HAL
open science

An asymptotic Reissner–Mindlin plate model

Christian Licht, Thibaut Weller

To cite this version:

Christian Licht, Thibaut Weller. An asymptotic Reissner–Mindlin plate model. Comptes Rendus Mécanique, 2018, 346 (6), pp.432 - 438. 10.1016/j.crme.2018.04.014 . hal-01787527

HAL Id: hal-01787527

https://hal.umontpellier.fr/hal-01787527v1

Submitted on 7 May 2018

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Distributed under a Creative Commons Attribution - NonCommercial - NoDerivatives 4.0 International License

C. R. Mecanique 346 (2018) 432–438

Contents lists available at ScienceDirect

Comptes Rendus Mecanique

www.sciencedirect.com

An asymptotic Reissner–Mindlin plate model

Christian Licht [a,b,c], Thibaut Weller [a,*]

[a] LMGC, Université de Montpellier, CNRS, Montpellier, France
[b] Department of Mathematics, Faculty of Science, Mahidol University, Bangkok 10400, Thailand
[c] Centre of Excellence in Mathematics, CHE, Bangkok 10400, Thailand

ARTICLE INFO

Article history:
Received 6 April 2018
Accepted 15 April 2018
Available online 30 April 2018

Keywords:
Reissner–Mindlin plate model
Periodic abutting of thin plates
Asymptotic modeling
Variational convergence
Space of bounded deformations

ABSTRACT

A mathematical study via variational convergence of a periodic distribution of classical linearly elastic thin plates softly abutted together shows that it is not necessary to use a different continuum model nor to make constitutive symmetry hypothesis as starting points to deduce the Reissner–Mindlin plate model.

© 2018 Académie des sciences. Published by Elsevier Masson SAS. This is an open access article under the CC BY-NC-ND license (http://creativecommons.org/licenses/by-nc-nd/4.0/).

1. Introduction

Due to its ability to account for shear effects, the Reissner–Mindlin plate model [1–3] is often preferred in the engineering literature (see [4]) over the Kirchhoff–Love plate model. So, as done for the Kirchhoff–Love plate model [5,6], it is challenging to proceed with a rigorous mathematical derivation of the Reissner–Mindlin plate model by studying the asymptotic behavior of a thin 3-dimensional elastic body when its thickness goes to zero. This was done in [7–9] by using a second gradient or Cosserat continuum for the body jointly with constitutive symmetry assumptions; here – being aware of the results of [10] on the bonding of thin plates – we prefer to consider a strongly heterogeneous *classical linearly elastic* body made of a periodic distribution of thin *anisotropic* plates abutted together.

Let $\{e_1, e_2, e_3\}$ be an orthonormal basis of $\mathbb{R}^3$ assimilated to the Euclidean physical space. For all $\xi = (\xi_1, \xi_2, \xi_3)$ in $\mathbb{R}^3$, $\hat{\xi}$ stands for (ξ_1, ξ_2). The space of all $(N \times N)$ symmetric matrices is denoted by $\mathbb{S}^N$ and equipped with the usual inner product and norm denoted by $\cdot$ and $|\cdot|$, as in $\mathbb{R}^3$. For all e in $\mathbb{S}^3$, we set

$$e = \hat{e} + e^{\perp} \tag{1}$$

where $(\hat{e})_{\alpha\beta} = e_{\alpha\beta}$ and $(e^{\perp})_{\alpha\beta} = 0$, $1 \le \alpha, \beta \le 2$, and $(\hat{e})_{i3} = 0$, $(e^{\perp})_{i3} = e_{i3}$, $1 \le i \le 3$. For all a, b in $\mathbb{R}^3$, $a \otimes_s b$ stands for the symmetrized tensor product of a by b. Moreover, for all subset $\mathcal{O}$ of $\mathbb{R}^N$, $\chi_{\mathcal{O}}$ is the characteristic function of $\mathcal{O}$. Finally, we will use the symbol $\mathfrak{h}_n$ to denote n-dimensional Hausdorff measure and the letter C to introduce various constants which may vary from line to line.

Let ω be a domain of $\mathbb{R}^2$ with a Lipschitz continuous boundary $\partial\omega$ and $2\eta_0$, ε_0 two positive real numbers lesser than 1. Let $Y := (0,1)^2$, $Y_\eta^{\text{ext}} := (-\eta, 1+\eta)^2$, $Y_\eta^{\text{int}} := (\eta, 1-\eta)^2$, $I_\varepsilon := \{i \in \mathbb{Z}^2 ; \varepsilon(i + Y_\eta^{\text{ext}}) \subset \omega\}$ for all η in $(0, \eta_0)$ and

* Corresponding author.
E-mail addresses: christian.licht@umontpellier.fr (C. Licht), thibaut.weller@umontpellier.fr (T. Weller).

https://doi.org/10.1016/j.crme.2018.04.014
1631-0721/© 2018 Académie des sciences. Published by Elsevier Masson SAS. This is an open access article under the CC BY-NC-ND license (http://creativecommons.org/licenses/by-nc-nd/4.0/).

$$\omega_{\eta,\varepsilon}^{\text{int}} := \bigcup_{i\in I_\varepsilon} \varepsilon(i + Y_\eta^{\text{int}}), \quad \omega_{\eta,\varepsilon}^{\text{ext}} := \omega \setminus \bigcup_{i\in I_\varepsilon} (i + \overline{Y_\eta^{\text{ext}}})$$

$$\omega_{\eta,\varepsilon} := \omega_{\eta,\varepsilon}^{\text{int}} \cup \omega_{\eta,\varepsilon}^{\text{ext}}, \quad b_{\eta,\varepsilon} := \omega \setminus \overline{\omega_{\eta,\varepsilon}}$$

Let h be a small positive number, we will consider a structure occupying $\Omega^h := \omega \times (-h, h)$ made of an εY-periodic distribution of thin linearly elastic plates inhabiting $\Omega_{\eta,\varepsilon}^{h\,\text{int}} := \omega_{\eta,\varepsilon}^{\text{int}} \times (-h, h)$ abutted together through a thin and narrow soft linearly elastic adhesive layer filling $B_{\eta,\varepsilon}^h := b_{\eta,\varepsilon} \times (-h, h)$ and surrounded by a thin linearly hollow plate occupying $\Omega_{\eta,\varepsilon}^{h\,\text{ext}} := \omega_{\eta,\varepsilon}^{\text{ext}} \times (-h, h)$. We set $\Omega_{\eta,\varepsilon}^h := \Omega_{\eta,\varepsilon}^{h\,\text{int}} \cup \Omega_{\eta,\varepsilon}^{h\,\text{ext}}$ and assume that all the constituents of the structure are perfectly bonded together.

For brevity and simplicity, we assume that the structure is subjected to body and surface forces on its upper/lower boundary $\Gamma_\pm^h := \omega \times \{\pm h\}$ of densities f^h, g^h, respectively, and, as in [7,8], is clamped on its lateral boundary $\Gamma_{\text{tat}}^h := \partial\omega \times (-h, h)$. The strain energy $\mathcal{W}_s$ of the body reads as:

$$\mathcal{W}_s(x, e) := \begin{cases} \mathcal{W}(e) \text{ a.e. } x \in \Omega_{\eta,\varepsilon}^h \\ \mu_\wedge \mathcal{W}_\wedge(\hat{e}) + \mu_\perp \mathcal{W}_\perp(e^\perp) \text{ a.e. } x \in B_{\eta,\varepsilon}^h \end{cases}$$

for all e in $\mathbb{S}^3$, $\mathcal{W}$, $\mathcal{W}_\wedge$, $\mathcal{W}_\perp$ being positive definite quadratic functions on $\mathbb{S}^3$.

Hence the equilibrium of the structure involves a quintuplet of data $s := (s', \varepsilon)$, $s' := (\mu_\wedge, \mu_\perp, \eta, h)$ and reads as:

$$(\mathcal{P}^s) \quad \text{Min} \left\{ \int_{\Omega^h} \mathcal{W}_s(x, e(v))\,\mathrm{d}x - \int_{\Omega^h} f^h(x) \cdot v(x)\,\mathrm{d}x - \int_{\Gamma_+^h \cup \Gamma_-^h} g^h(x) \cdot v(x)\,\mathrm{d}\mathfrak{h}_2 \,;\; v \in H^1_{\Gamma_{\text{tat}}^h}(\Omega^h; \mathbb{R}^3) \right\}$$

where $e(v)$ is the strain tensor associated with the displacement v and, in the sequel, for all domain $\mathcal{O}$ in $\mathbb{R}^N$ and all smooth part γ of its boundary $\partial\mathcal{O}$, $H^1_\gamma(\mathcal{O}; \mathbb{R}^N)$ denotes the subspace of $H^1(\mathcal{O}; \mathbb{R}^N)$ made of the elements with vanishing trace on γ. Clearly, if (f^h, g^h) belongs to $L^2(\Omega^h \times (\Gamma_+^h \cup \Gamma_-^h); \mathbb{R}^3)$, $(\mathcal{P}^s)$ has a unique solution u^s and, considering the data s as a parameter, we are interested in its asymptotic behavior when s takes values in a countable subset of $(0, +\infty)^2 \times (0, \eta_0) \times (0, +\infty) \times (0, \varepsilon_0)$ with 0 as a unique limit point. Like in the mathematical derivation of the Kirchhoff–Love theory of plates [5,6], it is convenient to introduce the linear mappings Π^h and S_h:

$$\xi = (\hat{\xi}, \xi_3) \in \mathbb{R}^3 \mapsto \Pi^h \xi = (\hat{\xi}, h\xi_3)$$

$$v \in L^1(\Omega^h; \mathbb{R}^3) \mapsto S_h v \in L^1(\Omega; \mathbb{R}^3) \text{ s.t. } (S_h v)(x) = \frac{1}{h} \Pi^h(v(\Pi^h x)),\ \forall x \in \Omega := \omega \times (-1, 1)$$

We make the following assumption on the loading:

$$(H_1) \quad : \quad \begin{cases} \exists (f, f', g) \in L^3(\Omega; \mathbb{R}^3) \times L^3(\Omega; \mathbb{R}^3) \times L^\infty(\Gamma^+ \cup \Gamma^-; \mathbb{R}^3) \text{ s.t.} \\ f^h(\Pi^h x) = h\Big(\chi_{\Omega_{\eta,\varepsilon}^h} \Pi^h f(x) + \chi_{B_{\eta,\varepsilon}^h} \Pi^h f'(x)\Big),\ \forall x \in \Omega \\ g^h(\Pi^h x) = h^2 \Pi^h \chi_{\Gamma_{\eta,\varepsilon,+} \cup \Gamma_{\eta,\varepsilon,-}} g(x),\ \forall x \in \Gamma_{\eta,\varepsilon,+} \cup \Gamma_{\eta,\varepsilon,-} \end{cases}$$

where $\Gamma_{\eta,\varepsilon,\pm} := \omega_{\eta,\varepsilon} \times \{\pm 1\}$. Therefore, $u_s := S_h u^s$ is the unique solution to

$$(\mathcal{P}_s) \quad \text{Min} \left\{ \mathcal{J}_s(v);\ v \in H^1_{\Gamma_{\text{tat}}}(\Omega; \mathbb{R}^3) \right\}$$

where

$$\mathcal{J}_s(v) := \int_\Omega \mathcal{W}_s(x, e(h, v)) - (\chi_{\Omega_{\eta,\varepsilon}} f + \chi_{B_{\eta,\varepsilon}} f') \cdot v\,\mathrm{d}x - \int_{\Gamma_{\eta,\varepsilon,+} \cup \Gamma_{\eta,\varepsilon,-}} g \cdot v\,\mathrm{d}\mathfrak{h}_2$$

$$e(h, v)_{\alpha\beta} := e(v)_{\alpha\beta}, \quad e(h, v)_{\alpha 3} := \frac{1}{h} e(v)_{\alpha 3},\ 1 \le \alpha, \beta \le 2, \quad e(h, v)_{33} := \frac{1}{h^2} e(v)_{33}$$

with Γ_{tat} the reciprocal image by Π^h of Γ_{tat}^h and, similarly, index h will be dropped for the image by $\Pi^{h^{-1}}$ of $\Omega_{\eta,\varepsilon}^h$, $B_{\eta,\varepsilon}^h$ and $\Gamma_\pm^h$.

2. A convergence result

Let the space of Kirchhoff–Love and Reissner–Mindlin displacements be respectively denoted by

$$V_{KL}(\Omega) := \Big\{ v \in H^1_{\Gamma_{\rm tat}}(\Omega;\mathbb{R}^3);\ \exists (v^M, v^F) \in H^1(\omega;\mathbb{R}^2)\times H^2(\omega) \text{ s.t. } \hat{v}(x) = v^M(\hat{x}) - x_3\widehat{\nabla v^F(\hat{x})},$$
$$v_3(x) = v^F(\hat{x}) \text{ a.e. } x\in\Omega\Big\}$$
$$= \Big\{ v \in H^1_{\Gamma_{\rm tat}}(\Omega;\mathbb{R}^3) \text{ s.t. } e_{i3}(v) = 0,\ 1\le i\le 3\Big\}$$
$$V_{RM}(\Omega) := \Big\{ v \in H^1_{\Gamma_{\rm tat}}(\Omega;\mathbb{R}^3);\ \exists (v^M, v^\theta, v^F) \in H^1(\omega;\mathbb{R}^2)\times H^1(\omega;\mathbb{R}^2)\times H^1(\omega) \text{ s.t. } \hat{v}(x) = v^M(\hat{x}) + x_3 v^\theta(\hat{x}),$$
$$v_3(x) = v^F(\hat{x}) \text{ a.e. } x\in\Omega\Big\}$$
$$= \Big\{ v \in H^1_{\Gamma_{\rm tat}}(\Omega;\mathbb{R}^3) \text{ s.t. } e_{33}(v) = 0,\ \partial_3 e_{\alpha 3}(v) = 0,\ 1\le\alpha\le 2\Big\}$$

We recall [11,12] that the space of bounded deformations is defined by

$$BD(\Omega) := \Big\{ v \in L^1(\Omega;\mathbb{R}^3);\ e(v) \in M_b(\Omega;\mathbb{S}^3)\Big\}$$

where $M_b(\Omega;\mathbb{S}^3)$ denotes the space of bounded $\mathbb{S}^3$-valued measures in Ω.

We assume

$$(H_2) \quad : \quad \exists\, (\bar{\mu}_\wedge, \bar{\mu}_\perp) \in (0,+\infty)^2 \text{ s.t. } (\bar{\mu}_\wedge, \bar{\mu}_\perp) := \lim_{s'\to 0} \frac{1}{2\eta}\Big(\mu_\wedge, \frac{1}{h^2}\mu_\perp\Big)$$

In the sequel, when no ambiguity ensues, we use the same symbol $\hat{e}$ for an element of $\mathbb{S}^2$, such that its entries are the non-vanishing ones of the element $\hat{e}$ of $\mathbb{S}^3$. For instance, for all (q,e) in $\mathbb{S}^2\times\mathbb{S}^3$, $q=\hat{e}$ means that $q_{\alpha\beta} = e_{\alpha\beta}$ for all α, β in $\{1,2\}$.

Let the two positive definite quadratic functions on $\mathbb{S}^2$ defined by:

$$\mathcal{W}_{KL}(q) = \text{Min}\Big\{\mathcal{W}(e);\ e\in\mathbb{S}^3 \text{ s.t. } \hat{e} = q\Big\}$$
$$\mathcal{W}_*(q) = \text{Min}\Big\{\mathcal{W}_{KL}(q^1) + \bar{\mu}_\wedge\mathcal{W}_\wedge(q^2) + \bar{\mu}_\wedge\mathcal{W}_\wedge(q^3);\ q^i\in\mathbb{S}^2,\ 1\le i\le 3,\ (q^2)_{22} = (q^3)_{11} = 0,\ q^1+q^2+q^3 = q\Big\}$$

Then we have the following theorem.

Theorem 2.1. *Under assumptions (H_1) and (H_2), as s' goes to 0 and then ε goes to 0, u_s converges weak $\star$ in $BD(\Omega)$ toward the unique solution u to*

$$(\mathcal{P}) \quad Min\Big\{\int_\Omega \mathcal{W}_*(\widehat{e(v)})\,\mathrm{d}x + \overline{\mu}_\perp\int_\Omega \mathcal{W}_\perp(e(v)^\perp)\,\mathrm{d}x - \int_\Omega f\cdot v\,\mathrm{d}x - \int_{\Gamma_+\cup\Gamma_-} g\cdot v\,\mathrm{d}\mathfrak{h}_2\,;\ v\in V_{RM}(\Omega)\Big\}$$

The proof, which uses a standard method of variational convergence, is achieved in three steps.

Step 1 (asymptotic behavior of u_s as s' goes to zero):

Let $\sigma_\varepsilon := \bigcup_{i\in I_\varepsilon}\varepsilon(i+\partial Y)$, $\Sigma_\varepsilon := \sigma_\varepsilon\times(-1,1)$, $\dot{\Omega}_\varepsilon := \Omega\setminus\Sigma_\varepsilon$, ν the unit normal to Σ_ε (or σ_ε) equal to either e_1 or e_2 and Σ^k_ε the part of Σ_ε such that $\nu = e_k$, $k=1,2$. For all w in $H^1(\dot{\Omega}_\varepsilon;\mathbb{R}^3)$, we denote by $[w]$ the jump of w across the surface Σ_ε oriented by ν.

Proposition 2.1. *When s' goes to 0, u_s (up to a not relabeled subsequence) converges weak $\star$ in $BD(\Omega)$ toward some u_ε such that*

$$u_\varepsilon \in V_{KL}(\dot{\Omega}_\varepsilon) := \Big\{ v\in H^1_{\Gamma_{\rm tat}}(\dot{\Omega}_\varepsilon;\mathbb{R}^3) \text{ s.t. } \dot{e}_\varepsilon(v)^\perp = 0\Big\}$$

where $\dot{e}_\varepsilon$ denotes the density with respect to the Lebesgue measure of the symmetrized gradient in the sense of distribution in $\dot{\Omega}_\varepsilon$, and with:

$$\mathcal{J}_\varepsilon(u_\varepsilon) := \int_{\dot\Omega_\varepsilon} \mathcal{W}_{KL}(\widehat{\dot e_\varepsilon(u_\varepsilon)})\,\mathrm{d}x + \frac{\bar\mu_\wedge}{\varepsilon}\int_{\Sigma_\varepsilon} \mathcal{W}_\wedge(\widehat{[u_\varepsilon]}\otimes_s \nu)\,\mathrm{d}\mathfrak{h}_2 + \frac{\bar\mu_\perp}{\varepsilon}\int_{\Sigma_\varepsilon} \mathcal{W}_\perp([u_\varepsilon]_3 e_3\otimes_s \nu)\,\mathrm{d}\mathfrak{h}_2$$
$$- \int_\Omega f\cdot u_\varepsilon\,\mathrm{d}x - \int_{\Gamma_+\cup\Gamma_-} g\cdot u_\varepsilon\,\mathrm{d}\mathfrak{h}_2 \le \varliminf_{s'\to 0}\mathcal{J}_s(u_s)$$

Proof. First, ellipticity assumptions on $\mathcal{W}$, $\mathcal{W}_\wedge$, $\mathcal{W}_\perp$, Hölder inequality and (H_1-H_2) imply that u_s is bounded in $LD(\Omega) = \{u\in L^1(\Omega;\mathbb{R}^3);\ e(u)\in L^1(\Omega;\mathbb{S}^3)\}$. Thus, up to a not relabeled subsequence, u_s weak $\star$ converges in $BD(\Omega)$ toward some u_ε, and consequently weakly in $L^{3/2}(\Omega;\mathbb{R}^3)$ and strongly in $L^p(\Omega;\mathbb{R}^3)$ for all $p<\dfrac{3}{2}$ (see [11,12]). Moreover, as there exists $\delta(\varepsilon)$ such that u_s is bounded in $H^1_{\Gamma_{\mathrm{tat}}}(\Omega_\delta;\mathbb{R}^3)$, $\Omega_\delta := \omega_\delta\times(-1,1)$, $\omega_\delta := \{\hat x\in\omega;\ \mathrm{dist}\,(\hat x,\partial\omega)<\delta\}$, the trace on Γ_{tat} of u_ε vanishes. As $\chi_{\Omega_{\eta,\varepsilon}} e(u_s)$ is bounded in $L^2(\Omega;\mathbb{S}^3)$, it converges, up to a not relabeled subsequence, to some q, which, by testing by any element of $\mathcal{D}(\dot\Omega_\varepsilon;\mathbb{S}^3)$, is clearly identified as $\dot e_\varepsilon(u_\varepsilon)$ with $\dot e_\varepsilon(u_\varepsilon)^\perp = 0$. Hence, u_ε belongs to $V_{KL}(\dot\Omega_\varepsilon)$ $(:=\{v\in H^1_{\Gamma_{\mathrm{tat}}}(\dot\Omega_\varepsilon;\mathbb{R}^3);\ \dot e_\varepsilon(v)^\perp = 0\})$. As for all τ in $\mathcal{D}(\Omega;\mathbb{S}^3)$

$$\int_{B_{\eta,\varepsilon}} \tau\cdot e(u_s)\,\mathrm{d}x = -\int_\Omega \Big(\mathrm{div}\,\tau\cdot u_s + \tau\cdot\chi_{\Omega_{\eta,\varepsilon}} e(u_s)\Big)\,\mathrm{d}x \tag{2}$$

we deduce:

$$e(u_\varepsilon) = \dot e_\varepsilon(u_\varepsilon)\,\mathrm{d}x + [u_\varepsilon]\otimes_s \nu\,\mathfrak{h}_2 \llcorner \Sigma_\varepsilon \tag{3}$$

Next let $\partial_\eta Y := \Big\{\hat x\in\partial Y \text{ s.t. } x_1 \text{ or } x_2\in(\eta,1-\eta)\Big\}$ and $\Sigma_{\eta,\varepsilon} := \bigcup_{i\in I_\varepsilon}\varepsilon(i+\partial_\eta Y)$. Jensen inequality implies

$$\varliminf_{s'\to 0}\mu_\wedge\int_{B_{\eta,\varepsilon}} \mathcal{W}_\wedge(\widehat{e(\varepsilon,u_s)})\,\mathrm{d}x \ge \varliminf_{s'\to 0}\frac{\mu_\wedge}{2\eta\varepsilon}\int_{\Sigma_{\eta,\varepsilon}} \mathcal{W}_\wedge\Big(\int_{-\eta\varepsilon}^{\eta\varepsilon}\widehat{e(u_s)}(x+t\nu)\,\mathrm{d}t\Big)\,\mathrm{d}\mathfrak{h}_2 \ge \frac{\bar\mu_\wedge}{\varepsilon}\int_{\Sigma_\varepsilon}\mathcal{W}_\wedge(\widehat{[u_\varepsilon]}\otimes_s\nu)\,\mathrm{d}\mathfrak{h}_2$$

$$\varliminf_{s'\to 0}\mu_\perp\int_{B_{\eta,\varepsilon}} \mathcal{W}_\perp(e(\varepsilon,u_s)^\perp)\,\mathrm{d}x \ge \varliminf_{s'\to 0}\frac{\mu_\perp}{2\eta\varepsilon h^2}\int_{\Sigma_{\eta,\varepsilon}} \mathcal{W}_\perp\Big(\int_{-\eta\varepsilon}^{\eta\varepsilon} e(u_s)^\perp(x+t\nu)\,\mathrm{d}t\Big)\,\mathrm{d}\mathfrak{h}_2 \ge \frac{\bar\mu_\perp}{\varepsilon}\int_{\Sigma_\varepsilon}\mathcal{W}_\perp([u_\varepsilon]_3 e_3\otimes_s\nu)\,\mathrm{d}\mathfrak{h}_2$$

because, by arguing similarly to (2), we get that $\chi_{\Sigma_{\eta,\varepsilon}}\displaystyle\int_{-\eta\varepsilon}^{\eta\varepsilon} e(u_s)(\cdot+t\nu)\,\mathrm{d}t$ weakly converges in $L^2(\Sigma_\varepsilon;\mathbb{S}^3)$ toward $[u_\varepsilon]\otimes_s\nu$. Moreover, arguing as in [13] through the introduction of cut-off functions and translations by $\pm\eta\varepsilon e_k$, $k=1,2$, and by $\pm\eta\varepsilon(e_1\pm e_2)$ yields $\displaystyle\lim_{s'\to 0}\int_{\Gamma_{\eta,\varepsilon,\pm}} g\cdot u_s\,\mathrm{d}\mathfrak{h}_2 = \int_{\Gamma_\pm} g\cdot u_\varepsilon\,\mathrm{d}\mathfrak{h}_2$. Hence, the desired result stems from the very definition of $\mathcal{W}_{KL}$, the weak convergence in $L^2(\Omega,\mathbb{S}^3)$ of $\chi_{\Omega_{\varepsilon,\eta}}\widehat{e(u_s)}$ toward $\widehat{\dot e_\varepsilon(u_\varepsilon)}$ and the weak $\star$ convergence in $BD(\Omega)$ of u_s. □

<u>Step 2</u> (asymptotic behavior of u_ε as ε goes to zero):

Let $Y^i_\varepsilon := \varepsilon(i+Y)$, $\Sigma^{k,i}_\varepsilon := \{x\in\Sigma^k_\varepsilon;\ x_k = \varepsilon(i_k+1),\ x_{3-k}\in\varepsilon\big(i_{3-k}, i_{3-k}+1\big)\}$ for all i in I_ε, and $\dot\Sigma^k_\varepsilon := \bigcup_{i\in I_\varepsilon}\Sigma^{k,i}_\varepsilon$. Then, for all ψ^k_ε in $L^2(\Sigma^k_\varepsilon)$ such that $\dfrac{1}{\varepsilon}\displaystyle\int_{\Sigma^k_\varepsilon}(\psi^k_\varepsilon)^2\,\mathrm{d}\mathfrak{h}_2 < C$, the function $\Psi^k_\varepsilon := \mathcal{L}^k_\varepsilon\psi^k_\varepsilon$ defined by $\mathcal{L}^k_\varepsilon(\psi^k_\varepsilon)(x) = \dfrac{1}{\varepsilon}\displaystyle\sum_{i\in I_\varepsilon}\psi^k_\varepsilon(x_{3-k},x_3)\chi_{Y^i_\varepsilon}(\hat x)$ is bounded in $L^2(\Omega)$. Note that, as $\mathfrak{h}_2(\Sigma^k_\varepsilon\setminus\dot\Sigma^k_\varepsilon)$ remains bounded, we have $\displaystyle\lim_{\varepsilon\to 0}\int_{\Sigma^k_\varepsilon}\tau\psi^k_\varepsilon\,\mathrm{d}\mathfrak{h}_2 = \lim_{\varepsilon\to 0}\int_\Omega\tau\Psi^k_\varepsilon\,\mathrm{d}x$ for all τ in $\mathcal{C}^0(\overline{\Omega})$.

Proposition 2.2. *When ε goes to zero, u_ε, up to a not relabeled subsequence, converges for the intermediate topology of $BD(\Omega)$ (see [11,12]) toward some u such that*

$$u\in V_{RM}(\Omega),\quad \mathcal{J}(u) := \int_\Omega \mathcal{W}_*(\widehat{e(u)})\,\mathrm{d}x + \bar\mu_\perp\int_\Omega\mathcal{W}(e(u)^\perp)\,\mathrm{d}x - \int_\Omega f\cdot u\,\mathrm{d}x - \int_{\Gamma_+\cup\Gamma_-} g\cdot u\,\mathrm{d}\mathfrak{h}_2 \le \varliminf_{\varepsilon\to 0}\mathcal{J}_\varepsilon(u_\varepsilon)$$

Proof. As $\mathcal{J}_\varepsilon(u_\varepsilon)$ is bounded, u_ε is bounded in $BD(\Omega)$ and then, up to a subsequence, weak $\star$ converges toward some u. Moreover as $u_\varepsilon \in V_{KL}(\dot{\Omega}_\varepsilon)$, $(u_\varepsilon^M, u_\varepsilon^F)$ converges weakly in $L^2(\omega;\mathbb{R}^2)\times L^2(\omega)$ and the measure $e(u_\varepsilon)$ has a narrow limit which can be identified with an element of $L^2(\Omega;\mathbb{S}^3)$ such that $e(u)_{33}=0$ and $\partial_3 e(u)^\perp = 0$. $\square$

Step 3 (identification of the limit field u):

For all $(\mathcal{Y},\mathcal{E})$ in $\{Y_\varepsilon^i, Y_\varepsilon^i+\varepsilon e_k\}\times\{\mathbb{R}^N,\mathbb{S}^N\}$ and ψ in $L^2(\mathcal{Y},\mathcal{E})$, we set $<\psi>_{\mathcal{Y}} = \dfrac{1}{\varepsilon^2}\displaystyle\int_{\mathcal{Y}}\psi(\hat{x})\mathrm{d}\hat{x}$ and let $\hat{x^i} = \varepsilon(\frac{1}{2},\frac{1}{2})+i$.

Proposition 2.3. *For all v in $V_{RM}(\Omega)$, there exists a sequence v_ε in $H^1_{\Gamma_{\mathrm{tat}}}(\Omega;\mathbb{R}^3)$ such that, when ε goes to zero, v_ε converges for the intermediate topology of $BD(\Omega)$ toward v with $\overline{\lim_{\varepsilon\to 0}}\,\mathcal{J}_\varepsilon(v_\varepsilon)\le\mathcal{J}(v)$, and there exists a sequence v_s in $H^1_{\Gamma_{\mathrm{tat}}}(\Omega;\mathbb{R}^3)$ such that, when s' goes to zero, $v_s := v_{s',\varepsilon}$ converges weak $\star$ in $BD(\Omega)$ toward v_ε with $\overline{\lim_{s'\to 0}}\,\mathcal{J}_s(v_s)\le\mathcal{J}_\varepsilon(v_\varepsilon)$.*

Proof. Through a classical density and diagonalization argument, we may assume that v^M, v^θ, v^F belong to $\mathcal{D}(\omega;\mathbb{R}^2)$ and

$$\mathcal{W}_*\big(\widehat{e(v)}\big) = \mathcal{W}_{KL}\big(\widehat{e(v)}-(\varphi^1+\varphi^2)\big)+\bar{\mu}_\wedge\mathcal{W}_\wedge(\varphi^1)+\bar{\mu}_\wedge\mathcal{W}_\wedge(\varphi^2)$$

with $q := \widehat{e(v)}-(\varphi^1+\varphi^2) = q^M(\hat{x})+x_3q^\theta(\hat{x})$ and q^M, q^θ in $\mathcal{D}(\omega;\mathbb{S}^2)$.

Let v_ε the element of $V_{KL}(\dot{\Omega}_\varepsilon)$ defined by:

$$\begin{aligned}
\widehat{v_\varepsilon}(x) &:= v_\varepsilon^M(\hat{x})+x_3v_\varepsilon^\theta(\hat{x})\\
v_\varepsilon^M(\hat{x}) &:= \sum_{i\in I_\varepsilon}\Big(<v^M>_{Y_\varepsilon^i}+<q^M>_{Y_\varepsilon^i}(\hat{x}-\hat{x^i})\Big)\chi_{Y_\varepsilon^i}(\hat{x})\\
v_\varepsilon^\theta(\hat{x}) &:= \sum_{i\in I_\varepsilon}\Big(<v^\theta>_{Y_\varepsilon^i}+<q^\theta>_{Y_\varepsilon^i}(\hat{x}-\hat{x^i})\Big)\chi_{Y_\varepsilon^i}(\hat{x})\\
\big(v_\varepsilon\big)_3(x) &:= \sum_{i\in I_\varepsilon}\Big[<v^F>_{Y_\varepsilon^i}-\Big(<v^\theta>_{Y_\varepsilon^i}+\frac{1}{2}<q^\theta>_{Y_\varepsilon^i}(\hat{x}-\hat{x^i})\Big)\cdot(\hat{x}-\hat{x^i})\Big]\chi_{Y_\varepsilon^i}(\hat{x})
\end{aligned}$$

The required smoothness of v^M, v^θ, v^F, q^M and q^θ implies that v_ε converges strongly in $L^2(\Omega;\mathbb{R}^2)$ toward v and that the measure $\widehat{e(v_\varepsilon)}$ satisfies

$$\begin{aligned}
\widehat{e(v_\varepsilon)} &= \sum_{i\in I_\varepsilon}<q>_{Y_\varepsilon^i}\ \chi_{Y_\varepsilon^i}\,\mathrm{d}x+\varphi_\varepsilon^1+\varphi_\varepsilon^2\\
\varphi_\varepsilon^k &:= [\widehat{v_\varepsilon}]\otimes_s e_k\,\mathfrak{h}_2\ \llcorner\ \Sigma_\varepsilon^k = \sum_{i\in I_\varepsilon}\varphi_\varepsilon^{k,i}\,\mathfrak{h}_2\ \llcorner\ \Sigma_\varepsilon^{k,i}
\end{aligned}$$

with

$$\begin{aligned}
\varphi_\varepsilon^{k,i} &= A_\varepsilon^{k,i}+B_\varepsilon^{k,i}(x_{3-k}-\frac{\varepsilon}{2}),\quad 1\le k\le 2\\
(A_\varepsilon^{k,i})_{k\alpha} &= \frac{(1+\delta_{k\alpha})}{2}\Big[<(v_\varepsilon)_\alpha>_{Y_\varepsilon^i+\varepsilon e_k}-<(v_\varepsilon)_\alpha>_{Y_\varepsilon^i}-\Big(<q_{k\alpha}>_{Y_\varepsilon^i}+<q_{k\alpha}>_{Y_\varepsilon^i+\varepsilon e_k}\Big)\frac{\varepsilon}{2}\Big],\quad \alpha=1,2\\
(A_\varepsilon^{k,i})_{3-k,3-k} &= 0\\
(B_\varepsilon^{k,i})_{k\alpha} &= \frac{(1+\delta_{k\alpha})}{2}\Big[<q_{k\alpha}>_{Y_\varepsilon^i+\varepsilon e_k}-<q_{k\alpha}>_{Y_\varepsilon^i}\Big]
\end{aligned}$$

where $\delta_{\alpha\beta}$ stands for the Kronecker delta. Clearly the smoothness of q and v implies that the measure $\widehat{e(v_\varepsilon)}$ converges narrowly to $(q+\varphi_1+\varphi_2)\,\mathrm{d}x$ because $\sum_{i\in I_\varepsilon}<q>_{Y_\varepsilon^i}\ \chi_{Y_\varepsilon^i}$ converges strongly in $L^2(\Omega;\mathbb{S}^2)$ toward q and the functions $\phi_\varepsilon^k = \mathcal{L}_\varepsilon^k(\varphi_\varepsilon^k)$ converges strongly in $L^2(\Omega;\mathbb{S}^2)$. As $e(v_\varepsilon)^\perp = \sum_{i\in I_\varepsilon}\varphi_\varepsilon^{\perp k,i}e_3\otimes_s e_k\mathfrak{h}_2\ \llcorner\ \Sigma_\varepsilon^{k,i}$, with

$$\varphi_\varepsilon^{\perp k,i} = <v^F>_{Y_\varepsilon^i+\varepsilon e_k}-<v^F>_{Y_\varepsilon^i}+\frac{\varepsilon}{2}\Big(<v_k^\theta>_{Y_\varepsilon^i}+<v_k^\theta>_{Y_\varepsilon^i+\varepsilon e_k}\Big)+\psi_\varepsilon^{\perp k,i}$$

where $|\psi_\varepsilon^{\perp k,i}|_{L^\infty(\Sigma_\varepsilon^{k,i})} \le C\varepsilon^2$ for all i in I_ε and all k in $\{1,2\}$, $\phi_\varepsilon^{\perp k,i} = \mathcal{L}_\varepsilon^k(\varphi_\varepsilon^{\perp k,i})$ converges strongly in $L^2(\Omega)$ toward $\partial_k v^F + v_k^\theta = 2e(v)_{k3}$. Hence one deduces that v_ε converges toward v for the intermediate topology of $BD(\Omega)$ and that $\lim_{\varepsilon\to 0} \mathcal{J}_\varepsilon(v_\varepsilon) = \mathcal{J}(v)$.

The last stage, $s' \to 0$, is a similar situation of bonding of thin plates as in [10] and is achieved by using a distribution and composition of smoothing operators along Σ_ε like in [14,13] acting on a classical suitable approximation of Kirchhoff–Love fields as in [5,6].

Hence, u is the unique solution to $(\mathcal{P})$ so that the whole sequences u_s and u_ε converge. □

This mathematical result can be rephrased in terms related to the genuine physical problem $(\mathcal{P}^s)$ (Theorem 2.2).

Theorem 2.2. *If $\bar{u}^s := S_h^{-1}u$, the field of displacement u^s is asymptotically equivalent to $\bar{u}^s$ solution to*

$$(\overline{\mathcal{P}}^s) \quad \mathrm{Min}\Big\{ \int_{\Omega^h} \mathcal{W}_*(\widehat{e(v)}) + \bar{\mu}_\perp \mathcal{W}_\perp(e(v)^\perp)\,\mathrm{d}x - \int_{\Omega^h} f^h \cdot v\,\mathrm{d}x - \int_{\Gamma_+^h \cup \Gamma_-^h} g^h \cdot v\,\mathrm{d}\mathfrak{h}_2\,;\ v \in V_{RM}(\Omega^h)\Big\}$$

$$V_{RM}(\Omega^h) := \Big\{ v \in H^1_{\Gamma^h_{\mathrm{tat}}}(\Omega^h;\mathbb{R}^3)\ s.t.\ e_{33}(v) = 0,\ \partial_3 e_{\alpha 3}(v) = 0,\ 1 \le \alpha \le 2 \Big\}$$

in the sense that

$$\lim_{s\to 0} \int_{\Omega^h} |\Big(\frac{1}{\varepsilon}(\widehat{u^s - \bar{u}^s}), u_3^s - \bar{u}_3^s\Big)|^p\,\mathrm{d}x = 0,\ \forall p < \frac{3}{2}$$

The limit problem $(\overline{\mathcal{P}}^s)$ describes the equilibrium of a thin linearly elastic plate clamped along its lateral boundary and subjected to body forces f^h and surface forces g^h on its upper and lower boundaries with an imposed Reissner–Mindlin kinematics. This problem is basically a two-dimensional one set on the middle surface ω; it accounts for shear effects and is easy to solve numerically by a meshing of the sole ω.

3. Concluding remarks

Obviously, the large discrepancy between the magnitudes of the stiffness coefficients $\hat{\mu}$ and $\mu_\perp$ of the adhesive generates the Reissner–Mindlin limit kinematics of the structure. Also such a result can be obtained by more general distributions of adhesives and heterogeneous adherents. As problem $(\mathcal{P})$ has a unique solution with (f, g) in $L^2(\Omega \times \{\Gamma_+ \cup \Gamma_-\}; \mathbb{R}^3)$, one can insert in s an additional parameter of approximation for the loading. It is worthwhile to note that our study also supplies a model for thin elastic masonries (see [15]).

Eventually, in another context, the stage ε goes to zero shows that a two-dimensional rotational flow can be the limit of a distribution of irrotational flows in Y_ε^i with suitable sliding conditions on the cells boundaries.

Acknowledgements

The authors wish to express their gratitude to the inspirational atmospheres of Tsakhkadzor, Adelaïde, les citadelles éternelles du Larzac, the eucalyptus trees of Ban Nong Hin, and Clarmont d'Erau, to mention but a few magic places where Poetry dovetails with Science.

References

[1] E. Reissner, On the theory of bending of elastic plates, J. Math. Phys. 23 (1945) 184–191.
[2] E. Reissner, The effect of transverse shear deformations on the bending of elastic plates, J. Appl. Mech. 12 (1945) A69–A77.
[3] R.D. Mindlin, Influence of rotary inertia and shear on flexural motions of isotropic elastic plate, J. Appl. Mech. 18 (1951) 31–38.
[4] D.N. Arnold, A.L. Madureira, S. Zhang, On the range of applicability of the Reissner–Mindlin and Kirchhoff–Love plate bending models, J. Elast. 67 (2002) 171–185.
[5] P.G. Ciarlet, Mathematical Elasticity, vol. II: Theory of Plates, North-Holland, Elsevier, 1997.
[6] O. Iosifescu, C. Licht, G. Michaille, Nonlinear boundary conditions in Kirchhoff–Love plate theory, J. Elast. 96 (2009) 57–79.
[7] R. Paroni, P. Podio-Guidugli, G. Tomassetti, A justification of the Reissner–Mindlin plate theory through variational convergence, Anal. Appl. 5 (2007) 165–182.
[8] M. Serpilli, F. Krasucki, G. Geymonat, An asymptotic strain gradient Reissner–Mindlin plate model, Meccanica 48 (2013) 2007–2018.
[9] P. Neff, K.-I. Hong, J. Jeong, The Reissner–Mindlin plate is the Γ-limit of Cosserat elasticity, Math. Models Methods Appl. Sci. 20 (2010) 1553–1590.
[10] C. Licht, Asymptotic modeling of assemblies of thin linearly elastic plates, C. R. Mecanique 335 (2007) 775–780.
[11] R. Temam, Problèmes Mathématiques en Plasticité, Gauthier-Villars, Paris, 1983.

[12] H. Attouch, G. Buttazzo, G. Michaille, Variational Analysis in Sobolev and BV Spaces: Applications to PDEs and Optimization, MPS–SIAM Series on Optimization, vol. 6, Society for Industrial and Applied Mathematics, Philadelphia, PA, USA, 2005.
[13] C. Licht, A. Léger, S. Orankitjaroen, A. Ould Khaoua, Dynamics of elastic bodies connected by a thin soft viscoelastic layer, J. Math. Pures Appl. 99 (2013) 685–703.
[14] C. Licht, G. Michaille, A modelling of elastic adhesive bonded joints, Adv. Math. Sci. Appl. 7 (1997) 711–740.
[15] A. Cecchi, K. Sab, A homogenized Reissner–Mindlin model for orthotropic periodic plates: application to brickwork panels, Int. J. Solids Struct. 44 (2007) 6055–6079.