



**HAL**  
open science

## An asymptotic Reissner–Mindlin plate model

Christian Licht, Thibaut Weller

► **To cite this version:**

Christian Licht, Thibaut Weller. An asymptotic Reissner–Mindlin plate model. *Comptes Rendus Mécanique*, 2018, 346 (6), pp.432 - 438. 10.1016/j.crme.2018.04.014 . hal-01787527

**HAL Id: hal-01787527**

**<https://hal.umontpellier.fr/hal-01787527>**

Submitted on 7 May 2018

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Distributed under a Creative Commons Attribution - NonCommercial - NoDerivatives 4.0 International License



# An asymptotic Reissner–Mindlin plate model

Christian Licht<sup>a,b,c</sup>, Thibaut Weller<sup>a,\*</sup>

<sup>a</sup> LMGC, Université de Montpellier, CNRS, Montpellier, France

<sup>b</sup> Department of Mathematics, Faculty of Science, Mahidol University, Bangkok 10400, Thailand

<sup>c</sup> Centre of Excellence in Mathematics, CHE, Bangkok 10400, Thailand

## ARTICLE INFO

### Article history:

Received 6 April 2018

Accepted 15 April 2018

Available online 30 April 2018

### Keywords:

Reissner–Mindlin plate model

Periodic abutting of thin plates

Asymptotic modeling

Variational convergence

Space of bounded deformations

## ABSTRACT

A mathematical study via variational convergence of a periodic distribution of classical linearly elastic thin plates softly abutted together shows that it is not necessary to use a different continuum model nor to make constitutive symmetry hypothesis as starting points to deduce the Reissner–Mindlin plate model.

© 2018 Académie des sciences. Published by Elsevier Masson SAS. This is an open access article under the CC BY-NC-ND license

(<http://creativecommons.org/licenses/by-nc-nd/4.0/>).

## 1. Introduction

Due to its ability to account for shear effects, the Reissner–Mindlin plate model [1–3] is often preferred in the engineering literature (see [4]) over the Kirchhoff–Love plate model. So, as done for the Kirchhoff–Love plate model [5,6], it is challenging to proceed with a rigorous mathematical derivation of the Reissner–Mindlin plate model by studying the asymptotic behavior of a thin 3-dimensional elastic body when its thickness goes to zero. This was done in [7–9] by using a second gradient or Cosserat continuum for the body jointly with constitutive symmetry assumptions; here – being aware of the results of [10] on the bonding of thin plates – we prefer to consider a strongly heterogeneous *classical linearly elastic* body made of a periodic distribution of thin *anisotropic* plates abutted together.

Let  $\{e_1, e_2, e_3\}$  be an orthonormal basis of  $\mathbb{R}^3$  assimilated to the Euclidean physical space. For all  $\xi = (\xi_1, \xi_2, \xi_3)$  in  $\mathbb{R}^3$ ,  $\hat{\xi}$  stands for  $(\xi_1, \xi_2)$ . The space of all  $(N \times N)$  symmetric matrices is denoted by  $\mathbb{S}^N$  and equipped with the usual inner product and norm denoted by  $\cdot$  and  $|\cdot|$ , as in  $\mathbb{R}^3$ . For all  $e$  in  $\mathbb{S}^3$ , we set

$$e = \hat{e} + e^\perp \quad (1)$$

where  $(\hat{e})_{\alpha\beta} = e_{\alpha\beta}$  and  $(e^\perp)_{\alpha\beta} = 0$ ,  $1 \leq \alpha, \beta \leq 2$ , and  $(\hat{e})_{i3} = 0$ ,  $(e^\perp)_{i3} = e_{i3}$ ,  $1 \leq i \leq 3$ . For all  $a, b$  in  $\mathbb{R}^3$ ,  $a \otimes_s b$  stands for the symmetrized tensor product of  $a$  by  $b$ . Moreover, for all subset  $\mathcal{O}$  of  $\mathbb{R}^N$ ,  $\chi_{\mathcal{O}}$  is the characteristic function of  $\mathcal{O}$ . Finally, we will use the symbol  $\mathfrak{h}_n$  to denote  $n$ -dimensional Hausdorff measure and the letter  $C$  to introduce various constants which may vary from line to line.

Let  $\omega$  be a domain of  $\mathbb{R}^2$  with a Lipschitz continuous boundary  $\partial\omega$  and  $2\eta_0, \varepsilon_0$  two positive real numbers lesser than 1. Let  $Y := (0, 1)^2$ ,  $Y_\eta^{\text{ext}} := (-\eta, 1 + \eta)^2$ ,  $Y_\eta^{\text{int}} := (\eta, 1 - \eta)^2$ ,  $I_\varepsilon := \{i \in \mathbb{Z}^2; \varepsilon(i + Y_\eta^{\text{ext}}) \subset \omega\}$  for all  $\eta$  in  $(0, \eta_0)$  and

\* Corresponding author.

E-mail addresses: [christian.licht@umontpellier.fr](mailto:christian.licht@umontpellier.fr) (C. Licht), [thibaut.weller@umontpellier.fr](mailto:thibaut.weller@umontpellier.fr) (T. Weller).

$$\omega_{\eta,\varepsilon}^{\text{int}} := \bigcup_{i \in I_\varepsilon} \varepsilon(i + Y_\eta^{\text{int}}), \quad \omega_{\eta,\varepsilon}^{\text{ext}} := \omega \setminus \bigcup_{i \in I_\varepsilon} (i + \overline{Y_\eta^{\text{ext}}})$$

$$\omega_{\eta,\varepsilon} := \omega_{\eta,\varepsilon}^{\text{int}} \cup \omega_{\eta,\varepsilon}^{\text{ext}}, \quad b_{\eta,\varepsilon} := \omega \setminus \overline{\omega_{\eta,\varepsilon}}$$

Let  $h$  be a small positive number, we will consider a structure occupying  $\Omega^h := \omega \times (-h, h)$  made of an  $\varepsilon Y$ -periodic distribution of thin linearly elastic plates inhabiting  $\Omega_{\eta,\varepsilon}^{h \text{ int}} := \omega_{\eta,\varepsilon}^{\text{int}} \times (-h, h)$  abutted together through a thin and narrow soft linearly elastic adhesive layer filling  $B_{\eta,\varepsilon}^h := b_{\eta,\varepsilon} \times (-h, h)$  and surrounded by a thin linearly hollow plate occupying  $\Omega_{\eta,\varepsilon}^{h \text{ ext}} := \omega_{\eta,\varepsilon}^{\text{ext}} \times (-h, h)$ . We set  $\Omega_{\eta,\varepsilon}^h := \Omega_{\eta,\varepsilon}^{h \text{ int}} \cup \Omega_{\eta,\varepsilon}^{h \text{ ext}}$  and assume that all the constituents of the structure are perfectly bonded together.

For brevity and simplicity, we assume that the structure is subjected to body and surface forces on its upper/lower boundary  $\Gamma_\pm^h := \omega \times \{\pm h\}$  of densities  $f^h, g^h$ , respectively, and, as in [7,8], is clamped on its lateral boundary  $\Gamma_{\text{tat}}^h := \partial\omega \times (-h, h)$ . The strain energy  $\mathcal{W}_s$  of the body reads as:

$$\mathcal{W}_s(x, e) := \begin{cases} \mathcal{W}(e) \text{ a.e. } x \in \Omega_{\eta,\varepsilon}^h \\ \mu_\wedge \mathcal{W}_\wedge(\hat{e}) + \mu_\perp \mathcal{W}_\perp(e^\perp) \text{ a.e. } x \in B_{\eta,\varepsilon}^h \end{cases}$$

for all  $e$  in  $\mathbb{S}^3$ ,  $\mathcal{W}, \mathcal{W}_\wedge, \mathcal{W}_\perp$  being positive definite quadratic functions on  $\mathbb{S}^3$ .

Hence the equilibrium of the structure involves a quintuplet of data  $s := (s', \varepsilon)$ ,  $s' := (\mu_\wedge, \mu_\perp, \eta, h)$  and reads as:

$$(\mathcal{P}^s) \quad \text{Min} \left\{ \int_{\Omega^h} \mathcal{W}_s(x, e(v)) \, dx - \int_{\Omega^h} f^h(x) \cdot v(x) \, dx - \int_{\Gamma_+^h \cup \Gamma_-^h} g^h(x) \cdot v(x) \, dh_2; v \in H_{\Gamma_{\text{tat}}^h}^1(\Omega^h; \mathbb{R}^3) \right\}$$

where  $e(v)$  is the strain tensor associated with the displacement  $v$  and, in the sequel, for all domain  $\mathcal{O}$  in  $\mathbb{R}^N$  and all smooth part  $\gamma$  of its boundary  $\partial\mathcal{O}$ ,  $H_\gamma^1(\mathcal{O}; \mathbb{R}^N)$  denotes the subspace of  $H^1(\mathcal{O}; \mathbb{R}^N)$  made of the elements with vanishing trace on  $\gamma$ . Clearly, if  $(f^h, g^h)$  belongs to  $L^2(\Omega^h \times (\Gamma_+^h \cup \Gamma_-^h); \mathbb{R}^3)$ ,  $(\mathcal{P}^s)$  has a unique solution  $u^s$  and, considering the data  $s$  as a parameter, we are interested in its asymptotic behavior when  $s$  takes values in a countable subset of  $(0, +\infty)^2 \times (0, \eta_0) \times (0, +\infty) \times (0, \varepsilon_0)$  with  $0$  as a unique limit point. Like in the mathematical derivation of the Kirchhoff–Love theory of plates [5,6], it is convenient to introduce the linear mappings  $\Pi^h$  and  $S_h$ :

$$\xi = (\hat{\xi}, \xi_3) \in \mathbb{R}^3 \mapsto \Pi^h \xi = (\hat{\xi}, h\xi_3)$$

$$v \in L^1(\Omega^h; \mathbb{R}^3) \mapsto S_h v \in L^1(\Omega; \mathbb{R}^3) \text{ s.t. } (S_h v)(x) = \frac{1}{h} \Pi^h(v(\Pi^h x)), \forall x \in \Omega := \omega \times (-1, 1)$$

We make the following assumption on the loading:

$$(H_1) \quad \begin{cases} \exists (f, f', g) \in L^3(\Omega; \mathbb{R}^3) \times L^3(\Omega; \mathbb{R}^3) \times L^\infty(\Gamma^+ \cup \Gamma^-; \mathbb{R}^3) \text{ s.t.} \\ f^h(\Pi^h x) = h(\chi_{\Omega_{\eta,\varepsilon}^h} \Pi^h f(x) + \chi_{B_{\eta,\varepsilon}^h} \Pi^h f'(x)), \forall x \in \Omega \\ g^h(\Pi^h x) = h^2 \Pi^h \chi_{\Gamma_{\eta,\varepsilon,+} \cup \Gamma_{\eta,\varepsilon,-}} g(x), \forall x \in \Gamma_{\eta,\varepsilon,+} \cup \Gamma_{\eta,\varepsilon,-} \end{cases}$$

where  $\Gamma_{\eta,\varepsilon,\pm} := \omega_{\eta,\varepsilon} \times \{\pm 1\}$ . Therefore,  $u_s := S_h u^s$  is the unique solution to

$$(\mathcal{P}_s) \quad \text{Min} \left\{ \mathcal{J}_s(v); v \in H_{\Gamma_{\text{tat}}}^1(\Omega; \mathbb{R}^3) \right\}$$

where

$$\mathcal{J}_s(v) := \int_{\Omega} \mathcal{W}_s(x, e(h, v)) - (\chi_{\Omega_{\eta,\varepsilon}} f + \chi_{B_{\eta,\varepsilon}} f') \cdot v \, dx - \int_{\Gamma_{\eta,\varepsilon,+} \cup \Gamma_{\eta,\varepsilon,-}} g \cdot v \, dh_2$$

$$e(h, v)_{\alpha\beta} := e(v)_{\alpha\beta}, \quad e(h, v)_{\alpha 3} := \frac{1}{h} e(v)_{\alpha 3}, \quad 1 \leq \alpha, \beta \leq 2, \quad e(h, v)_{33} := \frac{1}{h^2} e(v)_{33}$$

with  $\Gamma_{\text{tat}}$  the reciprocal image by  $\Pi^h$  of  $\Gamma_{\text{tat}}^h$  and, similarly, index  $h$  will be dropped for the image by  $\Pi^{h^{-1}}$  of  $\Omega_{\eta,\varepsilon}^h, B_{\eta,\varepsilon}^h$  and  $\Gamma_\pm^h$ .

## 2. A convergence result

Let the space of Kirchhoff–Love and Reissner–Mindlin displacements be respectively denoted by

$$\begin{aligned}
 V_{KL}(\Omega) &:= \left\{ v \in H^1_{\Gamma_{\text{tat}}}(\Omega; \mathbb{R}^3); \exists (v^M, v^F) \in H^1(\omega; \mathbb{R}^2) \times H^2(\omega) \text{ s.t. } \hat{v}(x) = v^M(\hat{x}) - x_3 \widehat{\nabla v^F}(\hat{x}), \right. \\
 &\qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \left. v_3(x) = v^F(\hat{x}) \text{ a.e. } x \in \Omega \right\} \\
 &= \left\{ v \in H^1_{\Gamma_{\text{tat}}}(\Omega; \mathbb{R}^3) \text{ s.t. } e_{i3}(v) = 0, 1 \leq i \leq 3 \right\} \\
 V_{RM}(\Omega) &:= \left\{ v \in H^1_{\Gamma_{\text{tat}}}(\Omega; \mathbb{R}^3); \exists (v^M, v^\theta, v^F) \in H^1(\omega; \mathbb{R}^2) \times H^1(\omega; \mathbb{R}^2) \times H^1(\omega) \text{ s.t. } \hat{v}(x) = v^M(\hat{x}) + x_3 v^\theta(\hat{x}), \right. \\
 &\qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \left. v_3(x) = v^F(\hat{x}) \text{ a.e. } x \in \Omega \right\} \\
 &= \left\{ v \in H^1_{\Gamma_{\text{tat}}}(\Omega; \mathbb{R}^3) \text{ s.t. } e_{33}(v) = 0, \partial_3 e_{\alpha 3}(v) = 0, 1 \leq \alpha \leq 2 \right\}
 \end{aligned}$$

We recall [11,12] that the space of bounded deformations is defined by

$$BD(\Omega) := \left\{ v \in L^1(\Omega; \mathbb{R}^3); e(v) \in M_b(\Omega; \mathbb{S}^3) \right\}$$

where  $M_b(\Omega; \mathbb{S}^3)$  denotes the space of bounded  $\mathbb{S}^3$ -valued measures in  $\Omega$ .

We assume

$$(H_2) \quad : \quad \exists (\bar{\mu}_\wedge, \bar{\mu}_\perp) \in (0, +\infty)^2 \text{ s.t. } (\bar{\mu}_\wedge, \bar{\mu}_\perp) := \lim_{s' \rightarrow 0} \frac{1}{2\eta} \left( \mu_\wedge, \frac{1}{h^2} \mu_\perp \right)$$

In the sequel, when no ambiguity ensues, we use the same symbol  $\hat{e}$  for an element of  $\mathbb{S}^2$ , such that its entries are the non-vanishing ones of the element  $\hat{e}$  of  $\mathbb{S}^3$ . For instance, for all  $(q, e)$  in  $\mathbb{S}^2 \times \mathbb{S}^3$ ,  $q = \hat{e}$  means that  $q_{\alpha\beta} = e_{\alpha\beta}$  for all  $\alpha, \beta$  in  $\{1, 2\}$ .

Let the two positive definite quadratic functions on  $\mathbb{S}^2$  defined by:

$$\begin{aligned}
 \mathcal{W}_{KL}(q) &= \text{Min} \left\{ \mathcal{W}(e); e \in \mathbb{S}^3 \text{ s.t. } \hat{e} = q \right\} \\
 \mathcal{W}_*(q) &= \text{Min} \left\{ \mathcal{W}_{KL}(q^1) + \bar{\mu}_\wedge \mathcal{W}_\wedge(q^2) + \bar{\mu}_\perp \mathcal{W}_\perp(q^3); q^i \in \mathbb{S}^2, 1 \leq i \leq 3, (q^2)_{22} = (q^3)_{11} = 0, q^1 + q^2 + q^3 = q \right\}
 \end{aligned}$$

Then we have the following theorem.

**Theorem 2.1.** *Under assumptions  $(H_1)$  and  $(H_2)$ , as  $s'$  goes to 0 and then  $\varepsilon$  goes to 0,  $u_\varepsilon$  converges weak  $\star$  in  $BD(\Omega)$  toward the unique solution  $u$  to*

$$(\mathcal{P}) \quad \text{Min} \left\{ \int_\Omega \mathcal{W}_*(\widehat{e(v)}) \, dx + \bar{\mu}_\perp \int_\Omega \mathcal{W}_\perp(e(v)^\perp) \, dx - \int_\Omega f \cdot v \, dx - \int_{\Gamma_+ \cup \Gamma_-} g \cdot v \, d\mathfrak{h}_2; v \in V_{RM}(\Omega) \right\}$$

The proof, which uses a standard method of variational convergence, is achieved in three steps.

Step 1 (asymptotic behavior of  $u_\varepsilon$  as  $s'$  goes to zero):

Let  $\sigma_\varepsilon := \bigcup_{i \in I_\varepsilon} \varepsilon(i + \partial Y)$ ,  $\Sigma_\varepsilon := \sigma_\varepsilon \times (-1, 1)$ ,  $\dot{\Omega}_\varepsilon := \Omega \setminus \Sigma_\varepsilon$ ,  $\nu$  the unit normal to  $\Sigma_\varepsilon$  (or  $\sigma_\varepsilon$ ) equal to either  $e_1$  or  $e_2$  and  $\Sigma_\varepsilon^k$  the part of  $\Sigma_\varepsilon$  such that  $\nu = e_k$ ,  $k = 1, 2$ . For all  $w$  in  $H^1(\dot{\Omega}_\varepsilon; \mathbb{R}^3)$ , we denote by  $[w]$  the jump of  $w$  across the surface  $\Sigma_\varepsilon$  oriented by  $\nu$ .

**Proposition 2.1.** *When  $s'$  goes to 0,  $u_\varepsilon$  (up to a not relabeled subsequence) converges weak  $\star$  in  $BD(\Omega)$  toward some  $u_\varepsilon$  such that*

$$u_\varepsilon \in V_{KL}(\dot{\Omega}_\varepsilon) := \left\{ v \in H^1_{\Gamma_{\text{tat}}}(\dot{\Omega}_\varepsilon; \mathbb{R}^3) \text{ s.t. } \dot{e}_\varepsilon(v)^\perp = 0 \right\}$$

where  $\dot{e}_\varepsilon$  denotes the density with respect to the Lebesgue measure of the symmetrized gradient in the sense of distribution in  $\dot{\Omega}_\varepsilon$ , and with:

$$\begin{aligned} \mathcal{J}_\varepsilon(u_\varepsilon) := & \int_{\hat{\Omega}_\varepsilon} \mathcal{W}_{KL}(\widehat{e}_\varepsilon(u_\varepsilon)) \, dx + \frac{\bar{\mu}_\wedge}{\varepsilon} \int_{\Sigma_\varepsilon} \mathcal{W}_\wedge([\widehat{u}_\varepsilon] \otimes_s \nu) \, dh_2 + \frac{\bar{\mu}_\perp}{\varepsilon} \int_{\Sigma_\varepsilon} \mathcal{W}_\perp([u_\varepsilon]_3 e_3 \otimes_s \nu) \, dh_2 \\ & - \int_{\Omega} f \cdot u_\varepsilon \, dx - \int_{\Gamma_+ \cup \Gamma_-} g \cdot u_\varepsilon \, dh_2 \leq \liminf_{s' \rightarrow 0} \mathcal{J}_s(u_s) \end{aligned}$$

**Proof.** First, ellipticity assumptions on  $\mathcal{W}$ ,  $\mathcal{W}_\wedge$ ,  $\mathcal{W}_\perp$ , Hölder inequality and  $(H_1 - H_2)$  imply that  $u_s$  is bounded in  $LD(\Omega) = \{u \in L^1(\Omega; \mathbb{R}^3); e(u) \in L^1(\Omega; \mathbb{S}^3)\}$ . Thus, up to a not relabeled subsequence,  $u_s$  weak  $\star$  converges in  $BD(\Omega)$  toward some  $u_\varepsilon$ , and consequently weakly in  $L^{3/2}(\Omega; \mathbb{R}^3)$  and strongly in  $L^p(\Omega; \mathbb{R}^3)$  for all  $p < \frac{3}{2}$  (see [11,12]). Moreover, as there exists  $\delta(\varepsilon)$  such that  $u_s$  is bounded in  $H^1_{\Gamma_{\text{tat}}}(\Omega_\delta; \mathbb{R}^3)$ ,  $\Omega_\delta := \omega_\delta \times (-1, 1)$ ,  $\omega_\delta := \{\hat{x} \in \omega; \text{dist}(\hat{x}, \partial\omega) < \delta\}$ , the trace on  $\Gamma_{\text{tat}}$  of  $u_\varepsilon$  vanishes. As  $\chi_{\Omega_{\eta,\varepsilon}} e(u_s)$  is bounded in  $L^2(\Omega; \mathbb{S}^3)$ , it converges, up to a not relabeled subsequence, to some  $q$ , which, by testing by any element of  $\mathcal{D}(\hat{\Omega}_\varepsilon; \mathbb{S}^3)$ , is clearly identified as  $\widehat{e}_\varepsilon(u_\varepsilon)$  with  $\widehat{e}_\varepsilon(u_\varepsilon)^\perp = 0$ . Hence,  $u_\varepsilon$  belongs to  $V_{KL}(\hat{\Omega}_\varepsilon)$  ( $:= \{v \in H^1_{\Gamma_{\text{tat}}}(\hat{\Omega}_\varepsilon; \mathbb{R}^3); \widehat{e}_\varepsilon(v)^\perp = 0\}$ ). As for all  $\tau$  in  $\mathcal{D}(\Omega; \mathbb{S}^3)$

$$\int_{B_{\eta,\varepsilon}} \tau \cdot e(u_s) \, dx = - \int_{\Omega} (\text{div } \tau \cdot u_s + \tau \cdot \chi_{\Omega_{\eta,\varepsilon}} e(u_s)) \, dx \tag{2}$$

we deduce:

$$e(u_\varepsilon) = \widehat{e}_\varepsilon(u_\varepsilon) \, dx + [u_\varepsilon] \otimes_s \nu \, dh_2 \llcorner \Sigma_\varepsilon \tag{3}$$

Next let  $\partial_\eta Y := \{\hat{x} \in \partial Y \text{ s.t. } x_1 \text{ or } x_2 \in (\eta, 1 - \eta)\}$  and  $\Sigma_{\eta,\varepsilon} := \bigcup_{i \in I_\varepsilon} \varepsilon(i + \partial_\eta Y)$ . Jensen inequality implies

$$\begin{aligned} \liminf_{s' \rightarrow 0} \mu_\wedge \int_{B_{\eta,\varepsilon}} \mathcal{W}_\wedge(\widehat{e}(\varepsilon, u_s)) \, dx & \geq \liminf_{s' \rightarrow 0} \frac{\mu_\wedge}{2\eta\varepsilon} \int_{\Sigma_{\eta,\varepsilon}} \mathcal{W}_\wedge \left( \int_{-\eta\varepsilon}^{\eta\varepsilon} \widehat{e}(u_s)(x + tv) \, dt \right) \, dh_2 \geq \frac{\bar{\mu}_\wedge}{\varepsilon} \int_{\Sigma_\varepsilon} \mathcal{W}_\wedge([\widehat{u}_\varepsilon] \otimes_s \nu) \, dh_2 \\ \liminf_{s' \rightarrow 0} \mu_\perp \int_{B_{\eta,\varepsilon}} \mathcal{W}_\perp(e(\varepsilon, u_s)^\perp) \, dx & \geq \liminf_{s' \rightarrow 0} \frac{\mu_\perp}{2\eta\varepsilon h^2} \int_{\Sigma_{\eta,\varepsilon}} \mathcal{W}_\perp \left( \int_{-\eta\varepsilon}^{\eta\varepsilon} e(u_s)^\perp(x + tv) \, dt \right) \, dh_2 \geq \frac{\bar{\mu}_\perp}{\varepsilon} \int_{\Sigma_\varepsilon} \mathcal{W}_\perp([u_\varepsilon]_3 e_3 \otimes_s \nu) \, dh_2 \end{aligned}$$

because, by arguing similarly to (2), we get that  $\chi_{\Sigma_{\eta,\varepsilon}} \int_{-\eta\varepsilon}^{\eta\varepsilon} e(u_s)(\cdot + tv) \, dt$  weakly converges in  $L^2(\Sigma_\varepsilon; \mathbb{S}^3)$  toward  $[u_\varepsilon] \otimes_s \nu$ . Moreover, arguing as in [13] through the introduction of cut-off functions and translations by  $\pm \eta \varepsilon e_k$ ,  $k = 1, 2$ , and by  $\pm \eta \varepsilon (e_1 \pm e_2)$  yields  $\lim_{s' \rightarrow 0} \int_{\Gamma_{\eta,\varepsilon,\pm}} g \cdot u_s \, dh_2 = \int_{\Gamma_\pm} g \cdot u_\varepsilon \, dh_2$ . Hence, the desired result stems from the very definition of  $\mathcal{W}_{KL}$ , the weak convergence in  $L^2(\Omega, \mathbb{S}^3)$  of  $\chi_{\Omega_{\varepsilon,\eta}} \widehat{e}(u_s)$  toward  $\widehat{e}_\varepsilon(u_\varepsilon)$  and the weak  $\star$  convergence in  $BD(\Omega)$  of  $u_s$ .  $\square$

Step 2 (asymptotic behavior of  $u_\varepsilon$  as  $\varepsilon$  goes to zero):

Let  $Y_\varepsilon^i := \varepsilon(i + Y)$ ,  $\Sigma_\varepsilon^{k,i} := \{x \in \Sigma_\varepsilon^k; x_k = \varepsilon(i_k + 1), x_{3-k} \in \varepsilon(i_{3-k}, i_{3-k} + 1)\}$  for all  $i$  in  $I_\varepsilon$ , and  $\dot{\Sigma}_\varepsilon^k := \bigcup_{i \in I_\varepsilon} \Sigma_\varepsilon^{k,i}$ . Then, for all  $\psi_\varepsilon^k$  in  $L^2(\Sigma_\varepsilon^k)$  such that  $\frac{1}{\varepsilon} \int_{\Sigma_\varepsilon^k} (\psi_\varepsilon^k)^2 \, dh_2 < C$ , the function  $\Psi_\varepsilon^k := \mathcal{L}_\varepsilon^k \psi_\varepsilon^k$  defined by  $\mathcal{L}_\varepsilon^k(\psi_\varepsilon^k)(x) = \frac{1}{\varepsilon} \sum_{i \in I_\varepsilon} \psi_\varepsilon^k(x_{3-k}, x_3) \chi_{Y_\varepsilon^i}(\hat{x})$  is bounded in  $L^2(\Omega)$ . Note that, as  $h_2(\Sigma_\varepsilon^k \setminus \dot{\Sigma}_\varepsilon^k)$  remains bounded, we have  $\lim_{\varepsilon \rightarrow 0} \int_{\Sigma_\varepsilon^k} \tau \psi_\varepsilon^k \, dh_2 = \lim_{\varepsilon \rightarrow 0} \int_{\Omega} \tau \Psi_\varepsilon^k \, dx$  for all  $\tau$  in  $\mathcal{C}^0(\overline{\Omega})$ .

**Proposition 2.2.** When  $\varepsilon$  goes to zero,  $u_\varepsilon$ , up to a not relabeled subsequence, converges for the intermediate topology of  $BD(\Omega)$  (see [11,12]) toward some  $u$  such that

$$u \in V_{RM}(\Omega), \quad \mathcal{J}(u) := \int_{\Omega} \mathcal{W}_*(\widehat{e}(u)) \, dx + \bar{\mu}_\perp \int_{\Omega} \mathcal{W}(e(u)^\perp) \, dx - \int_{\Omega} f \cdot u \, dx - \int_{\Gamma_+ \cup \Gamma_-} g \cdot u \, dh_2 \leq \liminf_{\varepsilon \rightarrow 0} \mathcal{J}_\varepsilon(u_\varepsilon)$$

**Proof.** As  $\mathcal{J}_\varepsilon(u_\varepsilon)$  is bounded,  $u_\varepsilon$  is bounded in  $BD(\Omega)$  and then, up to a subsequence, weak  $\star$  converges toward some  $u$ . Moreover as  $u_\varepsilon \in V_{KL}(\dot{\Omega}_\varepsilon)$ ,  $(u_\varepsilon^M, u_\varepsilon^F)$  converges weakly in  $L^2(\omega; \mathbb{R}^2) \times L^2(\omega)$  and the measure  $e(u_\varepsilon)$  has a narrow limit which can be identified with an element of  $L^2(\Omega; \mathbb{S}^3)$  such that  $e(u)_{33} = 0$  and  $\partial_3 e(u)^\perp = 0$ .  $\square$

Step 3 (identification of the limit field  $u$ ):

For all  $(\mathcal{Y}, \mathcal{E})$  in  $\{Y_\varepsilon^i, Y_\varepsilon^i + \varepsilon e_k\} \times \{\mathbb{R}^N, \mathbb{S}^N\}$  and  $\psi$  in  $L^2(\mathcal{Y}, \mathcal{E})$ , we set  $\langle \psi \rangle_{\mathcal{Y}} = \frac{1}{\varepsilon^2} \int_{\mathcal{Y}} \psi(\hat{x}) d\hat{x}$  and let  $\hat{x}^i = \varepsilon(\frac{1}{2}, \frac{1}{2}) + i$ .

**Proposition 2.3.** For all  $v$  in  $V_{RM}(\Omega)$ , there exists a sequence  $v_\varepsilon$  in  $H^1_{\Gamma_{\text{lat}}}(\Omega; \mathbb{R}^3)$  such that, when  $\varepsilon$  goes to zero,  $v_\varepsilon$  converges for the intermediate topology of  $BD(\Omega)$  toward  $v$  with  $\overline{\lim}_{\varepsilon \rightarrow 0} \mathcal{J}_\varepsilon(v_\varepsilon) \leq \mathcal{J}(v)$ , and there exists a sequence  $v_s$  in  $H^1_{\Gamma_{\text{lat}}}(\Omega; \mathbb{R}^3)$  such that, when  $s'$  goes to zero,  $v_s := v_{s', \varepsilon}$  converges weak  $\star$  in  $BD(\Omega)$  toward  $v_\varepsilon$  with  $\overline{\lim}_{s' \rightarrow 0} \mathcal{J}_s(v_s) \leq \mathcal{J}_\varepsilon(v_\varepsilon)$ .

**Proof.** Through a classical density and diagonalization argument, we may assume that  $v^M, v^\theta, v^F$  belong to  $\mathcal{D}(\omega; \mathbb{R}^2)$  and

$$\mathcal{W}_*(\widehat{e}(v)) = \mathcal{W}_{KL}(\widehat{e}(v) - (\varphi^1 + \varphi^2)) + \bar{\mu}_\wedge \mathcal{W}_\wedge(\varphi^1) + \bar{\mu}_\vee \mathcal{W}_\vee(\varphi^2)$$

with  $q := \widehat{e}(v) - (\varphi^1 + \varphi^2) = q^M(\hat{x}) + x_3 q^\theta(\hat{x})$  and  $q^M, q^\theta$  in  $\mathcal{D}(\omega; \mathbb{S}^2)$ .

Let  $v_\varepsilon$  the element of  $V_{KL}(\dot{\Omega}_\varepsilon)$  defined by:

$$\begin{aligned} \widehat{v}_\varepsilon(x) &:= v_\varepsilon^M(\hat{x}) + x_3 v_\varepsilon^\theta(\hat{x}) \\ v_\varepsilon^M(\hat{x}) &:= \sum_{i \in I_\varepsilon} \left( \langle v^M \rangle_{Y_\varepsilon^i} + \langle q^M \rangle_{Y_\varepsilon^i} (\hat{x} - \hat{x}^i) \right) \chi_{Y_\varepsilon^i}(\hat{x}) \\ v_\varepsilon^\theta(\hat{x}) &:= \sum_{i \in I_\varepsilon} \left( \langle v^\theta \rangle_{Y_\varepsilon^i} + \langle q^\theta \rangle_{Y_\varepsilon^i} (\hat{x} - \hat{x}^i) \right) \chi_{Y_\varepsilon^i}(\hat{x}) \\ (v_\varepsilon)_3(x) &:= \sum_{i \in I_\varepsilon} \left[ \langle v^F \rangle_{Y_\varepsilon^i} - \left( \langle v^\theta \rangle_{Y_\varepsilon^i} + \frac{1}{2} \langle q^\theta \rangle_{Y_\varepsilon^i} (\hat{x} - \hat{x}^i) \right) \cdot (\hat{x} - \hat{x}^i) \right] \chi_{Y_\varepsilon^i}(\hat{x}) \end{aligned}$$

The required smoothness of  $v^M, v^\theta, v^F, q^M$  and  $q^\theta$  implies that  $v_\varepsilon$  converges strongly in  $L^2(\Omega; \mathbb{R}^2)$  toward  $v$  and that the measure  $e(v_\varepsilon)$  satisfies

$$\begin{aligned} \widehat{e}(v_\varepsilon) &= \sum_{i \in I_\varepsilon} \langle q \rangle_{Y_\varepsilon^i} \chi_{Y_\varepsilon^i} dx + \varphi_\varepsilon^1 + \varphi_\varepsilon^2 \\ \varphi_\varepsilon^k &:= [\widehat{v}_\varepsilon] \otimes_s e_k \mathfrak{h}_2 \llcorner \Sigma_\varepsilon^k = \sum_{i \in I_\varepsilon} \varphi_\varepsilon^{k,i} \mathfrak{h}_2 \llcorner \Sigma_\varepsilon^{k,i} \end{aligned}$$

with

$$\begin{aligned} \varphi_\varepsilon^{k,i} &= A_\varepsilon^{k,i} + B_\varepsilon^{k,i} (x_{3-k} - \frac{\varepsilon}{2}), \quad 1 \leq k \leq 2 \\ (A_\varepsilon^{k,i})_{k\alpha} &= \frac{(1 + \delta_{k\alpha})}{2} \left[ \langle (v_\varepsilon)_\alpha \rangle_{Y_\varepsilon^i + \varepsilon e_k} - \langle (v_\varepsilon)_\alpha \rangle_{Y_\varepsilon^i} - \left( \langle q_{k\alpha} \rangle_{Y_\varepsilon^i} + \langle q_{k\alpha} \rangle_{Y_\varepsilon^i + \varepsilon e_k} \right) \frac{\varepsilon}{2} \right], \quad \alpha = 1, 2 \\ (A_\varepsilon^{k,i})_{3-k, 3-k} &= 0 \\ (B_\varepsilon^{k,i})_{k\alpha} &= \frac{(1 + \delta_{k\alpha})}{2} \left[ \langle q_{k\alpha} \rangle_{Y_\varepsilon^i + \varepsilon e_k} - \langle q_{k\alpha} \rangle_{Y_\varepsilon^i} \right] \end{aligned}$$

where  $\delta_{\alpha\beta}$  stands for the Kronecker delta. Clearly the smoothness of  $q$  and  $v$  implies that the measure  $\widehat{e}(v_\varepsilon)$  converges narrowly to  $(q + \varphi_1 + \varphi_2) dx$  because  $\sum_{i \in I_\varepsilon} \langle q \rangle_{Y_\varepsilon^i} \chi_{Y_\varepsilon^i}$  converges strongly in  $L^2(\Omega; \mathbb{S}^2)$  toward  $q$  and the functions  $\varphi_\varepsilon^k =$

$\mathcal{L}_\varepsilon^k(\varphi_\varepsilon^k)$  converges strongly in  $L^2(\Omega; \mathbb{S}^2)$ . As  $e(v_\varepsilon)^\perp = \sum_{i \in I_\varepsilon} \varphi_\varepsilon^{\perp k,i} e_3 \otimes_s e_k \mathfrak{h}_2 \llcorner \Sigma_\varepsilon^{k,i}$ , with

$$\varphi_\varepsilon^{\perp k,i} = \langle v^F \rangle_{Y_\varepsilon^i + \varepsilon e_k} - \langle v^F \rangle_{Y_\varepsilon^i} + \frac{\varepsilon}{2} \left( \langle v_k^\theta \rangle_{Y_\varepsilon^i} + \langle v_k^\theta \rangle_{Y_\varepsilon^i + \varepsilon e_k} \right) + \psi_\varepsilon^{\perp k,i}$$

where  $|\psi_\varepsilon^{\perp k,i}|_{L^\infty(\Sigma_\varepsilon^{k,i})} \leq C\varepsilon^2$  for all  $i$  in  $I_\varepsilon$  and all  $k$  in  $\{1, 2\}$ ,  $\phi_\varepsilon^{\perp k,i} = \mathcal{L}_\varepsilon^k(\varphi_\varepsilon^{\perp k,i})$  converges strongly in  $L^2(\Omega)$  toward  $\partial_k v^F + v_k^\theta = 2e(v)_{k3}$ . Hence one deduces that  $v_\varepsilon$  converges toward  $v$  for the intermediate topology of  $BD(\Omega)$  and that  $\lim_{\varepsilon \rightarrow 0} \mathcal{J}_\varepsilon(v_\varepsilon) = \mathcal{J}(v)$ .

The last stage,  $s' \rightarrow 0$ , is a similar situation of bonding of thin plates as in [10] and is achieved by using a distribution and composition of smoothing operators along  $\Sigma_\varepsilon$  like in [14,13] acting on a classical suitable approximation of Kirchhoff–Love fields as in [5,6].

Hence,  $u$  is the unique solution to  $(\mathcal{P})$  so that the whole sequences  $u_s$  and  $u_\varepsilon$  converge.  $\square$

This mathematical result can be rephrased in terms related to the genuine physical problem  $(\mathcal{P}^s)$  (Theorem 2.2).

**Theorem 2.2.** *If  $\bar{u}^s := S_\varepsilon^{-1}u$ , the field of displacement  $u^s$  is asymptotically equivalent to  $\bar{u}^s$  solution to*

$$(\bar{\mathcal{P}}^s) \quad \text{Min} \left\{ \int_{\Omega^h} \mathcal{W}_*(\widehat{e(v)}) + \bar{\mu}_\perp \mathcal{W}_\perp(e(v)^\perp) dx - \int_{\Omega^h} f^h \cdot v dx - \int_{\Gamma_+^h \cup \Gamma_-^h} g^h \cdot v dh_2; v \in V_{RM}(\Omega^h) \right\}$$

$$V_{RM}(\Omega^h) := \left\{ v \in H_{\Gamma_{\text{tat}}^h}^1(\Omega^h; \mathbb{R}^3) \text{ s.t. } e_{33}(v) = 0, \partial_3 e_{\alpha 3}(v) = 0, 1 \leq \alpha \leq 2 \right\}$$

in the sense that

$$\lim_{s \rightarrow 0} \int_{\Omega^h} \left| \left( \frac{1}{\varepsilon} (\widehat{u^s - \bar{u}^s}), u_3^s - \bar{u}_3^s \right) \right|^p dx = 0, \forall p < \frac{3}{2}$$

The limit problem  $(\bar{\mathcal{P}}^s)$  describes the equilibrium of a thin linearly elastic plate clamped along its lateral boundary and subjected to body forces  $f^h$  and surface forces  $g^h$  on its upper and lower boundaries with an imposed Reissner–Mindlin kinematics. This problem is basically a two-dimensional one set on the middle surface  $\omega$ ; it accounts for shear effects and is easy to solve numerically by a meshing of the sole  $\omega$ .

### 3. Concluding remarks

Obviously, the large discrepancy between the magnitudes of the stiffness coefficients  $\hat{\mu}$  and  $\mu_\perp$  of the adhesive generates the Reissner–Mindlin limit kinematics of the structure. Also such a result can be obtained by more general distributions of adhesives and heterogeneous adherents. As problem  $(\mathcal{P})$  has a unique solution with  $(f, g)$  in  $L^2(\Omega \times \{\Gamma_+ \cup \Gamma_-\}; \mathbb{R}^3)$ , one can insert in  $s$  an additional parameter of approximation for the loading. It is worthwhile to note that our study also supplies a model for thin elastic masonries (see [15]).

Eventually, in another context, the stage  $\varepsilon$  goes to zero shows that a two-dimensional rotational flow can be the limit of a distribution of irrotational flows in  $Y_\varepsilon^i$  with suitable sliding conditions on the cells boundaries.

### Acknowledgements

The authors wish to express their gratitude to the inspirational atmospheres of Tsakhkadzor, Adelaïde, les citadelles éternelles du Larzac, the eucalyptus trees of Ban Nong Hin, and Clarmont d'Erau, to mention but a few magic places where Poetry dovetails with Science.

### References

- [1] E. Reissner, On the theory of bending of elastic plates, *J. Math. Phys.* 23 (1945) 184–191.
- [2] E. Reissner, The effect of transverse shear deformations on the bending of elastic plates, *J. Appl. Mech.* 12 (1945) A69–A77.
- [3] R.D. Mindlin, Influence of rotary inertia and shear on flexural motions of isotropic elastic plate, *J. Appl. Mech.* 18 (1951) 31–38.
- [4] D.N. Arnold, A.L. Madureira, S. Zhang, On the range of applicability of the Reissner–Mindlin and Kirchhoff–Love plate bending models, *J. Elast.* 67 (2002) 171–185.
- [5] P.G. Ciarlet, *Mathematical Elasticity, vol. II: Theory of Plates*, North-Holland, Elsevier, 1997.
- [6] O. Iosifescu, C. Licht, G. Michaille, Nonlinear boundary conditions in Kirchhoff–Love plate theory, *J. Elast.* 96 (2009) 57–79.
- [7] R. Paroni, P. Podio-Guidugli, G. Tomassetti, A justification of the Reissner–Mindlin plate theory through variational convergence, *Anal. Appl.* 5 (2007) 165–182.
- [8] M. Serpilli, F. Krasucki, G. Geymonat, An asymptotic strain gradient Reissner–Mindlin plate model, *Meccanica* 48 (2013) 2007–2018.
- [9] P. Neff, K.-I. Hong, J. Jeong, The Reissner–Mindlin plate is the  $\Gamma$ -limit of Cosserat elasticity, *Math. Models Methods Appl. Sci.* 20 (2010) 1553–1590.
- [10] C. Licht, Asymptotic modeling of assemblies of thin linearly elastic plates, *C. R. Mecanique* 335 (2007) 775–780.
- [11] R. Temam, *Problèmes Mathématiques en Plasticité*, Gauthier-Villars, Paris, 1983.

- [12] H. Attouch, G. Buttazzo, G. Michaille, Variational Analysis in Sobolev and BV Spaces: Applications to PDEs and Optimization, MPS–SIAM Series on Optimization, vol. 6, Society for Industrial and Applied Mathematics, Philadelphia, PA, USA, 2005.
- [13] C. Licht, A. Léger, S. Orankitjaroen, A. Ould Khaoua, Dynamics of elastic bodies connected by a thin soft viscoelastic layer, *J. Math. Pures Appl.* 99 (2013) 685–703.
- [14] C. Licht, G. Michaille, A modelling of elastic adhesive bonded joints, *Adv. Math. Sci. Appl.* 7 (1997) 711–740.
- [15] A. Cecchi, K. Sab, A homogenized Reissner–Mindlin model for orthotropic periodic plates: application to brickwork panels, *Int. J. Solids Struct.* 44 (2007) 6055–6079.