$  is the logistic cdf (left) or Student  $F_{0.45}$  cdf (right) and after addition of non informative attributes, with median value (dashed line), 25% and 75% percentiles (resp. dotted and dashed-dotted line). The corresponding observed proportions are represented by crosses (stars and points), whose the size depends on the number of observations. The color indicates the chosen alternative (blue for air, red for bus, green for train and dark for car).