Asymptotic Analysis of Stratified Elastic Media in the Space of Functions with Bounded Deformation

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ASYMPTOTIC ANALYSIS OF STRATIFIED ELASTIC MEDIA IN THE SPACE OF FUNCTIONS WITH BOUNDED DEFORMATION.

MICHEL BELLIEUD * AND SHANE COOPER †

Abstract. We consider a heterogeneous elastic structure which is stratified in one direction. We derive the limit problem under the sole assumption that the Lamé coefficients and their inverses weakly* converge to some Radon measures.

Key words. Homogenization, Singular perturbations, Elasticity

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1. Introduction. In this paper, we study the asymptotic behavior of the three-dimensional isotropic linear elasticity problem

$$\begin{aligned} &-\mathbf{div}\,(\lambda_\varepsilon \operatorname{tr}(\boldsymbol{e}(\boldsymbol{u}))\boldsymbol{I} + 2\mu_\varepsilon \boldsymbol{e}(\boldsymbol{u})) = \boldsymbol{f} \quad \text{in } \Omega, \quad \boldsymbol{u} \in H^1_0(\Omega;\mathbb{R}^3), \\ &\boldsymbol{e}(\boldsymbol{u}) := \tfrac{1}{2}(\boldsymbol{\nabla}\boldsymbol{u} + \boldsymbol{\nabla}^T\boldsymbol{u}), \end{aligned} \tag{1.1}$$

when the Lamé coefficients λ_ε, μ_ε and their inverses are bounded in $L^1(\Omega)$ and weakly* converge to some measures. We determine the limit problem in terms of these measures in the case when λ_ε and μ_ε only depend on one variable. Our results have been announced in [13].

It is well known that, when the Lamé coefficients are uniformly bounded from above and below by positive constants, the sequence of the solutions to (1.1) converges, up to a subsequence, to the solution of a problem of the form $-\mathbf{div}\boldsymbol{a}^{eff}\boldsymbol{e}(\boldsymbol{u}) = \boldsymbol{f}$ (see [37, p. 374, 4^o]). Under suitable periodicity assumptions, the effective tensor $\boldsymbol{a}^{eff}$ can be characterized by means of the theory of homogenization [16], [29], [37], [44], [53]. Diverse asymptotic analyses of (1.1) and of the associated vibration problem have been performed under various hypotheses related to geometry and periodicity when the last mentioned boundedness assumptions fail [1], [7], [8], [9], [12], [14], [15], [20], [23], [40], [45], [46], [47], [48]. In this context, stratified media have recently been investigated in [11], where a two-phase medium comprising a distribution of possibly very stiff homothetical layers alternating with much softer ones is considered. An interesting aspect of this study resides in the possible emergence of higher order derivatives (resp. non local terms) in the effective equations when the Lamé coefficients (resp. their inverses) fail to be bounded in L^1. Let us also notice that spectral properties of high contrast stratified media have been studied in [21, 22], where, in the presence of a defect, unusual phenomena of 'super-exponential' localisation of eigenmodes to the vicinity of the defect are demonstrated.

In this paper, both the elasticity coefficients and their inverses are supposed to be bounded in L^1. Apart from these boundedness conditions, we make no assumption relating to the oscillatory behavior of these coefficients. In this respect, our analysis is much more far-reaching than that developed in [11]. Unlike [11], its range of application includes both homogenization and singular perturbations problems (see

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Remark 3.6). Indeed, most of our results do not fall within the scope of [11] (see Remark 3.5). We show that the last mentioned boundedness assumptions in L^1 preclude the appearance of higher order derivatives in the limit equations, and, in most cases, of non local effects. The sequence of the solutions to (1.1) is not, in general, bounded in $H^1(\Omega;\mathbb{R}^3)$. The natural functional space is the space of functions with bounded deformation, that is the set of elements $\boldsymbol{u}$ of $L^1(\Omega;\mathbb{R}^3)$ whose distributional symmetrized gradient $\boldsymbol{E}\boldsymbol{u}$ is a matrix-valued measure with finite total variation. This space, introduced in [39], [49] (see also [41]), has been intensively investigated in the literature [2], [4], [34], [38], [50], [54], [55]. A marking feature of our results is that the effective problem only depends on the limit measures of the elasticity coefficients and of their inverses, not on the sequences themselves, provided these measures have no common atom. Otherwise, the arbitrary choice of the converging sequences leads to infinitely many distinct limit problems, some exhibiting non-local terms (see Remark 3.7). Similar properties were already known for diffusion problems in stratified media [17] (see Remark 3.12). The generalization of such results to elasticity is anything but straightforward, because effective problems may take a much more complicated form. More precisely, the limit energy associated to a sequence of linear diffusion problems is always a Dirichlet form [41]. By contrast, the limit energy associated to a sequence of linear elasticity problems can be any non-negative lower semi-continuous quadratic form on $L^2(\Omega;\mathbb{R}^3)$ taking vanishing values on the set of rigid motions [18].

We now present our results in more details. For a given cylindrical bounded open subset $\Omega=(0,L)\times\Omega'$ of $\mathbb{R}^3$ with Lipschitz boundary, we consider the problem (1.1). The Lamé coefficients are assumed to depend only on the variable x_1. We suppose that $\lambda_\varepsilon=l\mu_\varepsilon$ for some non negative real l and that the following convergences hold

$$\mu_\varepsilon \overset{\star}{\rightharpoonup} m, \qquad (\mu_\varepsilon)^{-1} \overset{\star}{\rightharpoonup} \nu \quad \text{weakly* in } \mathcal{M}([0,L]). \tag{1.2}$$

Under (1.2), we prove that the solution $\boldsymbol{u}_\varepsilon$ to (1.1) weakly* converges in $BD(\Omega)$ to some function $\boldsymbol{u}$ with bounded deformation. This effective displacement is characterized by the emergence of jumps $\boldsymbol{u}^+-\boldsymbol{u}^-$ at the interfaces $\Sigma_t=\{t\}\times\Omega'$ corresponding to atoms $\{t\}$ of ν, giving rise, if m and ν have no common atom, to the following concentrations of elastic energy

$$\tfrac{1}{2}\nu(\{t\})^{-1}\int_{\Sigma_t}(\boldsymbol{u}^+-\boldsymbol{u}^-)\cdot\boldsymbol{A}(\boldsymbol{u}^+-\boldsymbol{u}^-)d\mathcal{H}^2, \tag{1.3}$$

where $\boldsymbol{A}$ is given by (3.13). Concentrations of elastic energy also appear on the planes Σ_t corresponding to atoms of m. These extra terms are similar to membrane stretching energy and take the form

$$\tfrac{1}{2}m(\{t\})\int_{\Sigma_t}\boldsymbol{a}^{\|}\boldsymbol{e}_{x'}(\boldsymbol{u}^\star):\boldsymbol{e}_{x'}(\boldsymbol{u}^\star)d\mathcal{H}^2,$$

where the operator $\boldsymbol{e}_{x'}$ and the fourth order tensor $\boldsymbol{a}^{\|}$ are given by (2.6) and (3.10), and $\boldsymbol{u}^\star$ stands for the precise representative of $\boldsymbol{u}$ (see (2.1)). A concentration of elastic energy also emerges on a set of fractal Hausdorf dimension comprised between 2 and 3. It is given in terms of the Cantor parts ν^c and m^c of the measures ν and m by

$$\tfrac{1}{2}\int_\Omega\boldsymbol{a}^{\perp}\frac{\boldsymbol{E}\boldsymbol{u}}{\nu^c\otimes\mathcal{L}^2}:\frac{\boldsymbol{E}\boldsymbol{u}}{\nu^c\otimes\mathcal{L}^2}d\nu^c\otimes\mathcal{L}^2+\tfrac{1}{2}\int_\Omega\boldsymbol{a}^{\|}\boldsymbol{e}_{x'}(\boldsymbol{u}^\star):\boldsymbol{e}_{x'}(\boldsymbol{u}^\star)dm^c\otimes\mathcal{L}^2,$$

the tensor $\boldsymbol{a}^\perp$ being given by (3.10). The effective displacement $\boldsymbol{u}$ is a function with bounded deformation, hence is approximately differentiable $\mathcal{L}^3$-almost everywhere in Ω (see Remark 3.4). The bulk effective energy takes the form of a classical linear elastic energy defined in terms of its approximate symmetric gradient $\boldsymbol{e}(\boldsymbol{u})$ by

$$\tfrac{1}{2}\int_\Omega \boldsymbol{a}\boldsymbol{e}(\boldsymbol{u}){:}\boldsymbol{e}(\boldsymbol{u})dx,$$

the effective tensor $\boldsymbol{a}$ being given by (3.13). The total elastic energy $F(\boldsymbol{u})$ is the sum of the terms mentioned above, which can be synthetized as follows:

$$F(\boldsymbol{u}) = \tfrac{1}{2}\int_\Omega \boldsymbol{a}^\perp \frac{\boldsymbol{E}\boldsymbol{u}}{\nu\otimes\mathcal{L}^2}\cdot\frac{\boldsymbol{E}\boldsymbol{u}}{\nu\otimes\mathcal{L}^2} d\nu\otimes\mathcal{L}^2 + \tfrac{1}{2}\int_\Omega \boldsymbol{a}^\parallel \boldsymbol{e}_{x'}(\boldsymbol{u}^\star)\cdot\boldsymbol{e}_{x'}(\boldsymbol{u}^\star)dm\otimes\mathcal{L}^2.$$

The effective displacement is the unique solution to the problem $\min_{BD_0^{\nu,m}(\Omega)} F(\boldsymbol{u}) - \int_\Omega \boldsymbol{f}\cdot\boldsymbol{u}dx$, where $BD_0^{\nu,m}(\Omega)$ is defined by (4.26). When the Cantor parts ν^c and m^c vanish and ν and m have a finite number of atoms, this limit problem is equivalent to the system of equations (3.14).

The paper is organised as follows: the notations are specified in Section 2 and the main results stated in Section 3. Section 4 is devoted to the analysis of the asymptotic behavior of the solution to (3.2), and Section 5 to technical results relating to partial mollification. The proof of the main result (Theorem 3.1) is situated in Section 6.

2. Notations. In this article, $\{\boldsymbol{e}_1,\boldsymbol{e}_2,\boldsymbol{e}_3\}$ stands for the canonical basis of $\mathbb{R}^3$. Points in $\mathbb{R}^3$ and real-valued functions are represented by symbols beginning with a lightface lowercase (example $x, i, \operatorname{tr}\boldsymbol{A},\dots$) while vectors and vector-valued functions by symbols beginning in boldface lowercase (examples: $\boldsymbol{u}$, $\boldsymbol{f}$, $\mathbf{div}\boldsymbol{\sigma}_\varepsilon$, …). Matrices and matrix-valued functions are represented by symbols beginning in boldface uppercase with the following exceptions: $\boldsymbol{\nabla}\boldsymbol{u}$ (displacement gradient), $\boldsymbol{e}(\boldsymbol{u})$ (linearized strain tensor). We denote by u_i or $(\boldsymbol{u})_i$ the components of a vector $\boldsymbol{u}$ and by A_{ij} or $(\boldsymbol{A})_{ij}$ those of a matrix $\boldsymbol{A}$ (that is $\boldsymbol{u} = \sum_{i=1}^3 u_i\boldsymbol{e}_i = \sum_{i=1}^3(\boldsymbol{u})_i\boldsymbol{e}_i$; $\boldsymbol{A} = \sum_{i,j=1}^3 A_{ij}\boldsymbol{e}_i\otimes\boldsymbol{e}_j = \sum_{i,j=1}^3(\boldsymbol{A})_{ij}\boldsymbol{e}_i\otimes\boldsymbol{e}_j$, where $\otimes$ stands for the tensor product). For any two vectors $\boldsymbol{a}$, $\boldsymbol{b}$ in $\mathbb{R}^3$, the symmetric product $\boldsymbol{a}\odot\boldsymbol{b}$ is the symmetric 3×3 matrix defined by $\boldsymbol{a}\odot\boldsymbol{b} := \frac{1}{2}(\boldsymbol{a}\otimes\boldsymbol{b}+\boldsymbol{b}\otimes\boldsymbol{a})$. We do not employ the usual repeated index convention for summation. We denote by $\boldsymbol{A}:\boldsymbol{B} = \sum_{i,j=1}^3 A_{ij}B_{ij}$ the inner product of two matrices, by $\mathbb{S}^3$ the set of all real symmetric matrices of order 3, by $\boldsymbol{I}$ the 3×3 identity matrix. We denote by $\mathcal{L}^n$ the Lebesgue measure in $\mathbb{R}^n$ and by $\mathcal{H}^k$ the k-dimensional Hausdorff measure. The letter C denotes constants whose precise values may vary from line to line. Let $\Omega := (0,L)\times\Omega'$ be a connected cylindrical open Lipschitz subset of $\mathbb{R}^3$. For any $\boldsymbol{\varphi}\in L^1_{loc}(\Omega;\mathbb{R}^3)$, we denote by $\boldsymbol{\varphi}^\star$ its precise representative, that is

$$\boldsymbol{\varphi}^\star(x) = \begin{cases} \lim\limits_{r\to 0} \fint_{B_r(x)} \boldsymbol{\varphi}(y)\ dy & \text{if this limit exists,} \\ 0 & \text{otherwise,} \end{cases} \tag{2.1}$$

where $B_r(x)$ is the open ball of radius r centered at x, and $\fint_{B_r(x)}\boldsymbol{\varphi}(y)\ dy := \frac{\int_{B_r(x)}\boldsymbol{\varphi}(y)\ dy}{\mathcal{L}^3(B_r(x))}$.

We also set

$$\boldsymbol{\varphi}^{\pm}(x) = \begin{cases} \lim\limits_{r\to 0} \fint_{B_r^{\pm}(x)} \boldsymbol{\varphi}(y)\, dy & \text{if this limit exists,} \\ 0 & \text{otherwise,} \end{cases} \tag{2.2}$$

where

$$B_r^{+}(x) := B_r(x) \cap \big((x_1, L)\times\Omega'\big), \quad B_r^{-}(x) := B_r(x) \cap \big((0, x_1)\times\Omega'\big). \tag{2.3}$$

The fields $\boldsymbol{\varphi}^{\star}$ and $\boldsymbol{\varphi}^{\pm}$ are Borel-measurable and take the same values on the Lebesgue points of $\boldsymbol{\varphi}$, thus

$$\boldsymbol{\varphi}^{\pm} = \boldsymbol{\varphi}^{\star} = \boldsymbol{\varphi} \quad \mathcal{L}^3\text{-a.e. in } \ \Omega. \tag{2.4}$$

We denote by $\boldsymbol{\varphi}'$ the element of $L^1_{loc}(\Omega;\mathbb{R}^3)$ defined by

$$\varphi_1' = 0, \qquad \varphi_\alpha' = \varphi_\alpha \quad \forall \alpha \in \{2,3\}, \tag{2.5}$$

and by $\widetilde{\boldsymbol{\varphi}}$ the extension of $\boldsymbol{\varphi}$ by 0 into $\mathbb{R}^3$. If φ_2, φ_3 admit weak derivatives with respect to x_2, x_3, we set

$$\boldsymbol{e}_{x'}(\boldsymbol{\varphi}) := \sum_{\alpha,\beta=2}^{3} \frac{1}{2}\left(\frac{\partial \varphi_\alpha}{\partial x_\beta} + \frac{\partial \varphi_\beta}{\partial x_\alpha}\right) \boldsymbol{e}_\alpha \otimes \boldsymbol{e}_\beta. \tag{2.6}$$

The symbol $\boldsymbol{D}\boldsymbol{\varphi}$ represents the distributional gradient of $\boldsymbol{\varphi}$ and $\boldsymbol{E}\boldsymbol{\varphi} := \frac{1}{2}\left(\boldsymbol{D}\boldsymbol{\varphi} + \boldsymbol{D}\boldsymbol{\varphi}^T\right)$ the symmetric distributional gradient of $\boldsymbol{\varphi}$. The space of functions with bounded deformation on Ω is defined by

$$BD(\Omega) := \left\{\boldsymbol{\varphi} \in L^1(\Omega;\mathbb{R}^3) : \ \boldsymbol{E}\boldsymbol{\varphi} \in \mathcal{M}(\Omega;\mathbb{S}^3)\right\}, \tag{2.7}$$

where $\mathcal{M}(\Omega;\mathbb{S}^3)$ stands for the space of $\mathbb{S}^3$-valued Radon measures on Ω with bounded total variation. For any $\varphi \in \mathbf{BD}(\boldsymbol{\Omega})$, we denote by $\widetilde{\boldsymbol{E}}\boldsymbol{\varphi}$ the extension of $\boldsymbol{E}\boldsymbol{\varphi}$ by 0 to $\overline{\Omega}$, that is the element of $\mathcal{M}(\overline{\Omega};\mathbb{S}^3)$ defined by

$$\widetilde{\boldsymbol{E}}\boldsymbol{\varphi}(A) := \boldsymbol{E}\boldsymbol{\varphi}(A\cap\Omega) \quad \text{ for all Borel subset } A \text{ of } \overline{\Omega}. \tag{2.8}$$

For any $x_1 \in [0, L]$, we set

$$\Sigma_{x_1} := \{x_1\} \times \Omega'. \tag{2.9}$$

The symbol $\frac{\lambda}{\theta}$ represents the Radon-Nikodým density of a (finite) vector valued Radon measure λ on Ω with respect to a positive Radon measure θ on Ω. For any Borel subset E of Ω, we denote by $\lambda_{\lfloor E}$ the Radon measure defined by $\lambda_{\lfloor E}(A) = \lambda(A\cap E)$.

3. Setting of the problem and results. Let $\Omega := (0, L)\times\Omega'$ be a bounded cylindrical Lipschitz domain of $\mathbb{R}^3$, let (λ_ε), (μ_ε) be two sequences in $L^\infty(0, L;\mathbb{R}^+)$ such that $\mu_\varepsilon^{-1} \in L^\infty(0, L;\mathbb{R}^+)$, and let

$$\nu_\varepsilon := \mu_\varepsilon^{-1}\mathcal{L}^1_{\lfloor [0,L]}; \qquad m_\varepsilon := \mu_\varepsilon \mathcal{L}^1_{\lfloor [0,L]}. \tag{3.1}$$

We are interested in the asymptotic analysis of the sequence of linear elasticity problems

$$(3.2)\qquad (\mathcal{P}_\varepsilon):\begin{cases} -\,\mathbf{div}(\boldsymbol{\sigma}_\varepsilon(\boldsymbol{u}_\varepsilon)) = \boldsymbol{f} \quad \text{in } \Omega,\\ \boldsymbol{\sigma}_\varepsilon(\boldsymbol{u}_\varepsilon)=\lambda_\varepsilon \mathrm{tr}(\boldsymbol{e}(\boldsymbol{u}_\varepsilon))\boldsymbol{I}+2\mu_\varepsilon \boldsymbol{e}(\boldsymbol{u}_\varepsilon), \quad \boldsymbol{e}(\boldsymbol{u}_\varepsilon)=\dfrac{1}{2}(\boldsymbol{\nabla}\boldsymbol{u}_\varepsilon+\boldsymbol{\nabla}^T\boldsymbol{u}_\varepsilon),\\ \boldsymbol{u}_\varepsilon\in H^1_0(\Omega;\mathbb{R}^3), \quad \boldsymbol{f}\in L^\infty(\Omega,\mathbb{R}^3),\end{cases}$$

under the hypotheses (see Remark 3.3)

$$(3.3)\qquad \begin{aligned} &\lambda_\varepsilon = l\mu_\varepsilon \quad (l\geq 0), \qquad \sup_{\varepsilon>0}\Big(||\mu_\varepsilon||_{L^1(0,L)}+\big|\big|\mu_\varepsilon^{-1}\big|\big|_{L^1(0,L)}\Big)<\infty,\\ &m_\varepsilon \overset{\star}{\rightharpoonup} m, \qquad \nu_\varepsilon \overset{\star}{\rightharpoonup} \nu \quad \text{weakly* in } \mathcal{M}([0,L]).\end{aligned}$$

We emphasize that λ_ε and μ_ε only depend on x_1. We suppose that ν and m have no common atom (see Remark 3.7), that is

$$(3.4)\qquad \mathcal{A}_\nu\cap\mathcal{A}_m=\emptyset,\ \mathcal{A}_\nu:=\{t\in[0,L];\nu(\{t\})>0\},\ \mathcal{A}_m:=\{t\in[0,L];m(\{t\})>0\},$$

and do not charge the boundary of $[0,L]$ (see Remark 3.8), namely

$$(3.5)\qquad m(\{0\})=m(\{L\})=\nu(\{0\})=\nu(\{L\})=0.$$

Under these assumptions, we prove that the sequence of the solutions to (3.2) weakly* converges in $BD(\Omega)$ to the unique solution to

$$(3.6)\qquad \min_{\boldsymbol{\varphi}\in BD_0^{\nu,m}(\Omega)} \tfrac{1}{2}a(\boldsymbol{\varphi},\boldsymbol{\varphi})-\int_\Omega \boldsymbol{f}\cdot\boldsymbol{\varphi}dx,$$

where $BD_0^{\nu,m}(\Omega)$ is the Hilbert space defined by (see (2.1))

$$(3.7)\qquad BD_0^{\nu,m}(\Omega):=\left\{\boldsymbol{\varphi}\in BD(\Omega)\ \middle|\ \begin{array}{l}\boldsymbol{E\varphi}\ll\nu\otimes\mathcal{L}^2,\ \frac{\boldsymbol{E\varphi}}{\nu\otimes\mathcal{L}^2}\in L^2_{\nu\otimes\mathcal{L}^2}(\Omega;\mathbb{S}^3)\\ \varphi^\star_\alpha\in L^2_m(0,L;H^1_0(\Omega'))\quad \alpha\in\{2,3\}\\ \boldsymbol{\varphi}=0 \ \text{ on } \ \partial\Omega\end{array}\right\},$$

$$(3.8)\qquad ||\boldsymbol{\varphi}||_{BD_0^{\nu,m}(\Omega)}:=\left(\int_\Omega\Big|\frac{\boldsymbol{E\varphi}}{\nu\otimes\mathcal{L}^2}\Big|^2\,d\nu\otimes\mathcal{L}^2\right)^{\frac{1}{2}}+\left(\int_\Omega|\boldsymbol{e}_{x'}(\boldsymbol{\varphi}^\star)|^2dm\otimes\mathcal{L}^2\right)^{\frac{1}{2}},$$

and $a(\cdot,\cdot)$ is the continuous coercive symmetric bilinear form on $BD_0^{\nu,m}(\Omega)$ given by

$$(3.9)\qquad a(\boldsymbol{\psi},\boldsymbol{\varphi}):=\int_\Omega \boldsymbol{a}^\perp\frac{\boldsymbol{E\psi}}{\nu\otimes\mathcal{L}^2}:\frac{\boldsymbol{E\varphi}}{\nu\otimes\mathcal{L}^2}d\nu\otimes\mathcal{L}^2+\int_\Omega \boldsymbol{a}^\parallel\boldsymbol{e}_{x'}(\boldsymbol{\psi}^\star):\boldsymbol{e}_{x'}(\boldsymbol{\varphi}^\star)dm\otimes\mathcal{L}^2,$$

in terms of the fourth order tensors $\boldsymbol{a}^\perp$ and $\boldsymbol{a}^\parallel$ defined by

$$\begin{aligned}
&\boldsymbol{a}^{\perp}\boldsymbol{\Xi} := \begin{pmatrix} l\,\mathrm{tr}\,\boldsymbol{\Xi} + 2\Xi_{11} & 2\Xi_{12} & 2\Xi_{13} \\ 2\Xi_{12} & \frac{l^2}{l+2}\,\mathrm{tr}\,\boldsymbol{\Xi} + \frac{2l}{l+2}\Xi_{11} & 0 \\ 2\Xi_{13} & 0 & \frac{l^2}{l+2}\,\mathrm{tr}\,\boldsymbol{\Xi} + \frac{2l}{l+2}\Xi_{11} \end{pmatrix}, \\
&\boldsymbol{a}^{\parallel}\boldsymbol{\Gamma} := \tfrac{2l}{l+2}\sum_{\beta=2}^{3}\Gamma_{\beta\beta}\sum_{\alpha=2}^{3}\boldsymbol{e}_\alpha\otimes\boldsymbol{e}_\alpha + 2\sum_{\alpha,\beta=2}^{3}\Gamma_{\alpha\beta}\boldsymbol{e}_\alpha\otimes\boldsymbol{e}_\beta.
\end{aligned} \tag{3.10}$$

Notice that

$$(\boldsymbol{a}^{\perp} + \boldsymbol{a}^{\parallel})\boldsymbol{\Xi} = l\,\mathrm{tr}\,\boldsymbol{\Xi}\boldsymbol{I} + 2\boldsymbol{\Xi}. \tag{3.11}$$

Equivalently, we have (see Remark 3.4)

$$\begin{aligned}
a(\boldsymbol{\psi},\boldsymbol{\varphi}) = &\int_\Omega \boldsymbol{a}\boldsymbol{e}(\boldsymbol{\psi}):\boldsymbol{e}(\boldsymbol{\varphi})dx + \sum_{t\in\mathcal{A}_\nu}\nu(\{t\})^{-1}\int_{\Sigma_t}(\boldsymbol{\psi}^+ - \boldsymbol{\psi}^-)\cdot\boldsymbol{A}(\boldsymbol{\varphi}^+ - \boldsymbol{\varphi}^-)d\mathcal{H}^2 \\
&+ \sum_{t\in\mathcal{A}_m} m(\{t\})\int_{\Sigma_t}\boldsymbol{a}^{\parallel}\boldsymbol{e}_{x'}(\boldsymbol{\psi}^\star):\boldsymbol{e}_{x'}(\boldsymbol{\varphi}^\star)d\mathcal{H}^2 \\
&+ \int_\Omega \boldsymbol{a}^{\perp}\frac{\boldsymbol{E}\boldsymbol{\psi}}{\nu^c\otimes\mathcal{L}^2}:\frac{\boldsymbol{E}\boldsymbol{\varphi}}{\nu^c\otimes\mathcal{L}^2}d\nu^c\otimes\mathcal{L}^2 + \int_\Omega \boldsymbol{a}^{\parallel}\boldsymbol{e}_{x'}(\boldsymbol{\psi}^\star):\boldsymbol{e}_{x'}(\boldsymbol{\varphi}^\star)dm^c\otimes\mathcal{L}^2,
\end{aligned} \tag{3.12}$$

where $\boldsymbol{\varphi}^{\pm}$, Σ_t are defined by (2.2), (2.9), ν^c (resp. m^c) stands for the Cantor part of ν (resp. m), $\boldsymbol{e}(\boldsymbol{\varphi})$ for the approximate symmetric differential of $\boldsymbol{\varphi}$, and

$$\boldsymbol{a} := \left(\tfrac{\nu}{\mathcal{L}^1}\right)^{-1}\boldsymbol{a}^{\perp} + \tfrac{m}{\mathcal{L}^1}\boldsymbol{a}^{\parallel}, \qquad \boldsymbol{A} := \begin{pmatrix} l+2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.13}$$

Theorem 3.1. *The space $BD_0^{\nu,m}(\Omega)$ defined by (3.7), endowed with the norm (3.8), is a Hilbert space. Under the assumptions (3.3), (3.4), and (3.5), the symmetric bilinear form $a(\cdot,\cdot)$ defined by (3.9), or (3.12), is coercive and continuous on $BD_0^{\nu,m}(\Omega)$. The sequence of the solutions to (3.2) weakly* converges in $BD(\Omega)$ to the unique solution to (3.6).*

We can derive the PDE system associated with (3.6) when ν and m have vanishing Cantor parts and a finite number of atoms.

Corollary 3.2. *If $\nu^c = m^c = 0$ and $\mathcal{A}_\nu$, $\mathcal{A}_m$ are finite, the problem (3.6) is equivalent to*

$$\begin{cases} -\,\mathbf{div}\boldsymbol{a}\boldsymbol{e}(\boldsymbol{u}) = \boldsymbol{f} \quad \text{in} \quad \Omega\setminus\Sigma, \quad \boldsymbol{u}\in BD_0^{\nu,m}(\Omega), \\ \nu(\{t\})^{-1}\boldsymbol{A}(\boldsymbol{u}^+ - \boldsymbol{u}^-) = (\boldsymbol{a}\boldsymbol{e}(\boldsymbol{u})\boldsymbol{e}_1)^- = (\boldsymbol{a}\boldsymbol{e}(\boldsymbol{u})\boldsymbol{e}_1)^+ \qquad \text{on } \Sigma_t,\ \forall t\in\mathcal{A}_\nu, \\ (\boldsymbol{a}\boldsymbol{e}(\boldsymbol{u})\boldsymbol{e}_1)^- - (\boldsymbol{a}\boldsymbol{e}(\boldsymbol{u})\boldsymbol{e}_1)^+ - m(\{t\})\mathbf{div}_{x'}\boldsymbol{a}^{\parallel}\boldsymbol{e}_{x'}(\boldsymbol{u}^\star) = 0 \qquad \text{on } \Sigma_t,\ \forall t\in\mathcal{A}_m, \end{cases} \tag{3.14}$$

where $(\boldsymbol{a}\boldsymbol{e}(\boldsymbol{u})\boldsymbol{e}_1)^+$ (resp. $(\boldsymbol{a}\boldsymbol{e}(\boldsymbol{u})\boldsymbol{e}_1)^-$) denotes the trace of $\boldsymbol{a}\boldsymbol{e}(\boldsymbol{u})\boldsymbol{e}_1$ on the right (resp. left) face of Σ_t, and

$$\Sigma := \Sigma_\nu \cup \Sigma_m, \qquad \Sigma_\nu := \bigcup_{t\in\mathcal{A}_\nu} \Sigma_t, \quad \Sigma_m := \bigcup_{t\in\mathcal{A}_m} \Sigma_t. \tag{3.15}$$

REMARK 3.3. *The conclusions of Theorem 3.1 are unchanged if the assumption $\lambda_\varepsilon = l\mu_\varepsilon$ in (3.3) is replaced by $\lambda_\varepsilon = l_\varepsilon\mu_\varepsilon$, where (l_ε) is a sequence of positive real numbers converging to some $l \in [0, +\infty)$.*

REMARK 3.4. *The equivalence between (3.9) and (3.12) derives from fine properties of functions with bounded deformations. More precisely, the symmetric distributional derivative $\boldsymbol{E}\boldsymbol{\varphi}$ of any $\boldsymbol{\varphi} \in BD(\Omega)$ can be decomposed into an absolutely continuous part $\boldsymbol{E}^a\boldsymbol{\varphi}$ with respect to $\mathcal{L}^3$, a jump part $\boldsymbol{E}^j\boldsymbol{\varphi}$ and a Cantor part $\boldsymbol{E}^c\boldsymbol{\varphi}$. The Cantor part vanishes on any Borel set which is σ-finite with respect to $\mathcal{H}^2$. Any element $\boldsymbol{\varphi}$ of $BD(\Omega)$ is approximately differentiable $\mathcal{L}^3$-almost everywhere in Ω [2, Theorem 7.4], [34]. This means that, for $\mathcal{L}^3$-a.e $x \in \Omega$, there exists a 3×3 matrix $\boldsymbol{\nabla}\boldsymbol{\varphi}(x)$ such that*

$$\lim_{r\to 0^+} \fint_{B_r(x)} \frac{|\boldsymbol{\varphi}(y) - \boldsymbol{\varphi}(x) - \boldsymbol{\nabla}\boldsymbol{\varphi}(x)(\boldsymbol{y}-\boldsymbol{x})|}{r} dy = 0.$$

The absolutely continuous part of $\boldsymbol{E}\boldsymbol{\varphi}$ with respect to $\mathcal{L}^3$ is given in terms of the approximate symmetric differential $\boldsymbol{e}(\boldsymbol{\varphi}) = \frac{1}{2}(\boldsymbol{\nabla}\boldsymbol{\varphi} + \boldsymbol{\nabla}^T\boldsymbol{\varphi})$ by

$$\boldsymbol{E}^a\boldsymbol{\varphi} = \boldsymbol{e}(\boldsymbol{\varphi})\mathcal{L}^3. \tag{3.16}$$

When $\boldsymbol{E}\boldsymbol{\varphi} \ll \mathcal{L}^3$, $\boldsymbol{e}(\boldsymbol{\varphi})$ is the weak symmetric gradient of $\boldsymbol{\varphi}$. The jump part takes the form $\boldsymbol{E}^j\boldsymbol{\varphi} = \boldsymbol{E}\boldsymbol{\varphi}_{\lfloor J_{\boldsymbol{\varphi}}}$, where the "jump set" $J_{\boldsymbol{\varphi}}$ is a countably $\mathcal{H}^2$-rectifiable subset of Ω (i.e. there exists countably many Lipschitz functions $f_i : \mathbb{R}^2 \to \Omega$ such that $\mathcal{H}^2\left(J_{\boldsymbol{\varphi}} \setminus \bigcup_{i=0}^{+\infty} f_i(\mathbb{R}^2)\right) = 0$, see [3, Definition 2.57]). For any countably $\mathcal{H}^2$-rectifiable Borel set $M \subset \Omega$, the following holds (see [54, Chapter II], [2, p.209 (3.2)])

$$\boldsymbol{E}\boldsymbol{\varphi}\lfloor_M = (\boldsymbol{\varphi}_M^+ - \boldsymbol{\varphi}_M^-) \odot \boldsymbol{n}_M \mathcal{H}^2_{\lfloor M}, \tag{3.17}$$

where $\boldsymbol{n}_M(x)$ is a unit normal to M at x and $\boldsymbol{\varphi}_M^\pm$ is deduced from (2.2) by substituting $B_r^\pm(x, \boldsymbol{n}_M) := \{y \in B_r(x), \pm(\boldsymbol{y}-\boldsymbol{x})\cdot \boldsymbol{n}_M(x) > 0\}$ for $B_r^\pm(x)$. In particular, we have

$$\boldsymbol{E}^j\boldsymbol{\varphi} = (\boldsymbol{\varphi}_{J_{\boldsymbol{\varphi}}}^+ - \boldsymbol{\varphi}_{J_{\boldsymbol{\varphi}}}^-) \odot \boldsymbol{n}_{J_{\boldsymbol{\varphi}}} \mathcal{H}^2_{\lfloor J_{\boldsymbol{\varphi}}}. \tag{3.18}$$

Due to its absolute continuity with respect to $\nu \otimes \mathcal{L}^2$, the symmetric distributional gradient of an element of $BD_0^{\nu,m}(\Omega)$ enjoys a specific decomposition. The measure ν (resp. m) can be split into an absolutely continuous part ν^a (resp. m^a) with respect to the Lebesgue measure, a singular part without atoms or Cantor part ν^c (resp. m^c), and a purely atomic part ν^{at}:

$$\begin{aligned} &\nu = \nu^a + \nu^c + \nu^{at}, \qquad \nu^{at} = \sum_{t\in\mathcal{A}_\nu} \nu(\{t\})\delta_t, \qquad \nu^a = \tfrac{\nu}{\mathcal{L}^1}\mathcal{L}^1, \\ &m = m^a + m^c + m^{at}, \quad m^{at} = \sum_{t\in\mathcal{A}_m} m(\{t\})\delta_t, \quad m^a = \tfrac{m}{\mathcal{L}^1}\mathcal{L}^1. \end{aligned} \tag{3.19}$$

We have $\nu^a\otimes\mathcal{L}^2\ll\mathcal{L}^3$ *and* $\nu^{at}\otimes\mathcal{L}^2\ll\mathcal{H}^2_{\lfloor\Sigma_\nu}$*, where* Σ_ν *is given by* (3.15). *The measures* $\nu^c\otimes\mathcal{L}^2$ *and* $\mathcal{L}^3$ *are mutually singular. If* A *is a Borel subset of* Ω *that is* σ*-finite with respect to* $\mathcal{H}^2$*, then by Fubini's theorem* $\nu^c\otimes\mathcal{L}^2(A)=\int_{(0,L)}\mathcal{H}^2(A\cap\Sigma_{x_1})d\nu^c=0$ *because* $\{x_1\in(0,L),\mathcal{H}^2(A\cap\Sigma_{x_1})>0\}$ *is at most countable, thus* ν^c*-negligible. Accordingly, there exists a Borel partition of* Ω*,* $\Omega=\Omega^a\cup\Omega^c\cup\Omega^{at}$ *with* $\Omega^{at}=\Sigma_\nu$ *(see* (3.15)*) such that*

$$\begin{aligned}&\nu^a\otimes\mathcal{L}^2=\nu\otimes\mathcal{L}^2_{\lfloor\Omega^a}=\tfrac{\nu}{\mathcal{L}^1}\mathcal{L}^3_{\lfloor\Omega^a},\quad \nu^c\otimes\mathcal{L}^2=\nu\otimes\mathcal{L}^2_{\lfloor\Omega^c},\\&\nu^{at}\otimes\mathcal{L}^2=\nu\otimes\mathcal{L}^2_{\lfloor\Sigma_\nu}=\sum_{t\in\mathcal{A}_\nu}\nu(\{t\})\mathcal{H}^2_{\lfloor\Sigma_t}.\end{aligned}\tag{3.20}$$

The condition $\boldsymbol{E}(\boldsymbol{\varphi})\ll(\nu^a+\nu^c+\nu^{at})\otimes\mathcal{L}^2$*, satisfied by any element* $\boldsymbol{\varphi}$ *of* $BD_0^{\nu,m}(\Omega)$*, implies* $\boldsymbol{E}^a\boldsymbol{\varphi}\ll\nu^a\otimes\mathcal{L}^2$*,* $\boldsymbol{E}^c\boldsymbol{\varphi}\ll\nu^c\otimes\mathcal{L}^2$*,* $\boldsymbol{E}^j\boldsymbol{\varphi}\ll\mathcal{H}^2_{\lfloor\Sigma_\nu}$*, and*

$$\begin{aligned}&\tfrac{\boldsymbol{E}\boldsymbol{\varphi}}{\nu\otimes\mathcal{L}^2}=\tfrac{\boldsymbol{E}^a\boldsymbol{\varphi}}{\nu^a\otimes\mathcal{L}^2}\mathbb{1}_{\Omega^a}+\tfrac{\boldsymbol{E}^c\boldsymbol{\varphi}}{\nu^c\otimes\mathcal{L}^2}\mathbb{1}_{\Omega^c}+\tfrac{\boldsymbol{E}^j\boldsymbol{\varphi}}{\nu^{at}\otimes\mathcal{L}^2}\mathbb{1}_{\Sigma_\nu}\quad \nu\otimes\mathcal{L}^2\text{-a.e. in }\ \Omega,\\&\boldsymbol{E}^a\boldsymbol{\varphi}=\tfrac{\boldsymbol{E}^a\boldsymbol{\varphi}}{\nu^a\otimes\mathcal{L}^2}\mathbb{1}_{\Omega^a}\nu^a\otimes\mathcal{L}^2=\tfrac{\boldsymbol{E}^a\boldsymbol{\varphi}}{\nu^a\otimes\mathcal{L}^2}\mathbb{1}_{\Omega^a}\tfrac{\nu}{\mathcal{L}^1}\mathcal{L}^3,\quad \boldsymbol{E}^c\boldsymbol{\varphi}=\tfrac{\boldsymbol{E}^c\boldsymbol{\varphi}}{\nu^c\otimes\mathcal{L}^2}\mathbb{1}_{\Omega^c}\nu^c\otimes\mathcal{L}^2,\\&\boldsymbol{E}^j\boldsymbol{\varphi}=\tfrac{\boldsymbol{E}^j\boldsymbol{\varphi}}{\nu^{at}\otimes\mathcal{L}^2}\mathbb{1}_{\Sigma_\nu}\nu^{at}\otimes\mathcal{L}^2=\sum_{t\in\mathcal{A}_\nu}\tfrac{\boldsymbol{E}^j\boldsymbol{\varphi}}{\nu^{at}\otimes\mathcal{L}^2}\mathbb{1}_{\Sigma_t}\nu(\{t\})\mathcal{H}^2_{\lfloor\Sigma_t}.\end{aligned}\tag{3.21}$$

In particular we have $J_{\boldsymbol{\varphi}}\subset\Sigma_\nu$*, therefore, by* (3.15), (3.18),

$$\boldsymbol{E}^j\boldsymbol{\varphi}=\boldsymbol{E}\boldsymbol{\varphi}_{\lfloor\Sigma_\nu}=\sum_{t\in\mathcal{A}_\nu}(\boldsymbol{\varphi}^+-\boldsymbol{\varphi}^-)\odot\boldsymbol{e}_1\mathcal{H}^2_{\lfloor\Sigma_t}.\tag{3.22}$$

Taking (3.16) *into account, we infer*

$$\boldsymbol{E}\boldsymbol{\varphi}=\boldsymbol{e}(\boldsymbol{\varphi})\mathcal{L}^3+\tfrac{\boldsymbol{E}\boldsymbol{\varphi}}{\nu^c\otimes\mathcal{L}^2}\nu^c\otimes\mathcal{L}^2+\sum_{t\in\mathcal{A}_\nu}(\boldsymbol{\varphi}^+-\boldsymbol{\varphi}^-)\odot\boldsymbol{e}_1\mathcal{H}^2_{\lfloor\Sigma_t}.\tag{3.23}$$

We deduce from (3.16), (3.21) *and* (3.22) *that* $\frac{\boldsymbol{E}^a\boldsymbol{\varphi}}{\nu^a\otimes\mathcal{L}^2}=\left(\frac{\nu}{\mathcal{L}^1}\right)^{-1}\boldsymbol{e}(\boldsymbol{\varphi})$ $\mathcal{L}^3$*-a.e. in* Ω*, and* $\frac{\boldsymbol{E}^j\boldsymbol{\varphi}}{\nu^{at}\otimes\mathcal{L}^2}=(\nu(\{t\}))^{-1}(\boldsymbol{\varphi}^+-\boldsymbol{\varphi}^-)\odot\boldsymbol{e}_1$ $\mathcal{H}^2$*-a.e. in* Σ_t*,* $\forall t\in\mathcal{A}_\nu$*, and then from* (3.21) *that*

$$\begin{aligned}\tfrac{\boldsymbol{E}\boldsymbol{\varphi}}{\nu\otimes\mathcal{L}^2}=\left(\tfrac{\nu}{\mathcal{L}^1}\right)^{-1}\boldsymbol{e}(\boldsymbol{\varphi})\mathbb{1}_{\Omega^a}+\tfrac{\boldsymbol{E}^c\boldsymbol{\varphi}}{\nu^c\otimes\mathcal{L}^2}\mathbb{1}_{\Omega^c}+\sum_{t\in\mathcal{A}_\nu}(\nu(\{t\}))^{-1}(\boldsymbol{\varphi}^+-\boldsymbol{\varphi}^-)\odot\boldsymbol{e}_1\mathbb{1}_{\Sigma_t}\\ \nu\otimes\mathcal{L}^2\text{-a.e. in }\ \Omega,\quad\forall\boldsymbol{\varphi}\in BD_0^{\nu,m}(\Omega).\end{aligned}\tag{3.24}$$

By (3.20), (3.24) *and the formula* $\boldsymbol{a}^\perp(\boldsymbol{b}\odot\boldsymbol{e}_1):(\boldsymbol{c}\odot\boldsymbol{e}_1)=\boldsymbol{c}\cdot\boldsymbol{A}\boldsymbol{b}$*,* $\forall\boldsymbol{b},\boldsymbol{c}\in\mathbb{R}^3$ *(see* (3.10) *and* (3.13)*), the following holds for* $\boldsymbol{\varphi},\boldsymbol{\psi}\in BD_0^{\nu,m}(\Omega)$*:*

$$\begin{aligned}&\int_\Omega\boldsymbol{a}^\perp\tfrac{\boldsymbol{E}\boldsymbol{\psi}}{\nu\otimes\mathcal{L}^2}:\tfrac{\boldsymbol{E}\boldsymbol{\varphi}}{\nu\otimes\mathcal{L}^2}d\nu\otimes\mathcal{L}^2\\&\quad=\int_{\Omega^a}\left(\tfrac{\nu}{\mathcal{L}^1}\right)^{-1}\boldsymbol{a}^\perp\boldsymbol{e}(\boldsymbol{\psi}):\boldsymbol{e}(\boldsymbol{\varphi})d\mathcal{L}^3+\int_{\Omega^c}\boldsymbol{a}^\perp\tfrac{\boldsymbol{E}^c\boldsymbol{\psi}}{\nu^c\otimes\mathcal{L}^2}:\tfrac{\boldsymbol{E}^c\boldsymbol{\varphi}}{\nu^c\otimes\mathcal{L}^2}d\nu^c\otimes\mathcal{L}^2\\&\qquad+\sum_{t\in\mathcal{A}_\nu}\int_{\Sigma_t}(\nu(\{t\}))^{-1}(\boldsymbol{\psi}^+-\boldsymbol{\psi}^-)\cdot\boldsymbol{A}(\boldsymbol{\varphi}^+-\boldsymbol{\varphi}^-)d\mathcal{H}^2.\end{aligned}\tag{3.25}$$

On the other hand we have, by (3.19),

$$
\begin{aligned}
&\int_\Omega \boldsymbol{a}^{\|}\boldsymbol{e}_{x'}(\boldsymbol{\psi}^\star):\boldsymbol{e}_{x'}(\boldsymbol{\varphi}^\star)dm\otimes\mathcal{L}^2\\
&\quad=\int_\Omega \tfrac{m}{\mathcal{L}^1}\boldsymbol{a}^{\|}\boldsymbol{e}_{x'}(\boldsymbol{\psi}^\star):\boldsymbol{e}_{x'}(\boldsymbol{\varphi}^\star)dx+\int_\Omega \boldsymbol{a}^{\|}\boldsymbol{e}_{x'}(\boldsymbol{\psi}^\star):\boldsymbol{e}_{x'}(\boldsymbol{\varphi}^\star)dm^c\otimes\mathcal{L}^2\\
&\qquad+\sum_{t\in\mathcal{A}_m} m(\{t\})\int_{\Sigma_t}\boldsymbol{a}^{\|}\boldsymbol{e}_{x'}(\boldsymbol{\psi}^\star):\boldsymbol{e}_{x'}(\boldsymbol{\varphi}^\star)d\mathcal{H}^2.
\end{aligned}
\tag{3.26}
$$

Combining (3.9), (3.25) and (3.26), noticing that, by (2.4) and (3.10),

$$
\int_\Omega \tfrac{m}{\mathcal{L}^1}\boldsymbol{a}^{\|}\boldsymbol{e}_{x'}(\boldsymbol{\psi}^\star):\boldsymbol{e}_{x'}(\boldsymbol{\varphi}^\star)dx=\int_\Omega \tfrac{m}{\mathcal{L}^1}\boldsymbol{a}^{\|}\boldsymbol{e}(\boldsymbol{\psi}):\boldsymbol{e}(\boldsymbol{\varphi})dx,
$$

taking (3.13) into account, we obtain (3.12). Notice that when ν^c vanishes, the space $BD_0^{\nu,m}(\Omega)$ is a subspace of the space of special functions with bounded deformation defined by $SBD(\Omega):=\{\boldsymbol{\varphi}\in BD(\Omega),\ \boldsymbol{E}^c\boldsymbol{\varphi}=0\}$ (see [2], [5], [6], [19]).

Remark 3.5 (Comparison with the results of [11]). *The paper [11] investigates the linear elastodynamic equations associated with (3.2) when μ_ε is given by*

$$
\mu_\varepsilon=\mu_{0\varepsilon}\mathbb{1}_{(0,L)\setminus C_\varepsilon}(x_1)+\mu_{1\varepsilon}\mathbb{1}_{C_\varepsilon}(x_1),\qquad C_\varepsilon=\bigcup_{a\in A_\varepsilon} a+r_\varepsilon\left(-\tfrac12;\tfrac12\right),
$$

where A_ε is a finite subset of $(0,L)$, r_ε is a small parameter satisfying $r_\varepsilon<\varepsilon:=\inf_{a,b\in A_\varepsilon,a\neq b}|b-a|$, and $(\mu_{0\varepsilon})$, $(\mu_{1\varepsilon})$ are two sequences of positive reals. Except in one case (see [11, Section 3.1, case $0<k<+\infty$]), this paper studies instances when one of the sequences (μ_ε) or (μ_ε^{-1}) is unbounded in $L^1(0,L)$. This case corresponds to $\mu_{\varepsilon 0}=\mu_0>0$, $r_\varepsilon\ll\varepsilon$, and $\lim_{\varepsilon\to0}\frac{r_\varepsilon}{\varepsilon}\mu_{1\varepsilon}=:k\in(0,+\infty)$. Then, the conclusions of Theorem 3.1 can be obtained in the context of [11]. More precisely, the sequence (μ_ε) (resp. (μ_ε^{-1})) weakly converges in $\mathcal{M}([0,L])$ to $m=(\mu_0+nk)\mathcal{L}^1$ (resp. $\nu=\frac{1}{\mu_0}\mathcal{L}^1$) for some function $n\in L^\infty(0,L)$, defined by [11, Formula (3.16)], which characterizes the rescaled effective number of sections of stiff layers per unit length in the $\mathbf{e}_1$ direction. By (3.4), (3.7) and (3.15), the following holds: $\mathcal{A}_\nu=\mathcal{A}_m=\emptyset$, $\Sigma=\emptyset$, $BD_0^{\nu,m}(\Omega)=H_0^1(\Omega;\mathbb{R}^3)$. The sequence of the solutions to (3.2) is bounded in $H_0^1(\Omega;\mathbb{R}^3)$ and weakly converges to the solution to the problem given, in accordance with (3.14), by*

$$
\begin{cases}
-\mathbf{div}\boldsymbol{a}\boldsymbol{e}(\boldsymbol{u})=\boldsymbol{f}\quad in\quad \Omega,\quad \boldsymbol{u}\in H_0^1(\Omega;\mathbb{R}^3),\\
\boldsymbol{a}=\mu_0(\boldsymbol{a}^\perp+\boldsymbol{a}^{\|})+nk\boldsymbol{a}^{\|}.
\end{cases}
$$

Taking (3.11) into account, setting $\lambda_0:=l\mu_0$, $\boldsymbol{\sigma}_0(\boldsymbol{u}):=\mu_0(\boldsymbol{a}^\perp+\boldsymbol{a}^{\|})\boldsymbol{e}(\boldsymbol{u})=\lambda_0\operatorname{tr}(\boldsymbol{e}(\boldsymbol{u}))\boldsymbol{I}+2\mu_0\boldsymbol{e}(\boldsymbol{u})$, $\boldsymbol{\sigma}_{x'}(\boldsymbol{u}):=\boldsymbol{a}^{\|}\boldsymbol{e}(\boldsymbol{u})$, this effective problem can be rewritten under the form

$$
\begin{cases}
-\mathbf{div}\boldsymbol{\sigma}_0(\boldsymbol{u})-nk\mathbf{div}\boldsymbol{\sigma}_{x'}(\boldsymbol{u}')=\boldsymbol{f}\qquad in\ \Omega,\\
\boldsymbol{u}\in H_0^1(\Omega;\mathbb{R}^3),
\end{cases}
$$

which corresponds to the stationary version of the limit problem obtained in [11, Equation (3.18)].

Remark 3.6 (Some applications). *(i) Our result can be applied to various problems of homogenization with high contrast which do not fall into the scope of [11]. As an* *example, let us fix two small parameters ε and r_ε such that $r_\varepsilon \ll \varepsilon$, and consider a two-phase ε-periodic composite comprising an alternation of possibly very soft elastic layers of thickness r_ε and of Lamé coefficients of order $\frac{r_\varepsilon}{\varepsilon}$, with stiffer layers of thickness of order ε and Lamé coefficients of order 1. More precisely, let us assume that*

$$\mu_\varepsilon = \mu_0 \mathbb{1}_{(0,L)\setminus C_\varepsilon} + \tfrac{r_\varepsilon}{\varepsilon}\mu_1 \mathbb{1}_{C_\varepsilon}, \quad \lambda_\varepsilon = l\mu_\varepsilon, \quad C_\varepsilon := \bigcup_{\{i\in\mathbb{Z}, (\varepsilon i + r_\varepsilon I)\subset(0,L)\}} (\varepsilon i + r_\varepsilon I)\,. \tag{3.27}$$

Then, the assumptions and convergences (3.3) hold with $m = \mu_0 \mathcal{L}^1$ and $\nu = \left(\frac{1}{\mu_0} + \frac{1}{\mu_1}\right)\mathcal{L}^1$. By (3.4) and (3.7), we have $\mathcal{A}_\nu = \mathcal{A}_m = \emptyset$, and $BD_0^{\nu,m}(\Omega) = H_0^1(\Omega;\mathbb{R}^3)$, and the limit problem as $\varepsilon \to 0$, deduced from (3.14), takes the form

$$\begin{cases} -\,\mathbf{div}\boldsymbol{\sigma}(\boldsymbol{u}) = \boldsymbol{f} & \text{in} \ \ \Omega, \quad \boldsymbol{u}\in H_0^1(\Omega;\mathbb{R}^3), \\ \boldsymbol{\sigma}(\boldsymbol{u}) = \left(\frac{\mu_0\mu_1}{\mu_0+\mu_1}\boldsymbol{a}^\perp + \mu_0 \boldsymbol{a}^\parallel\right)\boldsymbol{e}(\boldsymbol{u}), \end{cases} \tag{3.28}$$

where $\boldsymbol{a}^\perp$ and $\boldsymbol{a}^\parallel$ are defined by (3.10).
(ii) Besides homogenization, our result can be applied to various singular perturbation problems. By way of illustration, let us consider the case of an elastic homogeneous isotropic body reinforced by a single stiff layer of thickness ε and of Lamé coefficients of order $\frac{1}{\varepsilon}$. More precisely, let us assume that the Lamé coefficients take the form

$$\mu_\varepsilon = \mu_0 \mathbb{1}_{(0,L)\setminus C_\varepsilon} + \tfrac{1}{\varepsilon}\mu_1 \mathbb{1}_{C_\varepsilon}, \quad C_\varepsilon := \left(\tfrac{L}{2}-\tfrac{\varepsilon}{2}, \tfrac{L}{2}+\tfrac{\varepsilon}{2}\right), \quad \lambda_\varepsilon = l\mu_\varepsilon. \tag{3.29}$$

Under these hypotheses, the assumptions and convergences (3.3) hold with $m = \mu_0\mathcal{L}^1 + \mu_1\delta_{\frac{L}{2}}$ and $\nu = \frac{1}{\mu_0}\mathcal{L}^1$. By (3.4) and (3.7), we have $\mathcal{A}_\nu = \emptyset$, $\mathcal{A}_m = \{L/2\}$, and

$$BD_0^{\nu,m}(\Omega) = \{\boldsymbol{\varphi}\in H_0^1(\Omega;\mathbb{R}^3),\ \varphi^\star_\alpha(L/2,.)\in H_0^1(\Omega')\ \forall\alpha\in\{2;3\}\}.$$

Setting

$$\begin{aligned} &\boldsymbol{\sigma}_0(\boldsymbol{u}) = l\mu_0 \operatorname{tr}\boldsymbol{e}(\boldsymbol{u})\boldsymbol{I} + 2\mu_0\boldsymbol{e}(\boldsymbol{u}), \\ &\boldsymbol{\sigma}'_1((\boldsymbol{u}^\star)') = \tfrac{2l}{l+2}\mu_1 \operatorname{tr}(\boldsymbol{e}_{x'}((\boldsymbol{u}^\star)'))\boldsymbol{I}' + 2\mu_1\boldsymbol{e}_{x'}((\boldsymbol{u}^\star)'), \quad \boldsymbol{I}' := \begin{pmatrix} 0&0&0\\0&1&0\\0&0&1\end{pmatrix} \end{aligned} \tag{3.30}$$

the limit problem as $\varepsilon\to 0$, deduced from (3.14), takes the form

$$\begin{cases} -\,\mathbf{div}\boldsymbol{\sigma}_0(\boldsymbol{u}) = \boldsymbol{f} & \text{in} \ \ \Omega\setminus\Sigma_{L/2}, \\ (\boldsymbol{\sigma}_0(\boldsymbol{u})\boldsymbol{e}_1)^- - (\boldsymbol{\sigma}_0(\boldsymbol{u})\boldsymbol{e}_1)^+ - \mathbf{div}_{x'}\boldsymbol{\sigma}'_1((\boldsymbol{u}^\star)') = 0 & \text{on } \Sigma_{L/2}, \\ \boldsymbol{u}\in H_0^1(\Omega;\mathbb{R}^3), \quad u^\star_\alpha(L/2,.)\in H_0^1(\Omega')\ (\alpha\in\{2;3\}). \end{cases} \tag{3.31}$$

The field $(\boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{e}_1)^-$ (resp. $(\boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{e}_1)^+$) represents the supercicial density of forces exerted by the material occupying $\Omega\setminus\Sigma_{L/2}$ on the left (resp. right) face of $\Sigma_{L/2}$.

(iii) Assume now that the last-mentioned single layer is filled by a soft (instead of stiff) material of Lamé coefficients of order ε. More precisely, assume that μ_ε is defined by substituting $\varepsilon\mu_1$ for $\frac{1}{\varepsilon}\mu_1$ in (3.29). Then the assumptions and convergences (3.3) hold with $m = \mu_0\mathcal{L}^1$ and $\nu = \frac{1}{\mu_0}\mathcal{L}^1 + \frac{1}{\mu_1}\delta_{\frac{L}{2}}$. In this case, by (3.4) and (3.7), $\mathcal{A}_\nu = \{L/2\}$, $\mathcal{A}_m = \emptyset$, and

$$BD_0^{\nu,m}(\Omega) = \{\boldsymbol{\varphi} \in H^1(\Omega \setminus \Sigma_{L/2}; \mathbb{R}^3),\ \boldsymbol{\varphi} = 0 \text{ on } \partial\Omega\}.$$

By (3.14), the limit problem as $\varepsilon \to 0$ takes the form

$$\begin{cases} -\mathbf{div}\boldsymbol{\sigma}_0(\boldsymbol{u}) = \boldsymbol{f} & \text{in } \Omega \setminus \Sigma_{L/2}, \\ \mu_1 \boldsymbol{A}(\boldsymbol{u}^+ - \boldsymbol{u}^-) = (\boldsymbol{\sigma}_0(\boldsymbol{u})\boldsymbol{e}_1)^- = (\boldsymbol{\sigma}_0(\boldsymbol{u})\boldsymbol{e}_1)^+ & \text{on } \Sigma_{L/2}, \\ \boldsymbol{u} \in H^1(\Omega \setminus \Sigma_{L/2}; \mathbb{R}^3), \quad \boldsymbol{u} = 0 \text{ on } \partial\Omega, \end{cases}$$

where $\boldsymbol{A}$ (resp. $\boldsymbol{\sigma}_0(\boldsymbol{u})$) is defined by (3.13) (resp. (3.30)).

REMARK 3.7. *Assumption (3.4) is needed in the proof of Lemma 4.6 and in the proof of (4.42). This assumption is equivalent to (see [26, p. 300, Lemma 6.2]):*

$$\begin{aligned} &\forall \eta > 0, \quad \exists \delta > 0, \quad \exists \varepsilon_0 > 0, \quad \forall \varepsilon < \varepsilon_0, \\ &\int_{\{(s_1,t_1)\in(0,L)^2,\ |s_1-t_1|<\delta\}} \mu_\varepsilon^{-1}(s_1)\mu_\varepsilon(t_1)ds_1dt_1 < \varepsilon. \end{aligned} \tag{3.32}$$

When ν and m do not satisfy (3.4), the effective problem does not only depend on the couple (ν, m), but also on the choice of the sequence (μ_ε) satisfying (3.3). By way of illustration, let us choose two sequences of positive reals $(r_\varepsilon^{(1)})$, $(r_\varepsilon^{(2)})$ such that $r_\varepsilon^{(1)} \ll r_\varepsilon^{(2)} \ll 1$, set $I_{1\varepsilon} := \left(\frac{L}{2} - \frac{r_\varepsilon^{(1)}}{2}, \frac{L}{2} + \frac{r_\varepsilon^{(1)}}{2}\right)$, $I_{2\varepsilon} := \left(\frac{L}{2} - \frac{r_\varepsilon^{(2)}}{2}, \frac{L}{2} + \frac{r_\varepsilon^{(2)}}{2}\right)$, fix $\zeta \in \{-1, 1\}$, and consider the sequence (μ_ε) defined by

$$\mu_\varepsilon := \mathbb{1}_{(0,L)\setminus I_{2\varepsilon}} + \left(r_\varepsilon^{(2)}\right)^\zeta \mathbb{1}_{I_{2\varepsilon}\setminus I_{1\varepsilon}} + \left(r_\varepsilon^{(1)}\right)^{-\zeta} \mathbb{1}_{I_{1\varepsilon}}. \tag{3.33}$$

The convergences (3.3) are satisfied with $\nu = m = \delta_{\frac{L}{2}} + \mathcal{L}^1$. By adapting to the framework of elasticity the argument developed in [10, Chapter 4] in the context of the heat equation, one can prove that, under these assumptions, the solution $\boldsymbol{u}_\varepsilon$ to (3.2) strongly converges in $L^2(\Omega; \mathbb{R}^3)$ to the unique solution to

$$\begin{aligned} &(\mathcal{P}^{(\zeta)}): \quad \inf\left\{F^{(\zeta)}(\boldsymbol{\varphi}) - \int_\Omega \boldsymbol{f}\cdot\boldsymbol{\varphi}dx, \quad \boldsymbol{\varphi} \in \mathcal{D}\right\}, \\ &\mathcal{D} := \left\{\boldsymbol{\varphi} \in H^1(\Omega \setminus \Sigma_{L/2}),\ \boldsymbol{\varphi} = 0 \text{ on } \partial\Omega, \quad (\boldsymbol{\varphi}')^+, (\boldsymbol{\varphi}')^- \in H_0^1(\Sigma_{\frac{L}{2}}; \mathbb{R}^3)\right\}. \end{aligned} \tag{3.34}$$

If $\zeta = -1$, the effective energy is given by

$$\begin{aligned} F^{(-1)}(\boldsymbol{\varphi}) = {}& \tfrac{1}{2}\int_{\Omega\setminus\Sigma_{L/2}} \boldsymbol{\sigma}(\boldsymbol{\varphi}) : \boldsymbol{e}(\boldsymbol{\varphi})dx + \frac{l+2}{2}\int_{\Sigma_{L/2}} |\varphi_1^+ - \varphi_1^-|^2 d\mathcal{H}^2 \\ &+ \tfrac{1}{4}\int_{\Sigma_{L/2}} \boldsymbol{\sigma}_{x'}(\boldsymbol{\varphi}^-) : \boldsymbol{e}_{x'}(\boldsymbol{\varphi}^-)d\mathcal{H}^2 + \tfrac{1}{4}\int_{\Sigma_{L/2}} \boldsymbol{\sigma}_{x'}(\boldsymbol{\varphi}^+) : \boldsymbol{e}_{x'}(\boldsymbol{\varphi}^+)d\mathcal{H}^2 \\ &+ \tfrac{1}{2}\int_{\Sigma_{L/2}} \left|(\boldsymbol{\varphi}')^+ - (\boldsymbol{\varphi}')^-\right|^2 d\mathcal{H}^2. \end{aligned}$$

If $\zeta = 1$, the effective energy is is the non-local functional defined by

$$F^{(1)}(\boldsymbol{\varphi}) = \inf_{\boldsymbol{v}\in H^1_0(\Sigma_{L/2};\mathbb{R}^3)} \Phi(\boldsymbol{\varphi}, \boldsymbol{v}'), \tag{3.35}$$

where

$$\begin{aligned}\Phi(\boldsymbol{\varphi}, \boldsymbol{v}') := \tfrac{1}{2}\int_{\Omega\setminus\Sigma_{L/2}} \boldsymbol{\sigma}(\boldsymbol{\varphi}) : \boldsymbol{e}(\boldsymbol{\varphi})dx + \frac{l+2}{2}\int_{\Sigma_{L/2}} |\varphi_1^+ - \varphi_1^-|^2 d\mathcal{H}^2 \\ + \tfrac{1}{2}\int_{\Sigma_{L/2}} \boldsymbol{\sigma}_{x'}(\boldsymbol{v}') : \boldsymbol{e}_{x'}(\boldsymbol{v}')d\mathcal{H}^2 \\ + \int_{\Sigma_{L/2}} |\boldsymbol{v}' - (\boldsymbol{\varphi}')^+|^2 + |\boldsymbol{v}' - (\boldsymbol{\varphi}')^-|^2 d\mathcal{H}^2.\end{aligned}$$

Substituting $\frac{(\boldsymbol{\varphi}')^+ + (\boldsymbol{\varphi}')^-}{2}$ for $\boldsymbol{v}'$ in (3.35) and applying the two-dimensional Korn inequality in $H^1_0(\Sigma_{L/2};\mathbb{R}^2)$, we find

$$\begin{aligned}F^{(-1)}(\boldsymbol{\varphi}) &= \phi\left(\boldsymbol{\varphi}, \tfrac{(\boldsymbol{\varphi}')^+ + (\boldsymbol{\varphi}')^-}{2}\right) + \tfrac{1}{8}\int_{\Sigma_{L/2}} \boldsymbol{\sigma}_{x'}\left(\tfrac{(\boldsymbol{\varphi}')^+ - (\boldsymbol{\varphi}')^-}{2}\right) : \boldsymbol{e}_{x'}\left(\tfrac{(\boldsymbol{\varphi}')^+ - (\boldsymbol{\varphi}')^-}{2}\right) d\mathcal{H}^2 \\ &\geq \phi\left(\boldsymbol{\varphi}, \tfrac{(\boldsymbol{\varphi}')^+ + (\boldsymbol{\varphi}')^-}{2}\right) + C\int_{\Sigma_{L/2}} \left|\tfrac{(\boldsymbol{\varphi}')^+ - (\boldsymbol{\varphi}')^-}{2}\right|^2 d\mathcal{H}^2.\end{aligned}$$

Therefore, by (3.35), $F^{(-1)}(\boldsymbol{\varphi}) \geq F^{(1)}(\boldsymbol{\varphi})$, and the equality $F^{(-1)}(\boldsymbol{\varphi}) = F^{(1)}(\boldsymbol{\varphi})$ can only hold if

1. *$(\boldsymbol{\varphi}')^+ = (\boldsymbol{\varphi}')^-$ on $\Sigma_{L/2}$, which means that $\boldsymbol{\varphi}'^\star(L/2, x') = (\boldsymbol{\varphi}')^+(L/2, x') = (\boldsymbol{\varphi}')^-(L/2, x')$,*
2. *$\boldsymbol{v}' = \boldsymbol{\varphi}'^\star(L/2, x')$ is the solution to the infimum problem (3.35), which implies that $\boldsymbol{\varphi}'^\star = (\boldsymbol{\varphi}')^+ = (\boldsymbol{\varphi}')^- = 0$ in $\Sigma_{L/2}$.*

Such an occurence seems not likely to happen, in general, for the solution $\boldsymbol{\varphi}$ to (3.34): for instance, if we choose $\boldsymbol{f} = \boldsymbol{e}_2$ in (3.34), we intuitively expect that projections $(\boldsymbol{\varphi}')^+(L/2, x')$, $(\boldsymbol{\varphi}')^-(L/2, x')$ of the traces on $\Sigma_{L/2}$ of the solution $\boldsymbol{\varphi}$ to (3.34) do not vanish. Indeed, when (3.4) is not satisfied, one can prove the existence of infinitely many different limit problems associated to some sequence (μ_ε) satisfying (3.3).

REMARK 3.8. *If $\nu(\{0\}) > 0$, the effective displacement may fail to vanish on Σ_0, and the following concentration of elastic energy may appear on Σ_0:*

$$\tfrac{1}{2}\nu(\{0\})^{-1}\int_{\Sigma_0} \boldsymbol{u}^+ \cdot \boldsymbol{A}\boldsymbol{u}^+ d\mathcal{H}^2. \tag{3.36}$$

The extra term (3.36) is obtained by substituting $(0, 0)$ for $(t, \boldsymbol{u}^-)$ in (1.3). A similar contribution emerges on Σ_L if $\nu(\{L\}) > 0$. This phenomenon is related to the fact that the trace application is not weakly continuous on $BD(\Omega)$.*

REMARK 3.9. *Our method applies to the study of second-order elliptic systems of partial differential equations of the type*

$$(\mathcal{P}_\varepsilon):\ -\mathbf{div}(\mu_\varepsilon \boldsymbol{C}\,\boldsymbol{\nabla}\boldsymbol{u}_\varepsilon) = \boldsymbol{f} \quad \text{in } \Omega, \quad \boldsymbol{u}_\varepsilon \in H^1_0(\Omega;\mathbb{R}^n),\ \boldsymbol{f} \in L^\infty(\Omega;\mathbb{R}^n), \tag{3.37}$$

where $\Omega := (0,L)\times\Omega'$ is a cylindrical domain in $\mathbb{R}^d$ and $\boldsymbol{C}$ is a second order tensor on $\mathbb{R}^{n+d}$ satisfying the following assumptions of symmetry and ellipticity:

$$\begin{aligned}&\mathrm{C}_{ijpq} = \mathrm{C}_{pqij} \quad \forall((i,j),(p,q)) \in (\mathbb{R}^n\times\mathbb{R}^d)^2,\\ &\boldsymbol{C}\,\boldsymbol{\Xi} : \boldsymbol{\Xi} \geq c|\boldsymbol{\Xi}|^2 \quad \forall \boldsymbol{\Xi}\in\mathbb{R}^{n\times d} \qquad \textit{for some} \;\; c>0.\end{aligned} \tag{3.38}$$

We suppose that

$$\boldsymbol{T} := \sum_{i,p=1}^{n} \mathrm{C}_{i1p1}\boldsymbol{e}_i\otimes\boldsymbol{e}_p \quad \textit{is invertible.} \tag{3.39}$$

We denote by $BV(\Omega;\mathbb{R}^n)$ the space of $\mathbb{R}^n$-valued functions on Ω with bounded variation, that is

$$BV(\Omega;\mathbb{R}^n) := \left\{\boldsymbol{\varphi}\in L^1(\Omega;\mathbb{R}^n):\;\boldsymbol{D\varphi}\in\mathcal{M}(\Omega;\mathbb{R}^{n+d})\right\}. \tag{3.40}$$

Under these assumptions, the solution to (3.37) weakly converges in $BV(\Omega;\mathbb{R}^n)$ to the unique solution to the problem*

$$\min_{\boldsymbol{u}\in BV_0^{\nu,m}(\Omega)} \tfrac{1}{2}a(\boldsymbol{u},\boldsymbol{u}) - \int_\Omega \boldsymbol{f}\cdot\boldsymbol{u}dx \tag{3.41}$$

where $BV_0^{\nu,m}(\Omega)$ is the Hilbert space defined by

$$\begin{aligned}&BV_0^{\nu,m}(\Omega) := \left\{\boldsymbol{\varphi}\in BV(\Omega;\mathbb{R}^n)\;\middle|\;\begin{array}{l}\boldsymbol{D\varphi}\ll\nu\otimes\mathcal{L}^{d-1},\quad \boldsymbol{\varphi}=0\;\;\textit{on}\;\;\partial\Omega\\ \frac{\boldsymbol{D\varphi}}{\nu\otimes\mathcal{L}^{d-1}}\in L^2_{\nu\otimes\mathcal{L}^{d-1}}(\Omega;\mathbb{R}^n)\\ \boldsymbol{\varphi}^\star\in L^2_m(0,L;H^1_0(\Omega';\mathbb{R}^n))\end{array}\right\},\\ &\|\boldsymbol{\varphi}\|_{BV_0^{\nu,m}(\Omega)} := \left(\int_\Omega\left|\frac{\boldsymbol{D\varphi}}{\nu\otimes\mathcal{L}^{d-1}}\right|^2 d\nu\otimes\mathcal{L}^{d-1}\right)^{\frac{1}{2}} + \left(\int_\Omega|\boldsymbol{\nabla}_{x'}(\boldsymbol{\varphi}^\star)|^2 dm\otimes\mathcal{L}^{d-1}\right)^{\frac{1}{2}},\end{aligned} \tag{3.42}$$

and, setting

$$\boldsymbol{\nabla}_{x'}\boldsymbol{\varphi} := \sum_{i=1}^{n}\sum_{\alpha=2}^{d}\frac{\partial\varphi_i}{\partial x_\alpha}\boldsymbol{e}_i\otimes\boldsymbol{e}_\alpha, \tag{3.43}$$

a is the continuous coercive symmetric bilinear form on $BV_0^{\nu,m}(\Omega)$ given by

$$a(\boldsymbol{u},\boldsymbol{\varphi}) := \int_\Omega \boldsymbol{a}^\perp\frac{\boldsymbol{Du}}{\nu\otimes\mathcal{L}^2}:\frac{\boldsymbol{D\varphi}}{\nu\otimes\mathcal{L}^2}d\nu\otimes\mathcal{L}^{d-1} + \int_\Omega \boldsymbol{a}^\parallel\boldsymbol{\nabla}_{x'}(\boldsymbol{u}^\star):\boldsymbol{\nabla}_{x'}(\boldsymbol{\varphi}^\star)dm\otimes\mathcal{L}^{d-1}, \tag{3.44}$$

with

$$
\begin{aligned}
a^{\perp}_{ijkl} &:= \sum_{p,r=1}^{n} C_{ijp1}(\boldsymbol{T}^{-1})_{pr}C_{r1kl},\\
a^{\parallel}_{ijkl} &:= \sum_{p,r=1}^{n} (C_{ijp1}(\boldsymbol{T}^{-1})_{pr}C_{r1kl}+C_{ijkl})(1-\delta_{j1})(1-\delta_{l1}).
\end{aligned}
\tag{3.45}
$$

PROPOSITION 3.10. *The normed space $BV_0^{\nu,m}(\Omega)$ defined by (3.42) is a Hilbert space. Under the assumptions (3.3), (3.38), (3.39), the symmetric bilinear form $a(\cdot,\cdot)$ defined by (3.44) is coercive and continuous on $BV_0^{\nu,m}(\Omega)$, and the sequence $(\boldsymbol{u}_\varepsilon)$ of the solutions to (3.37) weakly* converges in $BV(\Omega;\mathbb{R}^n)$ to the unique solution $\boldsymbol{u}$ to (3.41).*

The proof of Proposition 3.10 is sketched in Section 6.4.

REMARK 3.11. *The particular case of the heat equation in a three-dimensional domain corresponds to the choice $(n,d)=(1,3)$ in (3.37). Setting $A_{jq}:=\mathrm{C}_{1j1q}$, we deduce from Proposition 3.10 that under the assumption (3.3), if $\boldsymbol{A}$ is definite positive and $A_{11}\neq 0$ (see (3.39)), the solution u_ε to*

$$
(\mathcal{P}_\varepsilon):\ -\mathbf{div}(\mu_\varepsilon \boldsymbol{A}\boldsymbol{\nabla} u_\varepsilon)=f \quad \text{in } \Omega, \quad u_\varepsilon\in H^1_0(\Omega), \quad f\in L^\infty(\Omega),
\tag{3.46}
$$

weakly converges in $BV(\Omega;\mathbb{R})$ to the unique solution to*

$$
\min_{u\in BD_0^{\nu,m}(\Omega)} \tfrac{1}{2}a(u,u)-\int_\Omega fu\,dx,
$$

where a is defined on $BV_0^{\nu,m}(\Omega)^2$ by

$$
a(u,\varphi):=\tfrac{1}{2}\int_\Omega \boldsymbol{A}^{\perp}\frac{\boldsymbol{D}u}{\nu\otimes\mathcal{L}^2}\cdot\frac{\boldsymbol{D}\varphi}{\nu\otimes\mathcal{L}^2}d\nu\otimes\mathcal{L}^2+\tfrac{1}{2}\int_\Omega \boldsymbol{A}^{\parallel}\boldsymbol{\nabla}_{x'}(u^\star)\cdot\boldsymbol{\nabla}_{x'}(\varphi^\star)dm\otimes\mathcal{L}^2,
$$

in terms of $\boldsymbol{A}^{\perp},\boldsymbol{A}^{\parallel}$ given by

$$
A^{\perp}_{ij}:=\tfrac{A_{i1}A_{1j}}{A_{11}}, \qquad A^{\parallel}_{ij}:=(\tfrac{A_{i1}A_{1j}}{A_{11}}+A_{ij})(1-\delta_{i1})(1-\delta_{j1}).
$$

Linear diffusion problems in stratified media with high contrast have also been studied in [25, 26, 27, 28, 30, 31, 32, 33, 35].

REMARK 3.12. *Let X, Y be separable reflexive Banach spaces such that $X\subset Y$ with dense and compact embedding, $f:[0,L]\times X\to[0,+\infty)$, $g:[0,L]\times Y\to[0,+\infty)$ be convex mappings with respect to the second variable with growth conditions of order strictly larger than 1, and (a_ε), (b_ε) be sequences in $L^\infty(0,L)$ such that $\frac{1}{a_\varepsilon}\overset{\star}{\rightharpoonup}\nu$ and $b_\varepsilon\overset{\star}{\rightharpoonup}m$ weakly* in $\mathcal{M}([0,L])$ for some couple (ν,m) satisfying (3.4), (3.5). Denoting by u' the distributional derivative of u, we set $W^{1,1}(0,L;Y,X):=\{u\in L^1(0,L;Y), u'\in L^1(0,L;X)\}$ and $BV(0,L;Y,X):=\{u\in L^1(0,L;Y), u'\in \mathcal{M}(0,L;X)\}$, where $\mathcal{M}(0,L;X)$ is the set of X-valued measures on $(0,L)$ with bounded total variation. Bouchitté and Picard have established in [17] the Γ-convergence (see [24]) with respect to the strong topology of $L^1(0,L;X)$ of the sequence of functional*

$$F_\varepsilon := \boldsymbol{u} \in L^1(0,L;X) \to \begin{cases} \displaystyle\int_0^L \tfrac{1}{a_\varepsilon} f(t, a_\varepsilon u'_\varepsilon)dt + \int_0^L b_\varepsilon G(t,u)dt \\ \qquad\qquad\qquad \text{if} \;\; u_\varepsilon \in W^{1,1}(0,L;Y,X), \\ +\infty \qquad\qquad \text{otherwise}, \end{cases}$$

to the functional

$$F := \boldsymbol{u} \in L^1(0,L;X) \to \begin{cases} \displaystyle\int_0^L f(t, \tfrac{u'}{\nu})d\nu + \int_0^L G(t,u^\star)dm \\ \qquad\qquad \text{if} \;\; u \in BV(0,L;Y,X) \;\text{and}\;\; u' \ll \nu, \\ +\infty \qquad\qquad \text{otherwise.} \end{cases}$$

As an application, setting $X = L^2(\Omega')$, $Y = H^1_0(\Omega')$, $f(t,u) = |u|^2_X$, $G(t,u) = |u|^2_Y$, $\boldsymbol{A}_\varepsilon := \begin{pmatrix} a_\varepsilon & 0 & 0 \\ 0 & b_\varepsilon & 0 \\ 0 & 0 & b_\varepsilon \end{pmatrix}$, *they deduce the convergence of the solution to* $-\mathbf{div}\boldsymbol{A}_\varepsilon \boldsymbol{\nabla} u_\varepsilon = f$, $u_\varepsilon \in H^1_0(\Omega)$, *to the solution to* $\min_{BV_0^{\nu,m}(\Omega)} F(u) - \int f u dx$, *where*

$$F(u) := \tfrac{1}{2}\int_\Omega \left|\frac{\boldsymbol{D}u}{\nu \otimes \mathcal{L}^2}\right|^2 d\nu \otimes \mathcal{L}^2 + \sum_{\alpha=2}^3 \tfrac{1}{2}\int_\Omega |\boldsymbol{\nabla}_{x'} u^\star|^2 \, dm \otimes \mathcal{L}^2.$$

Unlike ours, this approach does not apply to non diagonal conductivity matrices.

Remark 3.13. *When* (μ_ε) *and* (μ_ε^{-1}) *are uniformly bounded in* $L^\infty(0,L)$, *the solution* $\boldsymbol{u}_\varepsilon$ *to* (3.2) *weakly converges, up to a subsequence, to* $\boldsymbol{u}$ *in* $H^1_0(\Omega;\mathbb{R}^3)$ *and the sequence* $\boldsymbol{\sigma}_\varepsilon := \boldsymbol{\sigma}_\varepsilon(\boldsymbol{u}_\varepsilon)$ *weakly converges in* $L^2(\Omega;\mathbb{S}^3)$ *to some* $\boldsymbol{\sigma}$ *satisfying* $-\mathbf{div}\boldsymbol{\sigma} = \boldsymbol{f}$ *in* Ω. *The constitutive relation between* $\boldsymbol{\sigma}$ *and* $\boldsymbol{e} := \boldsymbol{e}(\boldsymbol{u})$ *can be deduced from classical layering arguments (see the early works of F. Murat and L. Tartar [43, 52], [53, p. 140], and also [29]). These arguments rest on the so-called "good" behavior of some components of* $\boldsymbol{\sigma}_\varepsilon$ *and* $\boldsymbol{e}_\varepsilon := \boldsymbol{e}(\boldsymbol{u}_\varepsilon)$, *which do not oscillate in* x_1 *in the following sense: a sequence* (g_ε) *that weakly converges in* $L^2(\Omega)$ *to* g *is said to not oscillate in* x_1 *if, for any sequence* $h_\varepsilon(x_1)$ *only depending on* x_1 *and weakly converging in* $L^2(0,L)$ *to* h, *the sequence* $(g_\varepsilon h_\varepsilon)$ *weakly* converges in* $\mathcal{M}(\Omega)$ *to* gh. *It turns out that* $(\sigma_{\varepsilon 1i})_{i\in\{1,2,3\}}$ *and* $(\boldsymbol{e}_{\varepsilon\alpha\beta})_{\alpha,\beta\in\{2,3\}}$ *are "good" components of* $\boldsymbol{\sigma}_\varepsilon$ *and* $\boldsymbol{e}_\varepsilon$: *for, denoting by* $\boldsymbol{\sigma}_\varepsilon^{(i)}$ *the* i^{th} *column of* $\boldsymbol{\sigma}_\varepsilon$, *noticing that* $-\mathrm{div}\boldsymbol{\sigma}_\varepsilon^{(i)} = f_i$ *and* $\mathbf{curl}\;(h_\varepsilon(x_1)\boldsymbol{e}_1) = 0$, *by the div-curl lemma (see [53, Lemma 7.2]) the sequence* $\boldsymbol{\sigma}_\varepsilon^{(i)} \cdot h_\varepsilon(x_1)\boldsymbol{e}_1 = \sigma_{\varepsilon 1i} h(x_1)$ *weakly* converges in* $\mathcal{M}(\Omega)$ *to* $\sigma_{1i}h$. *Likewise, since* $\mathbf{curl}\;\boldsymbol{\nabla} u_{\varepsilon\alpha} = 0$ *and* $\mathrm{div}(h_\varepsilon(x_1)\boldsymbol{e}_\beta) = 0$, *the sequence* $\boldsymbol{\nabla} u_{\varepsilon\alpha} \cdot h_\varepsilon(x_1)\boldsymbol{e}_\beta = \frac{\partial u_{\varepsilon\alpha}}{\partial x_\beta} h_\varepsilon(x_1)$ *weakly* converges in* $\mathcal{M}(\Omega)$ *to* $\frac{\partial u_\alpha}{\partial x_\beta} h(x_1)$. *The original idea of F. Murat and L. Tartar consists of transforming the constitutive equation* $\boldsymbol{\sigma}_\varepsilon = \boldsymbol{a}_\varepsilon(x_1)\boldsymbol{e}_\varepsilon$ *into an equation of the form* $\boldsymbol{O}_\varepsilon = \boldsymbol{b}_\varepsilon(x_1)\boldsymbol{G}_\varepsilon$, *where* $\boldsymbol{b}_\varepsilon = \boldsymbol{\phi}(\boldsymbol{a}_\varepsilon)$ *for some suitable fourth-order tensors' valued (non-linear) mapping* $\boldsymbol{\phi}$, *and* $\boldsymbol{G}_\varepsilon$ *(resp.* $\boldsymbol{O}_\varepsilon$*) is the matrix of the "good" components (resp. of the remaining, so-called "oscillatory" ones), namely*

$$\boldsymbol{G}_\varepsilon := \begin{pmatrix} \sigma_{\varepsilon 11} & \sigma_{\varepsilon 12} & \sigma_{\varepsilon 13} \\ \sigma_{\varepsilon 21} & e_{\varepsilon 22} & e_{\varepsilon 23} \\ \sigma_{\varepsilon 31} & e_{\varepsilon 32} & e_{\varepsilon 33} \end{pmatrix}, \qquad \boldsymbol{O}_\varepsilon := \begin{pmatrix} e_{\varepsilon 11} & e_{\varepsilon 12} & e_{\varepsilon 13} \\ e_{\varepsilon 21} & \sigma_{\varepsilon 22} & \sigma_{\varepsilon 23} \\ e_{\varepsilon 31} & \sigma_{\varepsilon 32} & \sigma_{\varepsilon 33} \end{pmatrix}.$$

Notice that $\boldsymbol{\sigma}_\varepsilon : \boldsymbol{e}_\varepsilon = \boldsymbol{O}_\varepsilon : \boldsymbol{G}_\varepsilon = \boldsymbol{b}_\varepsilon \boldsymbol{G}_\varepsilon : \boldsymbol{G}_\varepsilon$. *It turns out that, up to a subsequence,* $(\boldsymbol{b}_\varepsilon(x_1))$ *weakly converges to some* $\boldsymbol{b}$ *in* L^2, *hence we can pass to the limit in the equation* $\boldsymbol{O}_\varepsilon = \boldsymbol{b}_\varepsilon(x_1)\boldsymbol{G}_\varepsilon$ *in the weak* topology of* $\mathcal{M}(\Omega;\mathbb{S}^3)$. *We obtain the equation*

$$\boldsymbol{O} = \boldsymbol{b}\,\boldsymbol{G} \quad \textit{in } \Omega; \quad \boldsymbol{G} := \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & e_{22} & e_{23} \\ \sigma_{31} & e_{32} & e_{33} \end{pmatrix}, \quad \boldsymbol{O} := \begin{pmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & \sigma_{22} & \sigma_{23} \\ e_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix},$$

equivalent to the effective constitutive equation

$$\boldsymbol{\sigma} = \boldsymbol{a}\boldsymbol{e} \quad \textit{in } \Omega; \quad \boldsymbol{a} := \boldsymbol{\phi}^{-1}(\boldsymbol{b}).$$

The limit process yielding the effective elasticity tensor $\boldsymbol{a} = \boldsymbol{\phi}^{-1}(\lim_{L^2-weak} \boldsymbol{\phi}(\boldsymbol{a}_\varepsilon))$ *is called the* 1^**-convergence of the sequence* $(\boldsymbol{a}_\varepsilon)$ *(see [52, p.14]). Our proof is connected with these classical layering arguments insofar as, in order to pass to the limit as* $\varepsilon \to 0$ *in the variational formulation* (6.2), *we write* $\boldsymbol{\sigma}_\varepsilon(\boldsymbol{u}_\varepsilon) : \boldsymbol{e}(\boldsymbol{\varphi}_\varepsilon) = \boldsymbol{b}_\varepsilon \boldsymbol{G}_\varepsilon(\boldsymbol{u}_\varepsilon) : \boldsymbol{G}_\varepsilon(\boldsymbol{\varphi}_\varepsilon)$ *(see* (6.10)*) and establish that* $\boldsymbol{G}_\varepsilon(\boldsymbol{\varphi}_\varepsilon)$ *has a "good" behavior with respect to some suitable notion of strong convergence (see* (4.2), (6.9)*).*

4. Technical preliminaries and a priori estimates. This section is dedicated, essentially, to the analysis of the asymptotic behaviour of the solution $(\boldsymbol{u}_\varepsilon)$ to (3.2) and its stress $\boldsymbol{\sigma}_\varepsilon(\boldsymbol{u}_\varepsilon)$ in the limit $\varepsilon \to 0$. The following notion of convergence will take a crucial part in this study.

Definition 4.1. *Let* $\theta_\varepsilon, \theta$ *be positive Radon measures on a compact set* $K \subset \mathbb{R}^N$ *and let* f_ε, f *be Borel functions on* K. *We say that* (f_ε) *weakly converges to* f *with respect to the pair* $(\theta_\varepsilon, \theta)$ *if*

$$\begin{aligned} &\sup_\varepsilon \int_K |f_\varepsilon|^2 d\theta_\varepsilon < \infty, \quad f \in L^2_\theta(K) \\ &\theta_\varepsilon \overset{\star}{\rightharpoonup} \theta \quad \textit{and } f_\varepsilon\theta_\varepsilon \overset{\star}{\rightharpoonup} f\theta \qquad \textit{weakly* in } \mathcal{M}(K), \\ &(\textit{notation: } f_\varepsilon \overset{\theta_\varepsilon,\theta}{\rightharpoonup} f). \end{aligned} \tag{4.1}$$

We say that (f_ε) *strongly converges to* f *with respect to the pair* $(\theta_\varepsilon, \theta)$ *if*

$$f_\varepsilon \xrightarrow{\theta_\varepsilon,\theta} f \quad \textit{and} \quad \limsup_{\varepsilon\to 0} \int_K |f_\varepsilon|^2 d\theta_\varepsilon \le \int_K |f|^2 d\theta \quad (\textit{notation: } f_\varepsilon \xrightarrow{\theta_\varepsilon,\theta} f). \tag{4.2}$$

We now present the main statement of the section. For notational simplicity, the measures $(\nu_\varepsilon \otimes \mathcal{L}^2)_{\lfloor\overline{\Omega}}$ and $(m_\varepsilon \otimes \mathcal{L}^2)_{\lfloor\overline{\Omega}}$ are denoted by $\nu_\varepsilon \otimes \mathcal{L}^2$ and $m_\varepsilon \otimes \mathcal{L}^2$. We set (see (2.8))

$$\boldsymbol{\sigma}^\nu(\boldsymbol{\varphi}) := l \operatorname{tr}\left(\frac{\widetilde{\boldsymbol{E}}\boldsymbol{\varphi}}{\nu\otimes\mathcal{L}^2}\right)\boldsymbol{I} + 2\,\frac{\widetilde{\boldsymbol{E}}\boldsymbol{\varphi}}{\nu\otimes\mathcal{L}^2}. \tag{4.3}$$

Proposition 4.2.
Let $(\boldsymbol{u}_\varepsilon)$ *be the sequence of solutions to* (3.2). *Then* $\boldsymbol{u}_\varepsilon$ *is bounded in* $BD(\Omega)$ *and*

$$\sup_{\varepsilon>0}\int_\Omega |\boldsymbol{u}'_\varepsilon|^2 dm_\varepsilon\otimes\mathcal{L}^2+\int_\Omega|\boldsymbol{u}_\varepsilon|dx+\int_\Omega\mu_\varepsilon\,|\boldsymbol{e}(\boldsymbol{u}_\varepsilon)|^2\,dx<\infty. \tag{4.4}$$

Up to a subsequence, there exists $\boldsymbol{u}$ *such that*

$$\begin{aligned}
&\boldsymbol{u}_\varepsilon \overset{\star}{\rightharpoonup} \boldsymbol{u} \quad \textit{weakly* in } BD(\Omega), \qquad \boldsymbol{E}\boldsymbol{u}_\varepsilon \overset{\star}{\rightharpoonup} \widetilde{\boldsymbol{E}\boldsymbol{u}} \quad \textit{weakly* in } \mathcal{M}(\overline{\Omega};\mathbb{S}^3),\\
&\mu_\varepsilon\boldsymbol{e}(\boldsymbol{u}_\varepsilon) \xrightharpoonup{\nu_\varepsilon\otimes\mathcal{L}^2,\nu\otimes\mathcal{L}^2} \frac{\widetilde{\boldsymbol{E}\boldsymbol{u}}}{\nu\otimes\mathcal{L}^2}, \qquad \boldsymbol{\sigma}_\varepsilon(\boldsymbol{u}_\varepsilon) \xrightharpoonup{\nu_\varepsilon\otimes\mathcal{L}^2,\nu\otimes\mathcal{L}^2} \boldsymbol{\sigma}^\nu(\boldsymbol{u}),\\
&\boldsymbol{e}_{x'}(\boldsymbol{u}'_\varepsilon) \xrightharpoonup{m_\varepsilon\otimes\mathcal{L}^2,m\otimes\mathcal{L}^2} \boldsymbol{e}_{x'}((\boldsymbol{u}^\star)'), \quad \boldsymbol{u}\in BD_0^{\nu,m}(\Omega).
\end{aligned} \tag{4.5}$$

Before presenting the proof of Proposition 4.2, we establish some auxiliary results. The next lemma states some fundamental properties of convergence with respect to the pair $(\theta_\varepsilon,\theta)$, proved in [36, Theorem 4.4.2] in a more general context.

Lemma 4.3. *Let* (θ_ε) *be a sequence of positive Radon measures on a compact set* $K\subset\mathbb{R}^N$ *weakly* converging in* $\mathcal{M}(K)$ *to some positive Radon measure* θ*. Then,*
(i) any sequence (f_ε) *of Borel functions on* K *such that*

$$\sup_\varepsilon\int_K|f_\varepsilon|^2d\theta_\varepsilon<\infty, \tag{4.6}$$

has a weakly converging subsequence with respect to the pair $(\theta_\varepsilon,\theta)$*.*
(ii) If $f_\varepsilon\xrightharpoonup{\theta_\varepsilon,\theta} f$ *(resp.* $f_\varepsilon\xrightarrow{\theta_\varepsilon,\theta} f$*), then*

$$\liminf_{\varepsilon\to0}\int_K f_\varepsilon^2d\theta_\varepsilon\geq\int_K f^2d\theta \quad \left(\textit{resp.}\quad \lim_{\varepsilon\to0}\int_K f_\varepsilon^2 d\theta_\varepsilon=\int_K f^2d\theta\right). \tag{4.7}$$

(iii) If $f_\varepsilon\xrightharpoonup{\theta_\varepsilon,\theta} f$ *and* $g_\varepsilon\xrightarrow{\theta_\varepsilon,\theta} g$*, then*

$$\lim_{\varepsilon\to0}\int_K f_\varepsilon g_\varepsilon d\theta_\varepsilon=\int_K fgd\theta.$$

As a first application of Lemma 4.3, we obtain some relations between the measures ν, m, and $\mathcal{L}^1_{\lfloor[0,L]}$:

Lemma 4.4. *Under* (3.3)*, the following holds*

$$\begin{aligned}
&\mathcal{L}^1_{\lfloor[0,L]}\ll\nu; \quad \tfrac{\mathcal{L}^1}{\nu}\in L^2_\nu([0,L]); \quad \mathcal{L}^1_{\lfloor[0,L]}\ll m; \quad \tfrac{\mathcal{L}^1}{m}\in L^2_m([0,L]);\\
&\int_{[0,L]}|\tfrac{\mathcal{L}^1}{\nu}|^2d\nu\leq m([0,L]); \qquad \int_{[0,L]}|\tfrac{\mathcal{L}^1}{m}|^2dm\leq\nu([0,L]).
\end{aligned} \tag{4.8}$$

Proof. Noticing that, by (3.1) and (3.3), $\sup_\varepsilon\int_{[0,L]}|\mu_\varepsilon|^2d\nu_\varepsilon=\sup_\varepsilon m_\varepsilon([0,L])<\infty$ (resp. $\sup_\varepsilon\int_{[0,L]}|\mu_\varepsilon|^{-2}dm_\varepsilon=\sup_\varepsilon\nu_\varepsilon([0,L])<\infty$), we deduce from Lemma 4.3 that the sequence (μ_ε) (resp. (μ_ε^{-1})) has a converging subsequence with respect to the pair (ν_ε,ν) (resp. (m_ε,m)), and

$$\begin{aligned}
&\mu_\varepsilon\nu_\varepsilon\overset{\star}{\rightharpoonup} g\nu, \quad \mu_\varepsilon^{-1}m_\varepsilon\overset{\star}{\rightharpoonup} hm, \quad g\in L^2_\nu, \quad h\in L^2_m,\\
&\int|g|^2d\nu\leq\liminf_{\varepsilon\to0}\int|\mu_\varepsilon|^2d\nu_\varepsilon; \quad \int|h|^2dm\leq\liminf_{\varepsilon\to0}\int|\mu_\varepsilon|^{-2}dm_\varepsilon.
\end{aligned} \tag{4.9}$$

By (3.1) we have $\mu_\varepsilon\nu_\varepsilon = \mu_\varepsilon^{-1}m_\varepsilon = \mathcal{L}^1_{\lfloor[0,L]}$, $|\mu_\varepsilon|^2\nu_\varepsilon = m_\varepsilon$, $|\mu_\varepsilon|^{-2}m_\varepsilon = \nu_\varepsilon$, therefore $g\nu = hm = \mathcal{L}^1_{\lfloor[0,L]}$, $\mathcal{L}^1_{\lfloor[0,L]} \ll \nu$, $\mathcal{L}^1_{\lfloor[0,L]} \ll m$, $g = \frac{\mathcal{L}^1}{\nu}$, $h = \frac{\mathcal{L}^1}{m}$, and the convergences (3.3) and (4.9) imply

$$\begin{aligned} &\int_{[0,L]} |\tfrac{\mathcal{L}^1}{\nu}|^2 d\nu \le \limsup_{\varepsilon\to 0} m_\varepsilon([0,L]) \le m([0,L]), \\ &\int |\tfrac{\mathcal{L}^1}{m}|^2 dm \le \limsup_{\varepsilon\to 0} \nu_\varepsilon([0,L]) \le \nu([0,L]). \end{aligned} \tag{4.10}$$

Assertion (4.8) is proved. □

The following statement is proved in [17, Lemma 3.1].

Lemma 4.5. *Let (b_ε) be a bounded sequence in $L^1(0,L)$ that weakly* converges in $\mathcal{M}([0,L])$ to some Radon measure θ satisfying*

$$\theta(\{0\}) = \theta(\{L\}) = 0. \tag{4.11}$$

Let (w_ε) be a bounded sequence in $W^{1,1}(0,L)$ weakly converging in $BV(0,L)$ to some w. Assume that*

$$\theta(\{t\})Dw(\{t\}) = 0 \quad \forall t \in (0,L). \tag{4.12}$$

Then

$$\lim_{\varepsilon\to 0} \int_0^L \psi b_\varepsilon w_\varepsilon dx = \int_{(0,L)} \psi w^{(r)} d\theta = \int_{(0,L)} \psi w^{(l)} d\theta \qquad \forall \psi \in C([0,L]),$$

where $w^{(r)}$ (resp. $w^{(l)}$) denotes the right-continuous (resp. left-continuous) representative of w.

For any $\boldsymbol{\varphi} \in BD(\Omega)$, we denote by $\boldsymbol{\gamma}^{\pm}_{\Sigma_{x_1}}(\boldsymbol{\varphi})$ the trace of $\boldsymbol{\varphi}$ on both sides of Σ_{x_1} (see (2.9)). As shown in the next lemma, the mappings $x \to \boldsymbol{\gamma}^{\pm}_{\Sigma_{x_1}}(\boldsymbol{\varphi})(x)$ can be identified with the Borel fields $\boldsymbol{\varphi}^{\pm}$ defined by (2.2).

Lemma 4.6. *Let $\boldsymbol{\varphi} \in BD(\Omega)$ and let $\boldsymbol{\varphi}^\star$, $\boldsymbol{\varphi}^{\pm}$ be defined by (2.1), (2.2). Then*

$$\boldsymbol{\gamma}^{\pm}_{\Sigma_{x_1}}(\boldsymbol{\varphi})(x) = \boldsymbol{\varphi}^{\pm}(x) = \lim_{r\to 0} \fint_{B_r^{\pm}(x)} \boldsymbol{\varphi}(y) dy \quad \mathcal{H}^2\text{-}a.e.\ x \in \Sigma_{x_1},\ \forall x_1 \in (0,L). \tag{4.13}$$

Moreover, we have

$$\boldsymbol{\varphi}^\star = \frac{1}{2}(\boldsymbol{\varphi}^+ + \boldsymbol{\varphi}^-) \quad \mathcal{H}^2\text{-}a.e.\ on\ \Sigma_{x_1},\ \forall x_1 \in (0,L), \tag{4.14}$$

and

$$\boldsymbol{\varphi}^\star,\ \boldsymbol{\varphi}^{\pm} \in L^1_{\mathcal{H}^2}(\Sigma_{x_1}) \qquad \forall x_1 \in (0,L). \tag{4.15}$$

Furthermore, the following holds

$$\text{(4.16)}\quad \boldsymbol{\varphi}^+ = \boldsymbol{\varphi}^- = \boldsymbol{\varphi}^\star = \lim_{r\to 0} ⨍_{B_r^\pm(x)} \boldsymbol{\varphi}(y)dy \ \mathcal{H}^2\text{-}a.e. \ in \ \ \Sigma_{x_1} \quad \textit{if} \ \ |\boldsymbol{E\varphi}|(\Sigma_{x_1}) = 0,$$

and

$$\text{(4.17)}\ \ \widetilde{\boldsymbol{E}}\boldsymbol{\varphi} \ll \nu\otimes\mathcal{L}^2 \implies \boldsymbol{\varphi}^+ = \boldsymbol{\varphi}^- = \boldsymbol{\varphi}^\star \quad \mathcal{H}^2\text{-}a.e. \ on\ \Sigma_{x_1}, \ for\ m\text{-}a.e.\ x_1\in(0,L).$$

Proof. The traces of a function with bounded deformation on both side of a $\mathcal{C}^1$ hypersurface M contained in Ω is $\mathcal{H}^2$-a.e. equal to its one side Lebesgue limits (see [38, p. 84, Trace Theorem; p. 91, Proposition 2.2] or [2, p. 209 (ii)-(iii)]). Applying this to $M = \Sigma_{x_1}$, we obtain (4.13). Assertion (4.13) ensures that for all $x_1 \in (0, L)$, for $\mathcal{H}^2-$a.e. $x \in \Sigma_{x_1}$, the two limits in the first line of (2.2) exist and are finite. When they do, the limit in the first line of (2.1) also exists, and

$$\begin{aligned}\frac{1}{2}(\boldsymbol{\varphi}^+(x) + \boldsymbol{\varphi}^-(x)) &= \frac{1}{2}\left(\lim_{r\to 0} ⨍_{B_r^+(x)} \boldsymbol{\varphi}(y)\ dy + ⨍_{B_r^-(x)} \boldsymbol{\varphi}(y)\ dy\right)\\ &= \lim_{r\to 0} ⨍_{B_r(x)} \boldsymbol{\varphi}(y)\ dy = \boldsymbol{\varphi}^\star(x).\end{aligned}$$

Assertion (4.14) is proved. Assertion (4.15) results from (4.13), (4.14) and the fact that the traces of $\boldsymbol{\varphi}$ on each side of Σ_{x_1} belong to $L^1_{\mathcal{H}^2}(\Sigma_{x_1})$. Noticing that by (3.17) we have

$$|\boldsymbol{E\varphi}\lfloor_{\Sigma_{x_1}}| = \left|(\boldsymbol{\varphi}^+ - \boldsymbol{\varphi}^-)\odot \boldsymbol{e}_1\right| \mathcal{H}^2\lfloor_{\Sigma_{x_1}} \quad \forall x_1\in(0,L),$$

we deduce from the elementary inequality

$$\text{(4.18)}\qquad |\boldsymbol{a}| \le \sqrt{2}|\boldsymbol{a}\odot\boldsymbol{n}| \quad \text{ if } \ ||\boldsymbol{n}|| = 1,$$

that $\boldsymbol{\varphi}^+ = \boldsymbol{\varphi}^-$ $\mathcal{H}^2$-a.e. in Σ_{x_1} whenever $|\boldsymbol{E\varphi}|(\Sigma_{x_1}) = 0$. Assertion (4.16) then follows from (4.13) and (4.14). Assertion (4.17) is deduced from (4.16) by noticing that, by (3.4), $m(\mathcal{A}_\nu) = 0$ and that $\nu\otimes\mathcal{L}^2(\Sigma_{x_1}) = \nu(\{x_1\})\mathcal{L}^2(\Omega') = |\boldsymbol{E\varphi}|(\Sigma_{x_1}) = 0$ if $x_1 \notin \mathcal{A}_\nu$ and $\boldsymbol{E\varphi} \ll \nu\otimes\mathcal{L}^2$. ◻

Combined with Lemma 4.5, the following lemma will be used to prove a delicate identification relation (see (4.42)) in the proof of Proposition 4.8.

Lemma 4.7. *Let $\boldsymbol{\varphi} \in BD(\Omega)$ such that $\boldsymbol{\varphi} = 0$ on $\partial\Omega$, and let $\overline{\boldsymbol{\varphi}} \in L^1(0, L; \mathbb{R}^3)$ be the Borel function defined by*

$$\text{(4.19)}\qquad \overline{\boldsymbol{\varphi}}(x_1) := \int_{\Sigma_{x_1}} \boldsymbol{\varphi}^\star d\mathcal{H}^2 \quad \forall x_1 \in (0, L).$$

The following holds

$$\text{(4.20)}\quad \begin{aligned}&\overline{\boldsymbol{\varphi}}\in BV(0,L;\mathbb{R}^3),\ \ ||\overline{\boldsymbol{\varphi}}||_{L^1(0,L;\mathbb{R}^3)} \le ||\boldsymbol{\varphi}||_{L^1(\Omega)},\ \ ||\overline{\boldsymbol{\varphi}}||_{BV(0,L;\mathbb{R}^3)} \le \sqrt{2}||\boldsymbol{\varphi}||_{BD(\Omega)},\\ &D\overline{\boldsymbol{\varphi}} \ll |\boldsymbol{E\varphi}|(.\times\Omega'),\quad |D\overline{\boldsymbol{\varphi}}|(B) \le \sqrt{2}|\boldsymbol{E\varphi}|(B\times\Omega') \quad \forall B\in\mathcal{B}((0,L)),\end{aligned}$$

where $\mathcal{B}((0,L))$ denotes the Borel σ-algebra of $(0,L)$. Moreover, the left-continuous representative $\overline{\boldsymbol{\varphi}}^{(l)}$ (resp. right-continuous representative $\overline{\boldsymbol{\varphi}}^{(r)}$) of $\overline{\boldsymbol{\varphi}}$ is given by

$$
\begin{aligned}
&\overline{\boldsymbol{\varphi}}^{(l)}(x_1) = \int_{\Sigma_{x_1}} \boldsymbol{\varphi}^- d\mathcal{H}^2 \quad \forall x_1 \in (0,L). \\
&\left(\textit{resp.} \quad \overline{\boldsymbol{\varphi}}^{(r)}(x_1) = \int_{\Sigma_{x_1}} \boldsymbol{\varphi}^+ d\mathcal{H}^2 \quad \forall x_1 \in (0,L) \right).
\end{aligned}
\tag{4.21}
$$

Proof. Let $\mathrm{eV}(\overline{\boldsymbol{\varphi}},(0,L))$ denote the essential variation of $\overline{\boldsymbol{\varphi}}$ on $(0,L)$, that is

$$
\mathrm{eV}(\overline{\boldsymbol{\varphi}},(a,b)) := \inf_{\mathcal{L}^1(N)=0} \sup \left\{ \sum_{i=1}^n \left|\overline{\boldsymbol{\varphi}}(t_{i+1}) - \overline{\boldsymbol{\varphi}}(t_i)\right|, \, \middle| \, \begin{array}{l} t_1,\dots,t_n \in (a,b)\setminus N \\ a < t_1 < \dots < t_n < b \end{array} \right\}.
\tag{4.22}
$$

By [3, Proposition 3.6 and Theorem 3.27], the field $\overline{\boldsymbol{\varphi}}$ belongs to $BV(0,L;\mathbb{R}^3)$ if and only if $\mathrm{eV}(\overline{\boldsymbol{\varphi}},(0,L)) < \infty$ and in this case $\mathrm{eV}(\overline{\boldsymbol{\varphi}},(0,L)) = |\boldsymbol{D}\overline{\boldsymbol{\varphi}}|((0,L))$. Let a,b be two real numbers such that $0 \leq a < b \leq L$, $D := \{t \in (0,L), |\boldsymbol{E}\boldsymbol{\varphi}|(\Sigma_t) > 0\}$ and let $t_1,\dots,t_n \subset (a,b)\setminus D$ such that $0 < t_1 < \dots < t_n < L$. By (4.16), (4.18), and Green's formula in $BD(\Omega_i)$, where $\Omega_i := (t_i,t_{i+1}) \times \Omega'$, we have, since $\boldsymbol{\varphi} = 0$ on $\partial\Omega$,

$$
\begin{aligned}
\left|\overline{\boldsymbol{\varphi}}(t_{i+1}) - \overline{\boldsymbol{\varphi}}(t_i)\right| &= \left| \int_{\Sigma_{t_{i+1}}} \boldsymbol{\varphi}^- d\mathcal{H}^2 - \int_{\Sigma_{t_i}} \boldsymbol{\varphi}^+ d\mathcal{H}^2 \right| \\
&\leq \sqrt{2} \left| \left(\int_{\Sigma_{t_{i+1}}} \boldsymbol{\varphi}^- d\mathcal{H}^2 - \int_{\Sigma_{t_i}} \boldsymbol{\varphi}^+ d\mathcal{H}^2 \right) \odot \boldsymbol{e}_1 \right| \\
&= \sqrt{2} \left| \int_{\partial\Omega_i} \boldsymbol{\gamma}_i(\boldsymbol{\varphi}) \odot \boldsymbol{n} d\mathcal{H}^2 \right| = \sqrt{2} \left|\boldsymbol{E}\boldsymbol{\varphi}(\Omega_i)\right| \leq \sqrt{2} \left|\boldsymbol{E}\boldsymbol{\varphi}\right|(\Omega_i),
\end{aligned}
\tag{4.23}
$$

where $\boldsymbol{\gamma}_i(\boldsymbol{\varphi})$ denotes the trace on $\partial\Omega_i$ of the restriction of $\boldsymbol{\varphi}$ to Ω_i, therefore

$$
\sum_{i=1}^n \left|\overline{\boldsymbol{\varphi}}(t_{i+1}) - \overline{\boldsymbol{\varphi}}(t_i)\right| \leq \sum_{i=1}^n \sqrt{2} \left|\boldsymbol{E}\boldsymbol{\varphi}\right|(\Omega_i) \leq \sqrt{2} \left|\boldsymbol{E}\boldsymbol{\varphi}\right|((a,b)\times\Omega').
$$

By the arbitrary choice of $t_1,\dots,t_n$, noticing that D is at most countable thus $\mathcal{L}^1$-negligible, we infer that $\overline{\boldsymbol{\varphi}} \in BV(a,b;\mathbb{R}^3)$ and

$$
|\boldsymbol{D}\overline{\boldsymbol{\varphi}}|((a,b)) = \mathrm{eV}(\overline{\boldsymbol{\varphi}},(a,b)) \leq \sqrt{2} \left|\boldsymbol{E}\boldsymbol{\varphi}\right|((a,b)\times\Omega'),
\tag{4.24}
$$

yielding, by the arbitrariness of a,b, the second line of (4.20). The first line easily follows. Since $\overline{\boldsymbol{\varphi}} \in BV((0,L);\mathbb{R}^3)$, there exists a left-continuous (resp. right-continuous) representative $\overline{\boldsymbol{\varphi}}^{(l)}$ (resp. $\overline{\boldsymbol{\varphi}}^{(r)}$) of $\overline{\boldsymbol{\varphi}}$. Let us fix $x_1 \in (0,L)$. By (4.23), we have

$$
\begin{aligned}
\limsup_{t\to x_1^-, t\notin D} \left| \int_{\Sigma_{x_1}} \boldsymbol{\varphi}^- d\mathcal{H}^2 - \overline{\boldsymbol{\varphi}}(t) \right| &= \limsup_{t\to x_1^-, t\notin D} \left| \int_{\Sigma_{x_1}} \boldsymbol{\varphi}^- d\mathcal{H}^2 - \int_{\Sigma_t} \boldsymbol{\varphi}^+ d\mathcal{H}^2 \right| \\
&\leq \limsup_{t\to x_1^-, t\notin D} \sqrt{2} \left|\boldsymbol{E}\boldsymbol{\varphi}\right|((t,x_1)\times\Omega') = 0,
\end{aligned}
$$

therefore $\overline{\varphi}^{(l)}(x_1) = \int_{\Sigma_{x_1}} \varphi^- d\mathcal{H}^2$. The proof of the identity $\overline{\varphi}^{(r)}(x_1) = \int_{\Sigma_{x_1}} \varphi^+ d\mathcal{H}^2$ is similar. $\square$

In the next proposition, we study the asymptotic behavior of a sequence (φ_ε) satisfying the estimate

$$\sup_{\varepsilon>0} \int_\Omega |\varphi_\varepsilon| dx + \int_\Omega \mu_\varepsilon \left|e(\varphi_\varepsilon)\right|^2 dx < \infty. \tag{4.25}$$

This study will be applied to the sequence of the solutions to (3.2), and also to the sequence of test fields defined in Section 6 (see Proposition 6.1), which do not necessarily vanish on $\partial\Omega$. We are led to introduce the normed space $BD^{\nu,m}(\Omega)$ deduced from (3.7) by removing the boundary conditions, namely

$$\begin{aligned} &BD^{\nu,m}(\Omega) = \left\{ \varphi \in BD(\Omega) \;\middle|\; \begin{array}{l} E\varphi \ll \nu\otimes\mathcal{L}^2,\ \frac{E\varphi}{\nu\otimes\mathcal{L}^2} \in L^2_{\nu\otimes\mathcal{L}^2}(\Omega;\mathbb{R}^3) \\ (\varphi^\star)' \in L^2_m(0,L;H^1(\Omega';\mathbb{R}^3)) \end{array} \right\}, \\ &\|\varphi\|_{BD^{\nu,m}(\Omega)} := \int_\Omega |\varphi| dx + \left(\int_\Omega \left| \frac{E\varphi}{\nu\otimes\mathcal{L}^2} \right|^2 d\nu\otimes\mathcal{L}^2 \right)^{\frac{1}{2}} + \left(\int_\Omega |e_{x'}(\varphi^\star)|^2 dm\otimes\mathcal{L}^2 \right)^{\frac{1}{2}}. \end{aligned} \tag{4.26}$$

Proposition 4.8.
Let (φ_ε) be a sequence in $W^{1,1}(\Omega;\mathbb{R}^3)$ satisfying the estimate (4.25). Then (φ_ε) is bounded in $BD(\Omega)$ and, up to a subsequence,

$$\begin{aligned} &\varphi_\varepsilon \to \varphi && \textit{strongly in } L^p(\Omega;\mathbb{R}^3)\ \ \forall p \in \left[1, \tfrac{3}{2}\right), \\ &e(\varphi_\varepsilon)\mathcal{L}^3_{\lfloor\Omega} = E\varphi_\varepsilon \overset{\star}{\rightharpoonup} E\varphi && \textit{weakly* in } \mathcal{M}(\Omega;\mathbb{S}^3), \\ &e(\varphi_\varepsilon)\mathcal{L}^3_{\lfloor\overline{\Omega}} = \widetilde{E}\varphi_\varepsilon \overset{\star}{\rightharpoonup} \Upsilon && \textit{weakly* in } \mathcal{M}(\overline{\Omega};\mathbb{S}^3), \end{aligned} \tag{4.27}$$

for some $\varphi \in BD(\Omega)$, $\Upsilon \in \mathcal{M}(\overline{\Omega};\mathbb{S}^3)$. Moreover

$$\begin{aligned} &\Upsilon = \widetilde{E}\varphi, \\ &\widetilde{E}\varphi \ll \nu\otimes\mathcal{L}^2, \quad \frac{\widetilde{E}\varphi}{\nu\otimes\mathcal{L}^2} \in L^2_{\nu\otimes\mathcal{L}^2}(\overline{\Omega};\mathbb{S}^3), \\ &\mu_\varepsilon e(\varphi_\varepsilon) \xrightharpoonup{\nu_\varepsilon\otimes\mathcal{L}^2,\nu\otimes\mathcal{L}^2} \frac{\widetilde{E}\varphi}{\nu\otimes\mathcal{L}^2}, \qquad \sigma_\varepsilon(\varphi_\varepsilon) \xrightharpoonup{\nu_\varepsilon\otimes\mathcal{L}^2,\nu\otimes\mathcal{L}^2} \sigma^\nu(\varphi), \end{aligned} \tag{4.28}$$

where σ^ν is given by (4.3). Assume in addition

$$\sup_{\varepsilon>0} \int_\Omega |\varphi'_\varepsilon|^2 dm_\varepsilon \otimes \mathcal{L}^2 < \infty, \tag{4.29}$$

then

$$\begin{aligned} &(\varphi^\star)' \in L^2_m(0,L;H^1(\Omega';\mathbb{R}^3)), \qquad \varphi \in BD^{\nu,m}(\Omega), \\ &\varphi'_\varepsilon \xrightharpoonup{m_\varepsilon\otimes\mathcal{L}^2,m\otimes\mathcal{L}^2} (\varphi^\star)', \qquad e_{x'}(\varphi'_\varepsilon) \xrightharpoonup{m_\varepsilon\otimes\mathcal{L}^2,m\otimes\mathcal{L}^2} e_{x'}((\varphi^\star)'). \end{aligned} \tag{4.30}$$

Proof. By the Cauchy-Schwarz inequality and the estimates (3.3) and (4.25), we have

$$\begin{aligned}(4.31)\quad \int_\Omega |\boldsymbol{\varphi}_\varepsilon| dx + \int_\Omega |\boldsymbol{e}(\boldsymbol{\varphi}_\varepsilon)| dx &\leq \int_\Omega |\boldsymbol{\varphi}_\varepsilon| dx + \left(\int_\Omega \tfrac{1}{\mu_\varepsilon} dx\right)^{\frac{1}{2}} \left(\int_\Omega \mu_\varepsilon \left|\boldsymbol{e}(\boldsymbol{\varphi}_\varepsilon)\right|^2 dx\right)^{\frac{1}{2}} \\ &\leq C,\end{aligned}$$

thus the sequence $(\boldsymbol{\varphi}_\varepsilon)$ is bounded in $BD(\Omega)$ and weakly* converges in $BD(\Omega)$, up to a subsequence, to some $\boldsymbol{\varphi}$. From the compactness of the injection of $BD(\Omega)$ into $L^p(\Omega;\mathbb{R}^3)$ for $p \in \left[1, \frac{3}{2}\right)$ (see [54, Theorem 2.4, p. 153]), we deduce

$$(4.32)\qquad \begin{array}{ll} \boldsymbol{\varphi}_\varepsilon \to \boldsymbol{\varphi} & \text{strongly in } \ L^p(\Omega;\mathbb{R}^3) \ \ \forall p \in \left[1, \frac{3}{2}\right), \\ \boldsymbol{E}\boldsymbol{\varphi}_\varepsilon \overset{\star}{\rightharpoonup} \boldsymbol{E}\boldsymbol{\varphi} & \text{weakly* in } \ \mathcal{M}(\Omega;\mathbb{S}^3). \end{array}$$

The estimate (4.31) also implies that $(\boldsymbol{e}(\boldsymbol{\varphi}_\varepsilon)\mathcal{L}^3_{\lfloor\overline{\Omega}})$ is bounded in $\mathcal{M}(\overline{\Omega};\mathbb{S}^3)$, hence the following convergence holds, up to a subsequence, for some $\boldsymbol{\Upsilon} \in \mathcal{M}(\overline{\Omega};\mathbb{S}^3)$:

$$(4.33)\qquad \boldsymbol{e}(\boldsymbol{\varphi}_\varepsilon)\mathcal{L}^3_{\lfloor\overline{\Omega}} = \widetilde{\boldsymbol{E}}\boldsymbol{\varphi}_\varepsilon \overset{\star}{\rightharpoonup} \boldsymbol{\Upsilon} \qquad \text{weakly* in } \ \mathcal{M}(\overline{\Omega};\mathbb{S}^3).$$

By testing the convergences (4.32) (second line) and (4.33) with some arbitrary field $\boldsymbol{\Psi} \in D(\Omega;\mathbb{S}^3)$, we deduce that the following equation holds in $\mathcal{M}(\Omega;\mathbb{S}^3)$:

$$(4.34)\qquad \boldsymbol{\Upsilon}_{\lfloor\Omega} = \boldsymbol{E}\boldsymbol{\varphi}.$$

By (3.1) and (4.25), we have

$$(4.35)\qquad \sup_{\varepsilon>0} \int_\Omega \left|\mu_\varepsilon \boldsymbol{e}(\boldsymbol{\varphi}_\varepsilon)\right|^2 d\nu_\varepsilon \otimes \mathcal{L}^2 = \sup_{\varepsilon>0} \int_\Omega \mu_\varepsilon \left|\boldsymbol{e}(\boldsymbol{\varphi}_\varepsilon)\right|^2 dx < \infty.$$

Since the sequence $(\nu_\varepsilon \otimes \mathcal{L}^2)$ weakly* converges to $\nu \otimes \mathcal{L}^2$ in $\mathcal{M}(\overline{\Omega})$ (see (3.3)), we deduce from Lemma 4.3 and (4.3) that, up to a subsequence,

$$(4.36)\qquad \mu_\varepsilon \boldsymbol{e}(\boldsymbol{\varphi}_\varepsilon) \xrightharpoonup{\nu_\varepsilon\otimes\mathcal{L}^2, \nu\otimes\mathcal{L}^2} \boldsymbol{\Xi}, \qquad \boldsymbol{\sigma}_\varepsilon(\boldsymbol{\varphi}_\varepsilon) \xrightharpoonup{\nu_\varepsilon\otimes\mathcal{L}^2, \nu\otimes\mathcal{L}^2} l \,\mathrm{tr}\,(\boldsymbol{\Xi})\, \boldsymbol{I} + 2\,\boldsymbol{\Xi},$$

for some

$$(4.37)\qquad \boldsymbol{\Xi} \in L^2_{\nu\otimes\mathcal{L}^2}(\overline{\Omega};\mathbb{S}^3).$$

The first convergence in (4.36) implies, by Definition 4.1, that

$$(4.38)\qquad \boldsymbol{e}(\boldsymbol{\varphi}_\varepsilon)\mathcal{L}^3_{\lfloor\overline{\Omega}} = \widetilde{\boldsymbol{E}}\boldsymbol{\varphi}_\varepsilon \overset{\star}{\rightharpoonup} \boldsymbol{\Xi}\nu \otimes \mathcal{L}^2 \qquad \text{weakly* in } \ \mathcal{M}(\overline{\Omega};\mathbb{S}^3).$$

Taking (4.33) into account, we infer that the following equation holds in $\mathcal{M}(\overline{\Omega};\mathbb{R}^3)$:

$$(4.39)\qquad \Upsilon = \boldsymbol{\Xi}\nu \otimes \mathcal{L}^2.$$

Noticing that by (3.5) we have $\nu\otimes\mathcal{L}^2(\partial\Omega)=0$, we infer from (4.39) that $\Upsilon(\partial\Omega)=0$, and then from (2.8) and (4.34) that

$$\Upsilon=\Upsilon_{\lfloor\partial\Omega}+\Upsilon_{\lfloor\Omega}=\widetilde{E}\boldsymbol{\varphi}. \tag{4.40}$$

By (4.32), (4.33), (4.39), and (4.40), the assertions (4.27) and (4.28) are proved.

Let us now prove (4.30). By (3.3), the sequence $(m_\varepsilon\otimes\mathcal{L}^2)$ weakly* converges in $\mathcal{M}(\overline{\Omega})$ to $m\otimes\mathcal{L}^2$, and by (3.1), (4.29) and (4.35) we have

$$\sup_{\varepsilon>0}\int_\Omega |\boldsymbol{\varphi}'_\varepsilon|^2+|\boldsymbol{e}_{x'}(\boldsymbol{\varphi}'_\varepsilon)|^2\,dm_\varepsilon\otimes\mathcal{L}^2<+\infty.$$

Applying Lemma 4.3 we infer, up to a subsequence, the following convergences:

$$\begin{aligned}
&\boldsymbol{\varphi}'_\varepsilon \xrightarrow{m_\varepsilon\otimes\mathcal{L}^2,m\otimes\mathcal{L}^2} \boldsymbol{h}', && \mu_\varepsilon\boldsymbol{\varphi}'_\varepsilon \overset{\star}{\rightharpoonup} \boldsymbol{h}'m\otimes\mathcal{L}^2 && \text{weakly* in } \mathcal{M}(\overline{\Omega};\mathbb{R}^3),\\
&\boldsymbol{e}_{x'}(\boldsymbol{\varphi}'_\varepsilon) \xrightarrow{m_\varepsilon\otimes\mathcal{L}^2,m\otimes\mathcal{L}^2} \boldsymbol{\Gamma}, && \mu_\varepsilon\boldsymbol{e}_{x'}(\boldsymbol{\varphi}'_\varepsilon) \overset{\star}{\rightharpoonup} \boldsymbol{\Gamma}m\otimes\mathcal{L}^2 && \text{weakly* in } \mathcal{M}(\overline{\Omega};\mathbb{S}^3),
\end{aligned} \tag{4.41}$$

for some $\boldsymbol{h}'\in L^2_{m\otimes\mathcal{L}^2}(\Omega;\mathbb{R}^3)$, $\boldsymbol{\Gamma}\in L^2_{m\otimes\mathcal{L}^2}(\Omega;\mathbb{S}^3)$. The proof of (4.30) (and of Proposition 4.8) is achieved provided we show that

$$\boldsymbol{h}'=(\boldsymbol{\varphi}^\star)' \quad m\otimes\mathcal{L}^2\text{-a.e. in } \Omega, \tag{4.42}$$

$$(\boldsymbol{\varphi}^\star)'\in L^2_m(0,L;H^1(\Omega';\mathbb{R}^3)),\quad \boldsymbol{\Gamma}=\boldsymbol{e}_{x'}\left((\boldsymbol{\varphi}^\star)'\right)\quad m\otimes\mathcal{L}^2\text{-a.e. in } \Omega. \tag{4.43}$$

Proof of (4.42). Let us fix $\psi\in\mathcal{D}(\Omega)$. By (4.27), $(\psi\boldsymbol{\varphi}_\varepsilon)$ weakly* converges in $BD(\Omega)$ to $\psi\boldsymbol{\varphi}$, hence, by the estimates (4.20) established in Lemma 4.7, the sequence $(\overline{\psi\boldsymbol{\varphi}_\varepsilon})$ defined by (4.19) weakly* converges in $BV(0,L;\mathbb{R}^3)$ to $\overline{\psi\boldsymbol{\varphi}}$. By (4.8), (4.20) and (4.28) we have,

$$|D\overline{\psi\boldsymbol{\varphi}}|\ll|\boldsymbol{E}(\psi\boldsymbol{\varphi})|(.\times\Omega')=|\psi\boldsymbol{E}(\boldsymbol{\varphi})+\boldsymbol{\nabla}\psi\odot\boldsymbol{\varphi}\mathcal{L}^3|(.\times\Omega')\ll\nu,$$

therefore, by (3.4) and (3.5), the assumptions of Lemma 4.5 are satisfied by $(b_\varepsilon,w_\varepsilon):=(\mu_\varepsilon,\overline{\psi\boldsymbol{\varphi}_\varepsilon})$ and $(\theta,w):=(m,\overline{\psi\boldsymbol{\varphi}})$. Taking (4.17), (4.21) and (4.41) into account and applying Fubini's theorem, we deduce

$$\begin{aligned}
\int_\Omega \psi\boldsymbol{h}'dm\otimes\mathcal{L}^2 &= \lim_{\varepsilon\to0}\int_\Omega \mu_\varepsilon\psi\boldsymbol{\varphi}'_\varepsilon dx=\lim_{\varepsilon\to0}\int_\Omega \mu_\varepsilon\psi(\boldsymbol{\varphi}^\star)'_\varepsilon dx=\lim_{\varepsilon\to0}\int_0^L \mu_\varepsilon\overline{\psi\boldsymbol{\varphi}'_\varepsilon}dx_1\\
&=\int_{(0,L)}(\overline{\psi\boldsymbol{\varphi}'})^{(r)}dm=\int_{(0,L)}\left(\int_{\Sigma_{x_1}}\psi(\boldsymbol{\varphi}')^+d\mathcal{H}^2\right)dm\\
&=\int_\Omega\psi(\boldsymbol{\varphi}')^+dm\otimes\mathcal{L}^2=\int_\Omega\psi(\boldsymbol{\varphi}^\star)'dm\otimes\mathcal{L}^2.
\end{aligned}$$

By the arbitrary choice of ψ, Assertion (4.42) is proved.

Proof of (4.43). Let us fix $\boldsymbol{\Psi} \in \mathcal{D}(\Omega; \mathbb{S}^3)$. By (4.41) and (4.42), we have

$$\begin{aligned}\int_\Omega \boldsymbol{\Gamma} : \boldsymbol{\Psi}\, dm \otimes \mathcal{L}^2 &= \lim_{\varepsilon\to 0} \int_\Omega \mu_\varepsilon \boldsymbol{e}_{x'}(\boldsymbol{\varphi}_\varepsilon) : \boldsymbol{\Psi} dx = \lim_{\varepsilon\to 0} - \int_\Omega \mu_\varepsilon \boldsymbol{\varphi}'_\varepsilon \cdot \mathbf{div}_{x'} \boldsymbol{\Psi} dx \\ &= - \int_\Omega (\boldsymbol{\varphi}^\star)' \cdot \mathbf{div}_{x'} \boldsymbol{\Psi} dm \otimes \mathcal{L}^2,\end{aligned} \tag{4.44}$$

where $\mathbf{div}_{x'} \boldsymbol{\Psi} := \sum_{\alpha,\beta=2}^3 \frac{\partial \Psi_{\alpha\beta}}{\partial x_\beta} \boldsymbol{e}_\alpha$. By the arbitrary choice of $\boldsymbol{\Psi}$, we deduce that

$$\boldsymbol{e}_{x'}\left((\boldsymbol{\varphi}^\star)'\right) = \boldsymbol{\Gamma}, \qquad m \otimes \mathcal{L}^2\text{-a.e.},$$

yielding $\boldsymbol{e}_{x'}\left((\boldsymbol{\varphi}^\star)'\right) \in L^2_m(0, L; L^2(\Omega'; \mathbb{S}^3))$. This, along with (4.42) and the two-dimensional second Korn inequality in $H^1(\Omega'; \mathbb{R}^2)$, implies that $(\boldsymbol{\varphi}^\star)' \in L^2_m(0, L; H^1(\Omega'; \mathbb{S}^3))$. Assertion (4.43) is proved. □

We are now in a position to prove the main result of Section 4.

Proof. [**Proof of Proposition 4.2**] By multiplying (3.2) by $\boldsymbol{u}_\varepsilon$ and by integrating it by parts over Ω, we obtain $\int_\Omega \boldsymbol{\sigma}_\varepsilon(\boldsymbol{u}_\varepsilon) : \boldsymbol{e}(\boldsymbol{u}_\varepsilon) dx = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{u}_\varepsilon dx$, and deduce

$$\int_\Omega \mu_\varepsilon |\boldsymbol{e}(\boldsymbol{u}_\varepsilon)|^2 dx \le \int_\Omega \boldsymbol{\sigma}_\varepsilon(\boldsymbol{u}_\varepsilon) : \boldsymbol{e}(\boldsymbol{u}_\varepsilon) dx \le C ||\boldsymbol{f}||_{L^\infty(\Omega;\mathbb{R}^3)} \int_\Omega |\boldsymbol{u}_\varepsilon| dx. \tag{4.45}$$

The assumptions (3.3), Poincaré and Cauchy-Schwarz inequalities, imply

$$\begin{aligned}\int_\Omega |(\boldsymbol{u}_\varepsilon)_1| dx &\le C \int_\Omega \left| \frac{\partial (\boldsymbol{u}_\varepsilon)_1}{\partial x_1} \right| dx \le C \left(\int_\Omega \frac{1}{\mu_\varepsilon} dx \right)^{\frac{1}{2}} \left(\int_\Omega \mu_\varepsilon \left| \frac{\partial (\boldsymbol{u}_\varepsilon)_1}{\partial x_1} \right|^2 dx \right)^{\frac{1}{2}} \\ &\le C \left(\int_\Omega \mu_\varepsilon |\boldsymbol{e}(\boldsymbol{u}_\varepsilon)|^2 dx \right)^{\frac{1}{2}}.\end{aligned} \tag{4.46}$$

By Fubini's Theorem, Poincaré's inequality in $W^{1,1}_0(\Omega'; \mathbb{R}^2)$, Assertion (3.3), Cauchy-Schwarz and Jensen's inequalities, and Korn's inequality in $H^1_0(\Omega'; \mathbb{R}^2)$, we have

$$\begin{aligned}\int_\Omega |\boldsymbol{u}'_\varepsilon| dx &\le C \int_\Omega |\boldsymbol{\nabla}_{x'} \boldsymbol{u}'_\varepsilon| dx \le C \left(\int_0^L \tfrac{1}{\mu_\varepsilon} dx_1 \right)^{\frac{1}{2}} \left(\int_0^L \mu_\varepsilon \left(\int_{\Omega'} |\boldsymbol{\nabla}_{x'} \boldsymbol{u}'_\varepsilon|\, dx' \right)^2 dx_1 \right)^{\frac{1}{2}} \\ &\le C \left(\int_0^L \mu_\varepsilon \int_{\Omega'} |\boldsymbol{\nabla}_{x'} \boldsymbol{u}'_\varepsilon|^2 dx' dx_1 \right)^{\frac{1}{2}} \le C \left(\int_0^L \mu_\varepsilon \int_{\Omega'} |\boldsymbol{e}_{x'}(\boldsymbol{u}'_\varepsilon)|^2 dx' dx_1 \right)^{\frac{1}{2}}.\end{aligned} \tag{4.47}$$

We deduce from (4.45), (4.46), (4.47) that $\int_\Omega |\boldsymbol{u}_\varepsilon| dx \le C \left(\int_\Omega |\boldsymbol{u}_\varepsilon| dx \right)^{\frac{1}{2}}$, yielding

$$\int_\Omega |\boldsymbol{u}_\varepsilon| dx \le C. \tag{4.48}$$

On the other hand, by Korn's inequality in $H^1_0(\Omega'; \mathbb{R}^2)$, we have

$$
\begin{aligned}
\int_\Omega |\boldsymbol{u}'_\varepsilon|^2 dm_\varepsilon\otimes\mathcal{L}^2 &= \int_0^L \mu_\varepsilon\left(\int_{\Omega'}|\boldsymbol{u}'_\varepsilon|^2dx'\right)dx_1\\
&\le C\int_0^L\mu_\varepsilon\left(\int_{\Omega'}|\boldsymbol{e}_{x'}(\boldsymbol{u}'_\varepsilon)|^2dx'\right)dx_1\le C\int_\Omega\mu_\varepsilon|\boldsymbol{e}(\boldsymbol{u}_\varepsilon)|^2dx.
\end{aligned}
\tag{4.49}
$$

By (4.45), (4.48), and (4.49), the estimate (4.4) is proved. In other words, the field $\boldsymbol{\varphi}_\varepsilon=\boldsymbol{u}_\varepsilon$ satisfies (4.25) and (4.29). Therefore, by Proposition 4.8, the convergences stated in (4.5) hold for some $\boldsymbol{u}\in BD^{\nu,m}(\Omega)$. The proof of Proposition 4.2 is achieved provided we show that

$$
\boldsymbol{u}=0\quad \text{on}\ \ \partial\Omega,
\tag{4.50}
$$

(which is not straightforward, because the trace is not weakly* continuous on $BD(\Omega)$), and that

$$
(\boldsymbol{u}^\star)'=0\quad \mathcal{H}^1\otimes m\text{-a.e. on }\partial\Omega'\times(0,L).
\tag{4.51}
$$

Proof of (4.50). Let us fix $\boldsymbol{\Psi}\in C^\infty(\overline{\Omega};\mathbb{S}^3)$. By passing to the limit as $\varepsilon\to0$ in the integration by parts formula $\int_\Omega \boldsymbol{e}(\boldsymbol{u}_\varepsilon):\boldsymbol{\Psi}dx=-\int_\Omega\boldsymbol{u}_\varepsilon\cdot\mathbf{div}\boldsymbol{\Psi}dx$, taking the strong convergence of $\boldsymbol{u}_\varepsilon$ to $\boldsymbol{u}$ in $L^1(\Omega;\mathbb{R}^3)$ and the weak* convergence of $(\boldsymbol{e}(\boldsymbol{u}_\varepsilon))$ to $\widetilde{\boldsymbol{E}}(\boldsymbol{u})$ in $\mathcal{M}(\overline{\Omega};\mathbb{S}^3)$ into account (stated in (4.27), (4.28)), we obtain $\int_{\overline{\Omega}}\boldsymbol{\Psi}:d\widetilde{\boldsymbol{E}}\boldsymbol{u}=-\int_\Omega\boldsymbol{u}\cdot\mathbf{div}\boldsymbol{\Psi}dx$, and infer from (2.8) that $\int_\Omega\boldsymbol{\Psi}:d\boldsymbol{E}\boldsymbol{u}=-\int_\Omega\boldsymbol{u}\cdot\mathbf{div}\boldsymbol{\Psi}dx$. We then deduce from the Green Formula in $BD(\Omega)$

$$
\int_\Omega\boldsymbol{\Psi}:d\boldsymbol{E}\boldsymbol{u}=-\int_\Omega\boldsymbol{u}\cdot\mathbf{div}\boldsymbol{\Psi}dx+\int_{\partial\Omega}\boldsymbol{\Psi}:\boldsymbol{u}\odot\boldsymbol{n}d\mathcal{H}^2,
$$

that $\int_{\partial\Omega}\boldsymbol{\Psi}:\boldsymbol{u}\odot\boldsymbol{n}d\mathcal{H}^2(x)=0$. By the arbitrariness of $\boldsymbol{\Psi}$, taking (4.18) into account, Assertion (4.50) is proved.

Proof of (4.51). Let us fix $\boldsymbol{\Psi}\in C^\infty(\overline{\Omega};\mathbb{S}^3)$. Since $\boldsymbol{u}_\varepsilon=0$ on $\partial\Omega$, (4.44) holds for $\boldsymbol{\varphi}_\varepsilon=\boldsymbol{u}_\varepsilon$. We infer

$$
\int_\Omega\boldsymbol{e}_{x'}(\boldsymbol{u}'):\boldsymbol{\Psi}\,dm\otimes\mathcal{L}^2=-\int_{(0,L)}\left(\int_{\Omega'}(\boldsymbol{u}^\star)'\cdot\mathbf{div}_{x'}\boldsymbol{\Psi}dx'\right)dm(x_1).
\tag{4.52}
$$

By (4.30) applied to $\boldsymbol{\varphi}_\varepsilon:=\boldsymbol{u}_\varepsilon$, the field $(\boldsymbol{u}^\star)'$ belongs to $L^2_m(0,L;H^1(\Omega';\mathbb{R}^3))$, hence there exists an m-negligible subset N of $(0,L)$ such that $(\boldsymbol{u}^\star)'(x_1,.)\in H^1(\Omega';\mathbb{R}^3)$ for all $x_1\in(0,L)\setminus N$. By integration by parts, taking the symmetry of $\boldsymbol{\Psi}$ into account, we infer

$$
\int_{\Omega'}(\boldsymbol{u}^\star)'\cdot\mathbf{div}_{x'}\boldsymbol{\Psi}dx'=\int_{\partial\Omega'}(\boldsymbol{u}^\star)'\cdot\boldsymbol{\Psi}\boldsymbol{n}d\mathcal{H}^1(x')-\int_{\Omega'}\boldsymbol{e}_{x'}((\boldsymbol{u}^\star)'):\boldsymbol{\Psi}dx'\quad m\text{-a.e. }x_1.
\tag{4.53}
$$

It follows from (4.52) and (4.53) that $\int_{(0,L)\times\partial\Omega'}(\boldsymbol{u}^\star)'\cdot\boldsymbol{\Psi}\boldsymbol{n}dm\otimes\mathcal{H}^1=0$. By the arbitrary choice of $\boldsymbol{\Psi}$, Assertion (4.51) is proved. $\square$

5. Partial mollification in $BD^{\nu,m}(\Omega)$. For any two Borel functions $f,g:\Omega\to\overline{\mathbb{R}}$, we denote by $f*'g$ the partial convolution of g and f with respect to the variable x', defined by

$$f*'g(x):=\begin{cases}\displaystyle\int_{\mathbb{R}^2}\widetilde{f}(x_1,x'-y')\widetilde{g}(y')dy' & \text{if } \ \widetilde{f}(x_1,x'-.)\widetilde{g}(.)\in L^1(\mathbb{R}^2),\\ 0 & \text{otherwise.}\end{cases} \tag{5.1}$$

Given $\delta>0$, the symbol f^δ stands for the "partial mollification" of f with respect to x' given by

$$f^\delta:=f*'\eta_\delta, \tag{5.2}$$

where $\eta_\delta\in\mathcal{D}(\mathbb{R}^2)$ denotes the standard mollifier defined by

$$\eta(x'):=\begin{cases}C\exp\left(\dfrac{1}{|x'|^2-1}\right) & \text{if} \quad |x'|<1,\\ 0 & \text{otherwise,}\end{cases} \qquad \eta_\delta(x'):=\frac{1}{\delta^2}\eta\left(\frac{x'}{\delta}\right),$$

the constant C being chosen so that $\int_{\mathbb{R}^2}\eta dx'=1$. Some basic properties are stated in the next lemma.

LEMMA 5.1. *Let $f:\Omega\to\overline{\mathbb{R}}$ be a Borel function, θ a positive Radon measure on $[0,L]$, $\delta>0$, and $p\in[1,+\infty)$. Then f^δ is Borel measurable. If $f\in L^p_{\theta\otimes\mathcal{L}^2}(\Omega)$, the following estimates hold*

$$\int_{\Omega'}|f^\delta(x_1,x')|^p dx'\le\int_{\Omega'}|f(x_1,x')|^p dx' \qquad \forall x_1\in(0,L). \tag{5.3}$$

In particular, we have

$$f^\delta\in L^p_{\theta\otimes\mathcal{L}^2}(\Omega), \quad ||f^\delta||_{L^p_{\theta\otimes\mathcal{L}^2}(\Omega)}\le||f||_{L^p_{\theta\otimes\mathcal{L}^2}(\Omega)}. \tag{5.4}$$

Moreover, the next convergence holds

$$f^\delta\underset{\delta\to0}{\longrightarrow}f \quad \textit{strongly in} \ \ L^p_{\theta\otimes\mathcal{L}^2}(\Omega). \tag{5.5}$$

The following regularity assertion holds

$$f^\delta(x_1,.)\in C^\infty(\overline{\Omega}') \quad \forall x_1\in(0,L), \tag{5.6}$$

and

$$\begin{aligned}&\tfrac{\partial^{n+m}}{\partial x_2^n x_3^m}f^\delta=f*'\tfrac{\partial^{n+m}}{\partial x_2^n x_3^m}\eta_\delta\in L^p_{\theta\otimes\mathcal{L}^2}(\Omega), \quad \forall n,m\in\mathbb{N},\\ &\left|\left|\tfrac{\partial^{n+m}}{\partial x_2^n x_3^m}f^\delta\right|\right|_{L^p_{\theta\otimes\mathcal{L}^2}}\le\tfrac{C}{\delta^{n+m}}\,||f||_{L^p_{\theta\otimes\mathcal{L}^2}} \qquad \forall n,m\in\mathbb{N}.\end{aligned} \tag{5.7}$$

If $f \in L^p_{\theta\otimes\mathcal{L}^2}(\Omega)$ *and* $h \in L^{p'}_{\theta\otimes\mathcal{L}^2}(\Omega)$ $(\frac{1}{p}+\frac{1}{p'}=1)$, *then*

$$\int_\Omega f^\delta h d\theta\otimes\mathcal{L}^2 = \int_\Omega f h^\delta d\theta\otimes\mathcal{L}^2. \tag{5.8}$$

If $\psi \in C^1_c(\Omega)$, *then* $\psi^\delta \in C^1(\overline{\Omega})$ *and*

$$\frac{\partial\left(\psi^\delta\right)}{\partial x_k} = \left(\frac{\partial\psi}{\partial x_k}\right)^\delta, \ \forall k \in \{1,2,3\}. \tag{5.9}$$

Proof. By Fubini's theorem, the mappings $h^\pm(x) := \int_{\mathbb{R}^2}(\widetilde{f}(x_1, x'-y')\eta_\delta(y'))^\pm dy'$ (where $l^+(x) := \sup\{l(x), 0\}$) are Borel measurables and so is the set $A := \{x \in \Omega, \int_{\mathbb{R}^2}\left|\widetilde{f}(x_1,x'-y')\eta_\delta(y')\right|dy' < +\infty\}$, therefore, $f *' \eta_\delta = (h^+ - h^-)\mathbb{1}_A$ is Borel measurable. Assertion (5.3) follows from the classical properties of convolution in $\mathbb{R}^2$ (notice that $\int_{\mathbb{R}^2}\eta_\delta dx' = 1$). Assertion (5.4) is a straightforward consequence of (5.3). We have

$$\int_\Omega |f - f^\delta|^p d\theta\otimes\mathcal{L}^2 = \int_{[0,L]} d\theta(x_1)\int_{\Omega'}|f - f^\delta|^p(x_1,x')dx'.$$

By (5.3), the following holds $\int_{\Omega'}|f-f^\delta|^p(.,x')dx' \leq 2^{p-1}\int_{\Omega'}|f|^p(.,x')dx' \in L^1_\theta$, and by the properties of mollification in $L^p(\Omega')$, for all x_1 such that $f(x_1,.) \in L^p(\Omega')$, thus for θ-a.e. $x_1 \in [0,L]$, $\int_{\Omega'}|f - f^\delta|^p(x_1,x')dx'$ converges to 0. Assertion (5.5) then results from the dominated convergence theorem. Assertion (5.6) follows from well known properties of mollification and (5.7) is obtained by differentiation under the integral sign. Assertion (5.8) is proved by applying Fubini's theorem several times. Assertion (5.9) is obtained by noticing that $\widetilde{\psi} \in C^1_c(\mathbb{R}^2)$ and by differentiating under the integral sign. □

The next proposition specifies some properties of partial mollification when applied to elements of $BD_0^{\nu,m}(\Omega)$.

Proposition 5.2. *Let* $\boldsymbol{v} \in BD_0^{\nu,m}(\Omega)$ *and* $\delta > 0$. *Then,*

$$\boldsymbol{v}^\delta \in BD(\Omega), \quad \boldsymbol{E}\boldsymbol{v}^\delta \ll \nu\otimes\mathcal{L}^2, \quad \frac{\boldsymbol{E}(\boldsymbol{v}^\delta)}{\nu\otimes\mathcal{L}^2} = \left(\frac{\boldsymbol{E}\boldsymbol{v}}{\nu\otimes\mathcal{L}^2}\right)^\delta, \tag{5.10}$$

$$\begin{aligned} &(\boldsymbol{v}^\delta)^\pm = (\boldsymbol{v}^\pm)^\delta \quad \mathcal{H}^2\text{-a.e. on } \Sigma_{x_1}, \qquad \forall x_1 \in (0,L), \\ &(\boldsymbol{v}^\delta)^\star = (\boldsymbol{v}^\star)^\delta \quad \mathcal{H}^2\text{-a.e. on } \Sigma_{x_1}, \qquad \forall x_1 \in (0,L), \end{aligned} \tag{5.11}$$

$$\left((\boldsymbol{v}^\delta)^\star\right)' \in L^2_m(0,L;H^1(\Omega';\mathbb{R}^3)), \quad \boldsymbol{e}_{x'}\left((\boldsymbol{v}^\delta)^\star\right) = (\boldsymbol{e}_{x'}(\boldsymbol{v}^\star))^\delta, \tag{5.12}$$

$$\boldsymbol{v}^\delta \in BD^{\nu,m}(\Omega), \quad \lim_{\delta\to 0}\left|\left|\boldsymbol{v} - \boldsymbol{v}^\delta\right|\right|_{BD^{\nu,m}(\Omega)} = 0, \tag{5.13}$$

and the following holds for all $x \in \Omega$, $\alpha \in \{2,3\}$*:*

$$\begin{aligned} &\lim_{\kappa\to 0^+} (\boldsymbol{v}^\delta)^{\mp}(x_1 \pm \kappa, x') = (\boldsymbol{v}^\delta)^{\pm}(x), \\ &\lim_{\kappa\to 0^+} \tfrac{\partial}{\partial x_\alpha}(\boldsymbol{v}^\delta)^{\mp}(x_1 \pm \kappa, x') = \tfrac{\partial}{\partial x_\alpha}(\boldsymbol{v}^\delta)^{\pm}(x), \end{aligned} \tag{5.14}$$

$$\begin{aligned} \left(v_1^\delta\right)^+(x) &= \frac{1}{l+2}\int_{(0,x_1]} (\boldsymbol{\sigma}^\nu)_{11}(\boldsymbol{v}^\delta)(s_1,x')\, d\nu(s_1) \\ &\qquad - \sum_{\beta=2}^{3} \frac{l}{l+2}\int_0^{x_1} \frac{\partial v_\beta^\delta}{\partial x_\beta}(s_1,x') ds_1, \\ \left(v_\alpha^\delta\right)^+(x) &= \int_{(0,x_1]} (\boldsymbol{\sigma}^\nu)_{1\alpha}(\boldsymbol{v}^\delta)(s_1,x')\, d\nu(s_1) - \int_0^{x_1} \frac{\partial v_1^\delta}{\partial x_\alpha}(s_1,x')\, ds_1. \end{aligned} \tag{5.15}$$

Proof. By (5.4) we have $\boldsymbol{v}^\delta \in L^1(\Omega;\mathbb{R}^3)$ and $\int_\Omega |\boldsymbol{v}^\delta| dx \le \int_\Omega |\boldsymbol{v}| dx$. Let us fix $\boldsymbol{\Psi} \in \mathcal{D}(\Omega;\mathbb{S}^3)$. Then $\boldsymbol{\Psi}^\delta \in C^\infty(\overline{\Omega};\mathbb{S}^3)$, thus using (5.8), (5.9), Green's formula in $BD(\Omega)$, and the fact that $\boldsymbol{v} \in BD_0^{\nu,m}(\Omega)$, we obtain

$$\begin{aligned} \int_\Omega \boldsymbol{v}^\delta \cdot \mathbf{div}\boldsymbol{\Psi}\, dx &= \int_\Omega \boldsymbol{v}\cdot (\mathbf{div}\boldsymbol{\Psi})^\delta\, dx = \int_\Omega \boldsymbol{v}\cdot \mathbf{div}(\boldsymbol{\Psi}^\delta)\, dx = -\int_\Omega \boldsymbol{\Psi}^\delta : d\boldsymbol{E}\boldsymbol{v} \\ &= -\int_\Omega \boldsymbol{\Psi}^\delta : \tfrac{\boldsymbol{E}\boldsymbol{v}}{\nu\otimes\mathcal{L}^2} d\nu\otimes\mathcal{L}^2 = -\int_\Omega \boldsymbol{\Psi} : \left(\tfrac{\boldsymbol{E}\boldsymbol{v}}{\nu\otimes\mathcal{L}^2}\right)^\delta d\nu\otimes\mathcal{L}^2. \end{aligned}$$

By the arbitrary choice of $\boldsymbol{\Psi}$, the assertion (5.10) is proved. Similarly, applying Green's formula in $BD(\Omega)$ and using (5.8), (5.9), (5.10), we infer, for all $x_1 \in (0,L)$,

$$\begin{aligned} \int_{\Sigma_{x_1}} \boldsymbol{\Psi} : (\boldsymbol{v}^\delta)^- \odot \boldsymbol{e}_1 d\mathcal{H}^2 &= \int_{\partial((0,x_1)\times\Omega')} \boldsymbol{\Psi} : \boldsymbol{v}^\delta \odot \boldsymbol{n}\, d\mathcal{H}^2 = \int_{(0,x_1)\times\Omega'} \boldsymbol{\Psi} : d\boldsymbol{E}\boldsymbol{v}^\delta + \int_{(0,x_1)\times\Omega'} \mathbf{div}\boldsymbol{\Psi}\cdot \boldsymbol{v}^\delta dx \\ &= \int_{(0,x_1)\times\Omega'} \boldsymbol{\Psi} : \left(\tfrac{\boldsymbol{E}\boldsymbol{v}}{\nu\otimes\mathcal{L}^2}\right)^\delta d\nu\otimes\mathcal{L}^2 + \int_{(0,x_1)\times\Omega'} (\mathbf{div}\boldsymbol{\Psi})^\delta \cdot \boldsymbol{v} dx \\ &= \int_{(0,x_1)\times\Omega'} \boldsymbol{\Psi}^\delta : \tfrac{\boldsymbol{E}\boldsymbol{v}}{\nu\otimes\mathcal{L}^2} d\nu\otimes\mathcal{L}^2 + \int_{(0,x_1)\times\Omega'} \mathbf{div}(\boldsymbol{\Psi}^\delta)\cdot \boldsymbol{v} dx \\ &= \int_{\Sigma_{x_1}} \boldsymbol{\Psi}^\delta : \boldsymbol{v}^- \odot \boldsymbol{e}_1 d\mathcal{H}^2 = \int_{\Sigma_{x_1}} \boldsymbol{\Psi} : \left(\boldsymbol{v}^-\right)^\delta \odot \boldsymbol{e}_1 d\mathcal{H}^2. \end{aligned}$$

By the arbitrary nature of $\boldsymbol{\Psi}$ and x_1, we deduce that $\left(\boldsymbol{v}^\delta\right)^- \odot \boldsymbol{e}_1 = \left(\boldsymbol{v}^-\right)^\delta \odot \boldsymbol{e}_1$ and then, taking (4.18) into account, that $\left(\boldsymbol{v}^\delta\right)^- = \left(\boldsymbol{v}^-\right)^\delta$. Arguing in the same manner for $\left(\boldsymbol{v}^\delta\right)^+$, we find the first line of (5.11). By (4.14) and the last mentioned line, for all $x_1 \in (0,L)$ the following equalities hold $\mathcal{H}^2$-a.e. on Σ_{x_1}:

$$\left(\boldsymbol{v}^\star\right)^\delta = \tfrac{1}{2}\left(\boldsymbol{v}^+ + \boldsymbol{v}^-\right)^\delta = \tfrac{1}{2}\left(\left(\boldsymbol{v}^+\right)^\delta + \left(\boldsymbol{v}^-\right)^\delta\right) = \tfrac{1}{2}\left(\left(\boldsymbol{v}^\delta\right)^+ + \left(\boldsymbol{v}^\delta\right)^-\right) = \left(\boldsymbol{v}^\delta\right)^\star.$$

Assertion (5.11) is proved. To prove (5.12), we first notice that, by (3.7), (5.4) and (5.11), we have $\left(\left(\boldsymbol{v}^\delta\right)^\star\right)' \in L^2_m(0,L;\ L^2(\Omega';\mathbb{R}^3))$. Taking (5.8), (5.9), (5.11) into account and integrating by parts with respect to x' in $L^2_m(0,L;H^1_0(\Omega';\mathbb{R}^3))$, we find

$$\int_\Omega \left((\boldsymbol{v}^\delta)^\star\right)' \cdot \mathbf{div}\boldsymbol{\Psi} dm\otimes\mathcal{L}^2 = \int_\Omega \left((\boldsymbol{v}^\star)'\right)^\delta \cdot \mathbf{div}\boldsymbol{\Psi} dm\otimes\mathcal{L}^2 = \int_\Omega (\boldsymbol{v}^\star)' \cdot \mathbf{div}\left(\boldsymbol{\Psi}^\delta\right) dm\otimes\mathcal{L}^2$$
$$= -\int_\Omega \boldsymbol{e}_{x'}(\boldsymbol{v}^\star) : \boldsymbol{\Psi}^\delta dm\otimes\mathcal{L}^2 = -\int_\Omega \left(\boldsymbol{e}_{x'}(\boldsymbol{v}^\star)\right)^\delta : \boldsymbol{\Psi} dm\otimes\mathcal{L}^2,$$

yielding (5.12). Assertion (5.13) is a consequence of (4.26), (5.10), (5.12), and (5.5) applied for $f \in \left\{\frac{\boldsymbol{E}\boldsymbol{v}}{\nu\otimes\mathcal{L}^2}, \boldsymbol{e}_{x'}(\boldsymbol{v}), \boldsymbol{v}\right\}$ and $\theta \in \{\nu, m\}$. Let us fix $x \in \Omega$: by (4.18), (5.11) and Green's formula, denoting by $\boldsymbol{\gamma}$ the trace application on $BD((x_1, x_1+\kappa)\times\Omega')$, we have

$$\begin{aligned}
&\left|(\boldsymbol{v}^\delta)^-(x_1+\kappa, x') - (\boldsymbol{v}^\delta)^+(x)\right| \leq \sqrt{2}\left|\left((\boldsymbol{v}^-)^\delta(x_1+\kappa, x') - (\boldsymbol{v}^+)^\delta(x)\right)\odot \boldsymbol{e}_1\right| \\
&= \sqrt{2}\left|\int_{\partial((x_1,x_1+\kappa)\times\Omega')} \eta_\delta(x'-y')\boldsymbol{\gamma}(\boldsymbol{v})(s_1,y')\odot \boldsymbol{n} d\mathcal{H}^2(s_1,y')\right| \\
&= \sqrt{2}\left|\int_{(x_1,x_1+\kappa)\times\Omega'} \eta_\delta(x'-y') d\boldsymbol{E}\boldsymbol{v}(s_1,y') + \int_{(x_1,x_1+\kappa)\times\Omega'} \boldsymbol{v}\odot\boldsymbol{\nabla}_{x'}\eta_\delta(x'-y') ds_1 dy'\right| \\
&\leq C\left(|\boldsymbol{E}\boldsymbol{v}|\left((x_1,x_1+\kappa)\times\Omega'\right) + \int_{(x_1,x_1+\kappa)\times\Omega'} |\boldsymbol{v}| dx\right),
\end{aligned}$$

therefore $\lim_{\kappa\to 0^+}\left|(\boldsymbol{v}^\delta)^-(x_1+\kappa, x') - (\boldsymbol{v}^\delta)^+(x)\right| = 0$. We likewise find that $\lim_{\kappa\to 0^+}\left|(\boldsymbol{v}^\delta)^+(x_1-\kappa, x') - (\boldsymbol{v}^\delta)^-(x)\right| = 0$. The first line of (5.14) is proved. The second one is obtained by applying (5.7) and by substituting $\frac{\partial\eta_\delta}{\partial x_\alpha}$ for η_δ in the above computations. To prove (5.15), we fix $(x_1, x') \in \Omega$, $\kappa > 0$: by (5.10) and Green's formula, we have

$$\begin{aligned}
\int_{(0,x_1+\kappa)} \tfrac{E_{11}\boldsymbol{v}^\delta}{\nu\otimes\mathcal{L}^2}(s_1,x') d\nu(s_1) &= \int_{(0,x_1+\kappa)\times\Omega'} \tfrac{E_{11}\boldsymbol{v}}{\nu\otimes\mathcal{L}^2}(s_1,y')\eta_\delta(x'-y') d\nu\otimes\mathcal{L}^2(s_1,y') \\
&= \int_{(0,x_1+\kappa)\times\Omega'} \eta_\delta(x'-y') dE_{11}\boldsymbol{v}(s_1,y') \\
&= \int_{\Sigma_{x_1+\kappa}} \eta_\delta(x'-y') v_1^-(s_1,y') d\mathcal{H}^2(s_1,y') = \left(v_1^-\right)^\delta(x_1+\kappa, y').
\end{aligned}$$

Likewise, the following holds for $\beta \in \{2,3\}$,

$$\begin{aligned}
&\int_{(0,x_1+\kappa)} \tfrac{E_{\beta\beta}\boldsymbol{v}^\delta}{\nu\otimes\mathcal{L}^2}(s_1,x') d\nu(s_1) = \int_{(0,x_1+\kappa)\times\Omega'} \eta_\delta(x'-y') dE_{\beta\beta}\boldsymbol{v}(s_1,y') \\
&= -\int_{(0,x_1+\kappa)\times\Omega'} \frac{\partial}{\partial y_\beta}\left(\eta_\delta(x'-y')\right) v_\beta(s_1,y') d\mathcal{L}^3(s_1,y') \\
&= \int_0^{x_1+\kappa}\left(\frac{\partial}{\partial x_\beta}\int_{\Omega'} \eta_\delta(x'-y') v_\beta(s_1,y') dy'\right) ds_1 = \int_0^{x_1+\kappa} \frac{\partial v_\beta^\delta}{\partial x_\beta}(s_1,x') ds_1.
\end{aligned}$$

Passing to the limit as $\kappa \to 0^+$, taking (5.11) and (5.14) into account, we infer

$$
\begin{aligned}
&\int_{(0,x_1]} \tfrac{E_{11}\boldsymbol{v}^\delta}{\nu\otimes\mathcal{L}^2}(s_1,x')d\nu(s_1) = \left(v_1^\delta\right)^+(x_1,y'),\\
&\int_{(0,x_1]} \tfrac{E_{\beta\beta}\boldsymbol{v}^\delta}{\nu\otimes\mathcal{L}^2}(s_1,x')d\nu(s_1) = \int_0^{x_1} \frac{\partial v_\beta^\delta}{\partial x_\beta}(s_1,x')ds_1,
\end{aligned}
\tag{5.16}
$$

yielding, by (4.3),

$$
\begin{aligned}
\int_{(0,x_1]} (\boldsymbol{\sigma}^\nu)_{11}(\boldsymbol{v}^\delta)(s_1,x')\ d\nu(s_1) &= \int_{(0,x_1]} l\,\mathrm{tr}\left(\tfrac{\boldsymbol{E\varphi}}{\nu\otimes\mathcal{L}^2}\right) + 2\,\tfrac{E_{11}\boldsymbol{\varphi}}{\nu\otimes\mathcal{L}^2}\ d\nu(s_1)\\
&= (l+2)\left(v_1^\delta\right)^+(x_1,y') + l\sum_{\beta=2}^{3}\int_0^{x_1}\frac{\partial v_\beta^\delta}{\partial x_\beta}(s_1,x')ds_1.
\end{aligned}
$$

The first equation in (5.15) is proved. Similarly, by (5.10) and Green's formula, the following holds for $\alpha\in\{2,3\}$:

$$
\begin{aligned}
&\int_{(0,x_1+\kappa)} 2\tfrac{E_{1\alpha}\boldsymbol{v}^\delta}{\nu\otimes\mathcal{L}^2}(s_1,x')d\nu(s_1) = \int_{(0,x_1+\kappa)\times\Omega'} 2\eta_\delta(x'-y')dE_{1\alpha}\boldsymbol{v}(s_1,y')\\
&= \int_{\Sigma_{x_1+\kappa}} \eta_\delta(x'-y')v_\alpha^-(s_1,y')d\mathcal{H}^2(s_1,y') + \int_{(0,x_1+\kappa)\times\Omega'} v_1(s_1,y')\frac{\partial\eta_\delta}{\partial x_\alpha}(x'-y')ds_1dy'\\
&= \left(v_\alpha^\delta\right)^-(x_1+\kappa,x') + \int_0^{x_1+\kappa}\frac{\partial v_1^\delta}{\partial x_\alpha}(s_1,x')ds_1.
\end{aligned}
$$

Sending κ to 0^+, we infer from (5.14) that

$$
\int_{(0,x_1]} 2\tfrac{E_{1\alpha}\boldsymbol{v}^\delta}{\nu\otimes\mathcal{L}^2}(s_1,x')d\nu(s_1) = \left(v_\alpha^\delta\right)^+(x_1,x') + \int_0^{x_1}\frac{\partial v_1^\delta}{\partial x_\alpha}(s_1,x')ds_1,
\tag{5.17}
$$

and from (4.3) that

$$
\begin{aligned}
\int_{(0,x_1]} (\boldsymbol{\sigma}^\nu)_{1\alpha}(\boldsymbol{v}^\delta)(s_1,x')\ d\nu(s_1) &= \int_{(0,x_1]} 2\tfrac{E_{1\alpha}\boldsymbol{v}^\delta}{\nu\otimes\mathcal{L}^2}(s_1,x')d\nu(s_1)\\
&= \left(v_\alpha^\delta\right)^+(x_1,x') + \int_0^{x_1}\frac{\partial v_1^\delta}{\partial x_\alpha}(s_1,x')ds_1,
\end{aligned}
$$

yielding the second equation in (5.15). ☐

Proposition 5.3. *For all $\boldsymbol{v}\in BD_0^{\nu,m}(\Omega)$ and $\delta>0$, the following holds for some constant C independent of δ*

$$
\begin{aligned}
&\int_\Omega \left|\tfrac{\boldsymbol{E}\boldsymbol{v}^\delta}{\nu\otimes\mathcal{L}^2}\right|^2 d\nu\otimes\mathcal{L}^2 \le \int_\Omega \left|\tfrac{\boldsymbol{E}\boldsymbol{v}}{\nu\otimes\mathcal{L}^2}\right|^2 d\nu\otimes\mathcal{L}^2 < \infty,\\
&\int_\Omega \left|\tfrac{\partial}{\partial x_\alpha}\tfrac{\boldsymbol{E}\boldsymbol{v}^\delta}{\nu\otimes\mathcal{L}^2}\right|^2 d\nu\otimes\mathcal{L}^2 \le \frac{C}{\delta^2}\int_\Omega \left|\tfrac{\boldsymbol{E}\boldsymbol{v}}{\nu\otimes\mathcal{L}^2}\right|^2 d\nu\otimes\mathcal{L}^2 < \infty,
\end{aligned}
\tag{5.18}
$$

(5.19) $$\boldsymbol{v}^\delta,\ \frac{\partial \boldsymbol{v}^\delta}{\partial x_\alpha},\ \frac{\partial^2 \boldsymbol{v}^\delta}{\partial x_\alpha \partial x_\beta} \in L^2(\Omega;\mathbb{R}^3), \qquad \forall\ a,\beta \in \{2,3\}.$$

Proof. Assertion (5.18) follows from (5.4), (5.7) and (5.10). By Lemma 4.4, the Lebesgue measure on Ω is absolutely continuous with respect to $m \otimes \mathcal{L}^2$, thus, by (2.4) and (4.17),

(5.20) $$\left(\boldsymbol{v}^\delta\right)^+ = \left(\boldsymbol{v}^\delta\right)^- = \left(\boldsymbol{v}^\delta\right)^\star = \boldsymbol{v}^\delta \quad \mathcal{L}^3\text{-a.e. in } \Omega.$$

By (5.16), (5.18), (5.20), Cauchy-Schwarz inequality and Fubini Theorem, we have

$$\begin{aligned}\int_\Omega |v_1^\delta|^2 dx = \int_\Omega |(v_1^\delta)^+|^2 dx = \int_\Omega \left| \int_{(0,x_1]} \tfrac{E_{11}\boldsymbol{v}^\delta}{\nu\otimes\mathcal{L}^2}(s_1,x')\ d\nu(s_1)\right|^2 dx \\ \leq C\int_\Omega |\tfrac{E_{11}\boldsymbol{v}^\delta}{\nu\otimes\mathcal{L}^2}|^2 d\nu\otimes\mathcal{L}^2 \leq C\int_\Omega |\tfrac{E_{11}\boldsymbol{v}}{\nu\otimes\mathcal{L}^2}|^2 d\nu\otimes\mathcal{L}^2 < \infty,\end{aligned}$$

yielding, by (5.7)

$$\int_\Omega \left|\frac{\partial v_1^\delta}{\partial x_\alpha}\right|^2 dx \leq \frac{C}{\delta^2}\int_\Omega \left|v_1^\delta\right|^2 dx \leq \frac{C}{\delta^2}\int_\Omega |\tfrac{E_{11}\boldsymbol{v}}{\nu\otimes\mathcal{L}^2}|^2 d\nu\otimes\mathcal{L}^2 < \infty.$$

We deduce from (5.10), (5.17), (5.20) and the last inequalities that, for $\alpha \in \{2,3\}$,

$$\begin{aligned}\int_\Omega |v_\alpha^\delta|^2 dx \leq C\!\int_\Omega \left|\int_{(0,x_1]} \tfrac{E_{1\alpha}\boldsymbol{v}^\delta}{\nu\otimes\mathcal{L}^2}(s_1,x') d\nu(s_1)\right|^2 dx + C\!\int_\Omega \left|\int_0^{x_1} \frac{\partial v_1^\delta}{\partial x_\alpha}(s_1,x')\ ds_1\right|^2 dx \\ \leq C\int_\Omega |\tfrac{E_{1\alpha}\boldsymbol{v}}{\nu\otimes\mathcal{L}^2}|^2 d\nu\otimes\mathcal{L}^2 + C\int_\Omega \left|\frac{\partial v_1^\delta}{\partial x_\alpha}\right|^2 dx \leq \frac{C}{\delta^2}\int_\Omega |\tfrac{\boldsymbol{E}\boldsymbol{v}}{\nu\otimes\mathcal{L}^2}|^2 d\nu\otimes\mathcal{L}^2 < \infty,\end{aligned}$$

and then from (5.7) that, for $\alpha,\beta \in \{2,3\}$,

$$\begin{aligned}\int_\Omega \left|\frac{\partial \boldsymbol{v}^\delta}{\partial x_\alpha}\right|^2 dx \leq \frac{C}{\delta^2}\int_\Omega |\boldsymbol{v}_\alpha^\delta|^2 dx \leq \frac{C}{\delta^4}\int_\Omega |\tfrac{\boldsymbol{E}\boldsymbol{v}}{\nu\otimes\mathcal{L}^2}|^2 d\nu\otimes\mathcal{L}^2 < \infty, \\ \int_\Omega \left|\frac{\partial^2 \boldsymbol{v}^\delta}{\partial x_\alpha\partial x_\beta}\right|^2 dx \leq \frac{C}{\delta^2}\int_\Omega \left|\frac{\partial \boldsymbol{v}^\delta}{\partial x_\alpha}\right|^2 dx \leq \frac{C}{\delta^6}\int_\Omega |\tfrac{\boldsymbol{E}\boldsymbol{v}}{\nu\otimes\mathcal{L}^2}|^2 d\nu\otimes\mathcal{L}^2 < \infty.\end{aligned}$$

Assertion (5.19) is proved. □

6. Proof of Theorem 3.1. The proof of Theorem 3.1 rests on the choice of an appropriate sequence of test fields $(\boldsymbol{\varphi}_\varepsilon)$, which will be constructed from an arbitrarily chosen partially mollified element of $BD_0^{\nu,m}(\Omega)$, that is a field $\boldsymbol{\varphi}$ of the type

(6.1) $$\boldsymbol{\varphi} = \boldsymbol{v}^\delta, \quad \boldsymbol{v} \in BD_0^{\nu,m}(\Omega), \quad \delta > 0.$$

Let us briefly outline our approach. In the spirit of Tartar's method [51], we will multiply (3.2) by $\boldsymbol{\varphi}_\varepsilon$ and integrate by parts to obtain

$$\int_\Omega \boldsymbol{\sigma}_\varepsilon(\boldsymbol{u}_\varepsilon) : \boldsymbol{e}(\boldsymbol{\varphi}_\varepsilon)\ dx = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{\varphi}_\varepsilon dx. \tag{6.2}$$

By passing to the limit as $\varepsilon \to 0$ in accordance with the convergences established in propositions 4.2 and 6.1, we will find $a(\boldsymbol{u}, \boldsymbol{v}^\delta) = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}^\delta dx$, where $a(\cdot,\cdot)$ is the symmetric bilinear form on $BD^{\nu,m}(\Omega)$ defined by (3.9). Then, sending δ to 0, we will infer from Proposition 5.2 that $a(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} dx$. From Proposition 4.2, we will deduce that $\boldsymbol{u}$ belongs to $BD_0^{\nu,m}(\Omega)$, hence is a solution to (3.6). Next, we will prove that $BD_0^{\nu,m}(\Omega)$ is a Hilbert space and $a(\cdot,\cdot)$ is coercive and continuous on it, hence the solution to (3.6) is unique and the convergences established in Proposition 4.2 for subsequences, hold for the complete sequences.

The sequence $(\boldsymbol{\varphi}_\varepsilon)$ will be deduced from a family of sequences $((\boldsymbol{\varphi}_\varepsilon^k)_\varepsilon)_{k\in\mathbb{N}}$ by a diagonalization argument. Given $k \in \mathbb{N}$, the construction of $(\boldsymbol{\varphi}_\varepsilon^k)_\varepsilon$ is based on the choice of an appropriate finite partition $(I_j^k)_{j\in\{1,\dots,n_k\}}$ of $(0, L]$ defined as follows: since the set of the atoms of the measures ν and m is at most countable, we can fix a sequence $(A_k)_{k\in\mathbb{N}}$ of finite subsets of $[0, L]$ satisfying

$$\begin{cases} A_k = \left\{t_0^k, t_1^k, \dots, t_{n_k}^k\right\}, \quad A_k \subset A_{k+1} \quad \forall k \in \mathbb{N}, \\ 0 = t_0^k < t_1^k < t_2^k < \ldots < t_{n_{k-1}}^k < t_{n_k}^k = L, \\ \nu\left(\left\{t_j^k\right\}\right) = m\left(\left\{t_j^k\right\}\right) = 0 \quad \forall k \in \mathbb{N}, \quad \forall j \in \{0, \dots, n_k\}, \\ \lim\limits_{k\to\infty} \sup\limits_{j\in\{1,\dots,n_k\}} \left|t_j^k - t_{j-1}^k\right| = 0. \end{cases} \tag{6.3}$$

Setting

$$I_j^k := \left(t_{j-1}^k, t_j^k\right] \quad \forall\ k \in \mathbb{N},\ \forall j \in \{1, \dots, n_k\}, \tag{6.4}$$

we introduce the function $\phi_\varepsilon^k : (0, L) \to \mathbb{R}$ defined by

$$\phi_\varepsilon^k(x_1) := \sum_{j=1}^{n_k} \frac{\nu_\varepsilon((t_{j-1}^k, x_1))}{\nu_\varepsilon(I_j^k)} \mathbb{1}_{I_j^k}(x_1). \tag{6.5}$$

Note that the restriction of ϕ_ε^k to each I_j^k is absolutely continuous, and

$$\begin{aligned} &\frac{d\phi_\varepsilon^k}{dx_1}(x_1) = \frac{\mu_\varepsilon^{-1}(x_1)}{\nu_\varepsilon(I_j^k)} \quad \text{in} \quad \mathring{I}_j^k; \quad 0 \le \phi_\varepsilon^k \le 1 \ \text{ in } \ (0, L), \\ &\phi_\varepsilon^k((t_j^k)^-) = 1 \ \text{ and } \ \phi_\varepsilon^k((t_{j-1}^k)^+) = 0 \quad \forall j \in \{1, \dots, n_k\}. \end{aligned} \tag{6.6}$$

For all $j \in \{1, \dots, n_k\}$, $x \in I_j^k \times \Omega'$, $\alpha \in \{2, 3\}$, we set (see (4.3))

$$
\begin{aligned}
\varphi^k_{\varepsilon 1}(x) := \frac{\phi^k_\varepsilon(x_1)}{l+2}\int_{I^k_j}\sigma^\nu_{11}(\boldsymbol{\varphi})(s_1,x')d\nu(s_1)&\\
-\frac{l}{l+2}\sum_{\alpha=2}^{3}\int_{t^k_{j-1}}^{x_1}\frac{\partial\varphi_\alpha}{\partial x_\alpha}(s_1,x')ds_1+\varphi^+_1(t^k_{j-1},x'),&\\
\varphi^k_{\varepsilon\alpha}(x) := \phi^k_\varepsilon(x_1)\int_{I^k_j}\sigma^\nu_{1\alpha}(\boldsymbol{\varphi})(s_1,x')d\nu(s_1)-\int_{t^k_{j-1}}^{x_1}\frac{\partial\varphi_1}{\partial x_\alpha}(s_1,x')ds_1+\varphi^+_\alpha(t^k_{j-1},x').&
\end{aligned}
\tag{6.7}
$$

The sequence of test fields $(\boldsymbol{\varphi}_\varepsilon)$ is determined by the next proposition.

PROPOSITION 6.1. *Let $\boldsymbol{v}\in BD^{\nu,m}_0(\Omega)$, $\delta>0$, and $\boldsymbol{\varphi}$, $\boldsymbol{\varphi}^k_\varepsilon$ respectively given by (6.1), (6.7). There exists an increasing sequence (k_ε) of positive integers converging to ∞ such that $\boldsymbol{\varphi}_\varepsilon$ defined by*

$$
\boldsymbol{\varphi}_\varepsilon := \boldsymbol{\varphi}^{k_\varepsilon}_\varepsilon,
\tag{6.8}
$$

strongly converges to $\boldsymbol{\varphi}$ in $L^1(\Omega;\mathbb{R}^3)$ and satisfies the assumptions (4.25) and (4.29) of Proposition 4.8. In particular, the convergences and relations (4.27), (4.28) and (4.30) are satisfied. In addition, the following strong convergences in the sense of (4.2) hold:

$$
\boldsymbol{\sigma}_\varepsilon(\boldsymbol{\varphi}_\varepsilon)\boldsymbol{e}_1 \xrightarrow{\nu_\varepsilon\otimes\mathcal{L}^2,\nu\otimes\mathcal{L}^2} \boldsymbol{\sigma}^\nu(\boldsymbol{\varphi})\boldsymbol{e}_1,\quad \boldsymbol{e}_{x'}(\boldsymbol{\varphi}_\varepsilon) \xrightarrow{m_\varepsilon\otimes\mathcal{L}^2,m\otimes\mathcal{L}^2} \boldsymbol{e}_{x'}((\boldsymbol{\varphi}^\star)'),
\tag{6.9}
$$

where $\boldsymbol{\sigma}^\nu$ is given by (4.3).

Proposition 6.1 will be proved in Section 6.1. The next step consists in passing to the limit as $\varepsilon\to 0$ in (6.2). Expressing in (6.2), for $\boldsymbol{g}\in\{\boldsymbol{u}_\varepsilon,\boldsymbol{\varphi}_\varepsilon\}$, the scalar fields $e_{11}(\boldsymbol{g})$, $\sigma_{\varepsilon 22}(\boldsymbol{g})$, $\sigma_{\varepsilon 33}(\boldsymbol{g})$ in terms of the components of $\boldsymbol{\sigma}_\varepsilon(\boldsymbol{g})\boldsymbol{e}_1$ and $\boldsymbol{e}_{x'}(\boldsymbol{g})$ (the details of this computation are situated at the end of the section), leads to the following equation:

$$
\begin{aligned}
&\int_\Omega \tfrac{1}{l+2}\sigma_{\varepsilon 11}(\boldsymbol{u}_\varepsilon)\sigma_{\varepsilon 11}(\boldsymbol{\varphi}_\varepsilon)+\sum_{\alpha=2}^{3}\sigma_{\varepsilon 1\alpha}(\boldsymbol{u}_\varepsilon)\sigma_{\varepsilon 1\alpha}(\boldsymbol{\varphi}_\varepsilon)\ d\nu_\varepsilon\otimes\mathcal{L}^2\\
&+\int_\Omega 4e_{23}(\boldsymbol{u}_\varepsilon)e_{23}(\boldsymbol{\varphi}_\varepsilon)+\tfrac{4(l+1)}{l+2}\sum_{\alpha=2}^{3}e_{\alpha\alpha}(\boldsymbol{u}_\varepsilon)e_{\alpha\alpha}(\boldsymbol{\varphi}_\varepsilon)\,dm_\varepsilon\otimes\mathcal{L}^2\\
&\quad+\int_\Omega \tfrac{2l}{l+2}\Big(e_{22}(\boldsymbol{u}_\varepsilon)e_{33}(\boldsymbol{\varphi}_\varepsilon)+e_{33}(\boldsymbol{u}_\varepsilon)e_{22}(\boldsymbol{\varphi}_\varepsilon)\Big)dm_\varepsilon\otimes\mathcal{L}^2=\int_\Omega \boldsymbol{f}\cdot\boldsymbol{\varphi}_\varepsilon dx.
\end{aligned}
\tag{6.10}
$$

By (4.5), the next weak convergences in the sense of (4.1) hold

$$
\boldsymbol{\sigma}_\varepsilon(\boldsymbol{u}_\varepsilon)\boldsymbol{e}_1 \xrightarrow{\nu_\varepsilon\otimes\mathcal{L}^2,\nu\otimes\mathcal{L}^2} \boldsymbol{\sigma}^\nu(\boldsymbol{u})\boldsymbol{e}_1,\quad \boldsymbol{e}_{x'}(\boldsymbol{u}'_\varepsilon) \xrightarrow{m_\varepsilon\otimes\mathcal{L}^2,m\otimes\mathcal{L}^2} \boldsymbol{e}_{x'}((\boldsymbol{u}^\star)').
\tag{6.11}
$$

By passing to the limit as $\varepsilon\to 0$ in (6.10), by virtue of (6.9) , (6.11) and Lemma 4.3 (iii), we obtain

$$\begin{aligned}
&\int_\Omega \tfrac{1}{l+2}\sigma^\nu_{11}(\boldsymbol{u})\sigma^\nu_{11}(\boldsymbol{\varphi}) + \sum_{\alpha=2}^{3} \sigma^\nu_{1\alpha}(\boldsymbol{u})\sigma^\nu_{1\alpha}(\boldsymbol{\varphi})\ d\nu\otimes\mathcal{L}^2\\
&+\int_\Omega 4\, e_{23}(\boldsymbol{u}^\star)e_{23}(\boldsymbol{\varphi}^\star) + \tfrac{4(l+1)}{l+2}\sum_{\alpha=2}^{3} e_{\alpha\alpha}(\boldsymbol{u}^\star)e_{\alpha\alpha}(\boldsymbol{\varphi}^\star)dm\otimes\mathcal{L}^2\\
&+\int_\Omega \tfrac{2l}{l+2}\big(e_{22}(\boldsymbol{u}^\star)e_{33}(\boldsymbol{\varphi}^\star)+e_{33}(\boldsymbol{u}^\star)e_{22}(\boldsymbol{\varphi}^\star)\big)dm\otimes\mathcal{L}^2 = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{\varphi}\ dx.
\end{aligned}\tag{6.12}$$

An elementary computation yields

$$\begin{aligned}
&\int_\Omega \tfrac{1}{l+2}\sigma^\nu_{11}(\boldsymbol{u})\sigma^\nu_{11}(\boldsymbol{\varphi}) + \sum_{\alpha=2}^{3} \sigma^\nu_{1\alpha}(\boldsymbol{u})\sigma^\nu_{1\alpha}(\boldsymbol{\varphi}) d\nu\otimes\mathcal{L}^2 = \int_\Omega \boldsymbol{a}^\perp \tfrac{\boldsymbol{E}\boldsymbol{u}}{\nu\otimes\mathcal{L}^2} : \tfrac{\boldsymbol{E}\boldsymbol{\varphi}}{\nu\otimes\mathcal{L}^2} d\nu\otimes\mathcal{L}^2,\\
&\int_\Omega 4\, e_{23}(\boldsymbol{u}^\star)e_{23}(\boldsymbol{\varphi}^\star) + \int_\Omega \tfrac{2l}{l+2}\big(e_{22}(\boldsymbol{u}^\star)e_{33}(\boldsymbol{\varphi}^\star)+ e_{33}(\boldsymbol{u}^\star)e_{22}(\boldsymbol{\varphi}^\star)\big)dm\otimes\mathcal{L}^2\\
&\quad + \tfrac{4(l+1)}{l+2}\sum_{\alpha=2}^{3} e_{\alpha\alpha}(\boldsymbol{u}^\star)e_{\alpha\alpha}(\boldsymbol{\varphi}^\star)dm\otimes\mathcal{L}^2 = \int_\Omega \boldsymbol{a}^\parallel \boldsymbol{e}_{x'}(\boldsymbol{u}^\star) : \boldsymbol{e}_{x'}(\boldsymbol{\varphi}^\star)\ dm\otimes\mathcal{L}^2,
\end{aligned}\tag{6.13}$$

where $\boldsymbol{a}^\perp$ and $\boldsymbol{a}^\parallel$ are given by (3.10). We infer from (6.12) and (6.13) that

$$a(\boldsymbol{u},\boldsymbol{\varphi}) = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{\varphi}\ dx,$$

where $a(\cdot,\cdot)$ is the continuous symmetric bilinear form on $BD^{\nu,m}(\Omega)$ defined by (3.9). Substituting $\boldsymbol{v}^\delta$ for $\boldsymbol{\varphi}$ (see (6.1)) and letting δ converge to 0, we deduce from the strong convergence in $BD^{\nu,m}(\Omega)$ of $\boldsymbol{v}^\delta$ to $\boldsymbol{v}$ stated in (5.13) that

$$a(\boldsymbol{u},\boldsymbol{v}) = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\ dx \quad \forall \boldsymbol{v}\in BD^{\nu,m}_0(\Omega).\tag{6.14}$$

Since, by Proposition 4.2, the field $\boldsymbol{u}$ belongs to $BD^{\nu,m}_0(\Omega)$, we conclude that $\boldsymbol{u}$ is a solution to (3.6).

Let us prove that $BD^{\nu,m}_0(\Omega)$ is a Hilbert space. By the Poincaré inequality in $\{\boldsymbol{v}\in BD(\Omega),\ \boldsymbol{v}=0 \text{ on } \partial\Omega\}$ (see [54, Remark 2.5 (ii) p. 156]), we have

$$\begin{aligned}
&\int_\Omega |\boldsymbol{v}|dx \le C\int_\Omega d|\boldsymbol{E}\boldsymbol{v}| = C\int_\Omega \left|\tfrac{\boldsymbol{E}\boldsymbol{v}}{\nu\otimes\mathcal{L}^2}\right| d\nu\otimes\mathcal{L}^2\\
&\qquad \le C\left(\int_\Omega \left|\tfrac{\boldsymbol{E}\boldsymbol{v}}{\nu\otimes\mathcal{L}^2}\right|^2 d\nu\otimes\mathcal{L}^2\right)^{\frac12} \le C||\boldsymbol{v}||_{BD^{\nu,m}_0(\Omega)} \quad \forall\ \boldsymbol{v}\in BD^{\nu,m}_0(\Omega),
\end{aligned}\tag{6.15}$$

hence the semi-norm $||.||_{BD^{\nu,m}_0(\Omega)}$ defined by (3.8) is a norm on $BD^{\nu,m}_0(\Omega)$. On the other hand, Fubini's Theorem and Korn's inequality in $H^1_0(\Omega';\mathbb{R}^2)$ imply

$$\begin{aligned}
&\int_\Omega |(\boldsymbol{v}')^\star|^2 dm\otimes\mathcal{L}^2 = \int_0^L dm(x_1)\int_{\Omega'} |(\boldsymbol{v}')^\star|^2 dx'\\
&\qquad \le C\int_0^L dm(x_1)\int_{\Omega'} |\boldsymbol{e}_{x'}(\boldsymbol{v}^\star)|^2 dx' \le C||\boldsymbol{v}||^2_{BD^{\nu,m}_0(\Omega)} \quad \forall\ \boldsymbol{v}\in BD^{\nu,m}_0(\Omega).
\end{aligned}\tag{6.16}$$

Let $(\boldsymbol{v}_n)$ be a Cauchy sequence in $BD_0^{\nu,m}(\Omega)$. By (6.15) and (6.16), the sequences $(\boldsymbol{v}_n)$, $((\boldsymbol{v}_n')^\star)$, $(\frac{\boldsymbol{E}\boldsymbol{v}_n}{\nu\otimes\mathcal{L}^2})$ are Cauchy sequences in $BD(\Omega)$, $L^2_m(0,L;H^1_0(\Omega;\mathbb{R}^3))$, $L^2_{\nu\otimes\mathcal{L}^2}(\Omega;\mathbb{S}^3)$ respectively, hence the following convergences hold

$$
\begin{aligned}
&\boldsymbol{v}_n \to \boldsymbol{v} && \text{strongly in} \quad BD(\Omega),\\
&(\boldsymbol{v}_n')^\star \to \boldsymbol{w}' && \text{strongly in} \quad L^2_m(0,L;H^1_0(\Omega';\mathbb{R}^3)),\\
&\tfrac{\boldsymbol{E}\boldsymbol{v}_n}{\nu\otimes\mathcal{L}^2} \to \boldsymbol{\Xi} && \text{strongly in} \quad L^2_{\nu\otimes\mathcal{L}^2}(\Omega;\mathbb{S}^3),
\end{aligned}
\tag{6.17}
$$

for some $\boldsymbol{v}$, $\boldsymbol{w}'$, $\boldsymbol{\Xi}$. We prove below that

$$
\boldsymbol{E}(\boldsymbol{v}) \ll \nu\otimes\mathcal{L}^2, \quad \boldsymbol{\Xi} = \tfrac{\boldsymbol{E}\boldsymbol{v}}{\nu\otimes\mathcal{L}^2}, \quad \boldsymbol{v}=0 \ \text{ on } \ \partial\Omega, \tag{6.18}
$$

$$
\boldsymbol{w}' = (\boldsymbol{v}')^\star \quad m\otimes\mathcal{L}^2\text{-a.e.}. \tag{6.19}
$$

It follows from (6.17)-(6.19) that $\boldsymbol{v}\in BD_0^{\nu,m}(\Omega)$ and $(\boldsymbol{v}_n)$ strongly converges to $\boldsymbol{v}$ in $BD_0^{\nu,m}(\Omega)$, hence $BD_0^{\nu,m}(\Omega)$ is a Hilbert space. The proof of Theorem 3.1 is achieved provided we establish that the form $a(\cdot,\cdot)$ is continous and coercive on $BD_0^{\nu,m}(\Omega)$. The continuity is straightforward. The coercivity of $a(\cdot,\cdot)$ results from Lemma 6.2 stated below.

Proof of (6.18)**.** As $\boldsymbol{v}_n = 0$ on $\partial\Omega$, by (6.17) and Green's formula we have, for $\boldsymbol{\Psi}\in C^1(\overline{\Omega};\mathbb{S}^3)$,

$$
\begin{aligned}
\int_\Omega \boldsymbol{v}\cdot \mathbf{div}\boldsymbol{\Psi} dx &= \lim_{n\to\infty}\int_\Omega \boldsymbol{v}_n\cdot\mathbf{div}\boldsymbol{\Psi} dx = -\lim_{n\to\infty}\int_\Omega \boldsymbol{\Psi} d\boldsymbol{E}\boldsymbol{v}_n\\
&= -\lim_{n\to\infty}\int_\Omega \tfrac{\boldsymbol{E}\boldsymbol{v}_n}{\nu\otimes\mathcal{L}^2} : \boldsymbol{\psi} d\nu\otimes\mathcal{L}^2 = -\int_\Omega \boldsymbol{\Xi} : \boldsymbol{\psi} d\nu\otimes\mathcal{L}^2.
\end{aligned}
$$

We deduce from Green's formula that

$$
-\int_\Omega \boldsymbol{\Psi} : d\boldsymbol{E}(\boldsymbol{v}) + \int_{\partial\Omega} \boldsymbol{v}\odot\boldsymbol{n} : \boldsymbol{\Psi} d\mathcal{H}^2 = -\int_\Omega \boldsymbol{\Xi} : \boldsymbol{\psi} d\nu\otimes\mathcal{L}^2.
$$

By the arbitrary choice of $\boldsymbol{\psi}$, we infer (6.18).

Proof of (6.19)**.** By (6.17), $\lim_{n\to+\infty}\int_\Omega |(\boldsymbol{v}_n')^\star - \boldsymbol{w}'|^2 dm\otimes\mathcal{L}^2 = 0$, hence there exists a m-negligible subset N of $(0,L)$ such that

$$
\lim_{n\to+\infty}\int_{\Sigma_{x_1}} |(\boldsymbol{v}_n')^\star - \boldsymbol{w}'|^2 d\mathcal{H}^2 = 0 \qquad \forall x_1\in(0,L)\setminus N. \tag{6.20}
$$

On the other hand, since $(\boldsymbol{v}_n)$ strongly converges to $\boldsymbol{v}$ in $BD(\Omega)$, the traces $\boldsymbol{\gamma}^\pm_{\Sigma_{x_1}}(\boldsymbol{v}_n)$ on both side of Σ_{x_1} strongly converges to $\boldsymbol{\gamma}^\pm_{\Sigma_{x_1}}(\boldsymbol{v})$ in $L^1_{\mathcal{H}^2}(\Sigma_{x_1})$ for all $x_1\in(0,L)$. By (4.13), (4.17), and (6.18), $\boldsymbol{v}^\star(x_1,.) = \boldsymbol{\gamma}^+_{\Sigma_{x_1}}(\boldsymbol{v}) = \boldsymbol{\gamma}^-_{\Sigma_{x_1}}(\boldsymbol{v})$ $\mathcal{H}^2$-a.e. on Σ_{x_1} for m-a.e. $x_1\in(0,L)$. Accordingly, there exists a m-negligible subset N_1 of $(0,L)$ such that

$$
\lim_{n\to+\infty}\int_{\Sigma_{x_1}} |(\boldsymbol{v}_n)^\star - \boldsymbol{v}^\star| d\mathcal{H}^2 = 0 \qquad \forall x_1\in(0,L)\setminus N_1. \tag{6.21}
$$

Let us fix $x_1 \in (0,L) \setminus (N \cup N_1)$. By (6.20) there exists a subsequence of $(\boldsymbol{v}'_n)^\star$ converging $\mathcal{H}^2$-a.e. on Σ_{x_1} to $\boldsymbol{w}'$. By (6.21), there exists a further subsequence converging $\mathcal{H}^2$-a.e. on Σ_{x_1} to $(\boldsymbol{v}')^\star$. Hence $\boldsymbol{w}' = (\boldsymbol{v}')^\star$ $\mathcal{H}^2$-a.e. on Σ_{x_1} for m-a.e. $x_1 \in (0,L)$. Setting $A := \{x \in \Omega,\ \boldsymbol{w}'(x) \neq (\boldsymbol{v}')^\star(x)\}$, $A_{x_1} := A \cap \Sigma_{x_1}$, we infer that $\mathcal{H}^2(A_{x_1}) = 0$ for all $x_1 \in (0,L) \setminus (N \cup N_1)$. It then follows from Fubini's theorem that $m \otimes \mathcal{L}^2(A) = \int_{(0,L)} \mathcal{H}^2(A_{x_1}) dm(x_1) = 0$.

Lemma 6.2. *For all $\boldsymbol{v} \in BD_0^{\nu,m}(\Omega)$, $\alpha, \beta \in \{2,3\}$, we have*

$$\int_\Omega \left| \frac{E_{\alpha\beta}\boldsymbol{v}}{\nu\otimes\mathcal{L}^2} \right|^2 \, d\nu \otimes \mathcal{L}^2 \leq \int_\Omega \left| e_{\alpha\beta}((\boldsymbol{v}^\star)') \right|^2 \, dm \otimes \mathcal{L}^2. \tag{6.22}$$

Proof. Let $\boldsymbol{v} \in BD_0^{\nu,m}(\Omega)$, $\delta > 0$, and $\boldsymbol{\varphi}_\varepsilon$ defined by (6.1), (6.8). By Proposition 6.1, the convergence (4.28) holds, hence by Lemma 4.3 (ii), we have for $\alpha, \beta \in \{2,3\}$,

$$\int_\Omega \left| \frac{E_{\alpha\beta}\boldsymbol{\varphi}}{\nu\otimes\mathcal{L}^2} \right|^2 \, d\nu \otimes \mathcal{L}^2 \leq \liminf_{\varepsilon\to 0} \int_\Omega \mu_\varepsilon \left| \boldsymbol{e}_{\alpha\beta}(\boldsymbol{\varphi}_\varepsilon) \right|^2 \, dx.$$

As, on the other hand, by (4.2) and (6.9), the following holds

$$\lim_{\varepsilon\to 0} \int_\Omega \mu_\varepsilon \left| \boldsymbol{e}_{\alpha\beta}(\boldsymbol{\varphi}_\varepsilon) \right|^2 \, dx = \int_\Omega \left| \boldsymbol{e}_{\alpha\beta}((\boldsymbol{\varphi}^\star)') \right|^2 \, dm \otimes \mathcal{L}^2,$$

we deduce that

$$\int_\Omega \left| \frac{E_{\alpha\beta}\boldsymbol{\varphi}}{\nu\otimes\mathcal{L}^2} \right|^2 \, d\nu \otimes \mathcal{L}^2 \leq \int_\Omega \left| \boldsymbol{e}_{\alpha\beta}((\boldsymbol{\varphi}^\star)') \right|^2 \, dm \otimes \mathcal{L}^2.$$

Substituting $\boldsymbol{v}^\delta$ for $\boldsymbol{\varphi}$ and passing to the limit as $\delta \to 0$, taking (4.26), (5.13) into account, we obtain (6.22). ◻

Justification of (6.10). We fix $\boldsymbol{e}, \widetilde{\boldsymbol{e}} \in \mathbb{S}^3$ and set $\boldsymbol{\sigma} := l(\operatorname{tr}\boldsymbol{e})\boldsymbol{I} + 2\boldsymbol{e}$, $\widetilde{\boldsymbol{\sigma}} := l(\operatorname{tr}\widetilde{\boldsymbol{e}})\boldsymbol{I} + 2\widetilde{\boldsymbol{e}}$. We have

$$\boldsymbol{\sigma} : \widetilde{\boldsymbol{e}} = \sum_{i=1}^{3} \sigma_{ii}\widetilde{e}_{ii} + \sigma_{12}\widetilde{\sigma}_{12} + \sigma_{13}\widetilde{\sigma}_{13} + 4e_{23}\widetilde{e}_{23}. \tag{6.23}$$

Noticing that

$$\begin{aligned}
&e_{11} = \tfrac{1}{l+2}(\sigma_{11} - le_{22} - le_{33}), \quad \widetilde{e}_{11} = \tfrac{1}{l+2}(\widetilde{\sigma}_{11} - l\widetilde{e}_{22} - l\widetilde{e}_{33}),\\
&\sigma_{22} = le_{11} + (l+2)e_{22} + le_{33} = \tfrac{l}{l+2}(\sigma_{11} - le_{22} - le_{33}) + (l+2)e_{22} + le_{33},\\
&\sigma_{33} = le_{11} + le_{22} + (l+2)e_{33} = \tfrac{l}{l+2}(\sigma_{11} - le_{22} - le_{33}) + le_{22} + (l+2)e_{33},
\end{aligned}$$

we obtain, by substitution,

$$\begin{aligned}
\sum_{i=1}^{3}\sigma_{ii}\widetilde{e}_{ii} &= \sigma_{11}\tfrac{1}{l+2}(\widetilde{\sigma}_{11} - l\widetilde{e}_{22} - l\widetilde{e}_{33}) + \Big(\tfrac{l}{l+2}(\sigma_{11} - le_{22} - le_{33}) + (l+2)e_{22} + le_{33}\Big)\widetilde{e}_{22}\\
&\qquad + \Big(\tfrac{l}{l+2}(\sigma_{11} - le_{22} - le_{33}) + le_{22} + (l+2)e_{33}\Big)\widetilde{e}_{33}\\
&= \tfrac{1}{l+2}\sigma_{11}\widetilde{\sigma}_{11} + \tfrac{4(l+1)}{l+2}(e_{22}\widetilde{e}_{22} + e_{33}\widetilde{e}_{33}) + \tfrac{2l}{l+2}(e_{22}\widetilde{e}_{33} + e_{33}\widetilde{e}_{22}),
\end{aligned}$$

yielding, by (6.23),

$$\begin{aligned}\boldsymbol{\sigma} : \widetilde{\boldsymbol{e}} = \tfrac{1}{l+2}\sigma_{11}\widetilde{\sigma}_{11} + +2\sigma_{12}\widetilde{e}_{12} + 2\sigma_{13}\widetilde{e}_{13} \\ + 4e_{23}\widetilde{e}_{23} + \tfrac{4(l+1)}{l+2}(e_{22}\widetilde{e}_{22} + e_{33}\widetilde{e}_{33}) + \tfrac{2l}{l+2}(e_{22}\widetilde{e}_{33} + e_{33}\widetilde{e}_{22}).\end{aligned}$$

Substituting $\boldsymbol{e}(\boldsymbol{u}_\varepsilon)$, $\boldsymbol{e}(\boldsymbol{\varphi}_\varepsilon)$, $\frac{1}{\mu_\varepsilon}\boldsymbol{\sigma}_\varepsilon(\boldsymbol{u}_\varepsilon)$, $\frac{1}{\mu_\varepsilon}\boldsymbol{\sigma}_\varepsilon(\boldsymbol{\varphi}_\varepsilon)$, respectively, for $\boldsymbol{e}$, $\widetilde{\boldsymbol{e}}$, $\boldsymbol{\sigma}$, $\widetilde{\boldsymbol{\sigma}}$, we infer (6.10).

6.1. Proof of Proposition 6.1. The proof of Proposition 6.1 lies in the asymptotic analysis of the family of sequences $\left((\boldsymbol{\varphi}_\varepsilon^k)_\varepsilon\right)_{k\in\mathbb{N}}$, the results of which are presented in the next proposition whose proof is located in Section 6.2.

PROPOSITION 6.3. *Let $\boldsymbol{v} \in BD_0^{\nu,m}(\Omega)$, $\delta > 0$, $\boldsymbol{\sigma}^\nu$ defined by (4.3), and $\boldsymbol{\varphi}$, $\boldsymbol{\varphi}_\varepsilon^k$ respectively given by (6.1), (6.7). Then $\boldsymbol{\varphi}_\varepsilon^k$ belongs to $H^1(\Omega;\mathbb{R}^3)$ and satisfies*

$$\sup_{k\in\mathbb{N};\ \varepsilon>0} \int_\Omega |\boldsymbol{\varphi}_\varepsilon^k|^2 dm_\varepsilon \otimes \mathcal{L}^2 < \infty, \tag{6.24}$$

$$\lim_{k\to\infty} \sup_{\varepsilon>0} \int_\Omega |\boldsymbol{\varphi}_\varepsilon^k - \boldsymbol{\varphi}|\ dx = 0, \tag{6.25}$$

$$\limsup_{k\to\infty} \limsup_{\varepsilon\to 0} \int_\Omega \left|\boldsymbol{\sigma}_\varepsilon(\boldsymbol{\varphi}_\varepsilon^k)\boldsymbol{e}_1\right|^2 d\nu_\varepsilon \otimes \mathcal{L}^2 \le \int_\Omega |\boldsymbol{\sigma}^\nu(\boldsymbol{\varphi})\boldsymbol{e}_1|^2\, d\nu \otimes \mathcal{L}^2, \tag{6.26}$$

$$\limsup_{k\to\infty} \limsup_{\varepsilon\to 0} \int_\Omega \left|\boldsymbol{e}_{x'}((\boldsymbol{\varphi}_\varepsilon^k)')\right|^2 dm_\varepsilon \otimes \mathcal{L}^2 \le \int_\Omega |\boldsymbol{e}_{x'}((\boldsymbol{\varphi}^\star)')|^2\, dm \otimes \mathcal{L}^2. \tag{6.27}$$

Let us fix a decreasing sequence of positive reals $(\alpha_k)_{k\in\mathbb{N}}$ converging to 0. By Proposition 6.3, there exists a decreasing sequence of positive reals $(\varepsilon_k)_{k\in\mathbb{N}}$ converging to 0 as $k \to \infty$ and such that, for all $\varepsilon < \varepsilon_k$,

$$\begin{aligned}&\int_\Omega |\boldsymbol{\varphi}_\varepsilon^k - \boldsymbol{\varphi}|\ dx \le \alpha_k,\\ &\int_\Omega \left|\boldsymbol{\sigma}_\varepsilon(\boldsymbol{\varphi}_\varepsilon^k)\boldsymbol{e}_1\right|^2 d\nu_\varepsilon \otimes \mathcal{L}^2 \le \int_\Omega |\boldsymbol{\sigma}^\nu(\boldsymbol{\varphi})\boldsymbol{e}_1|^2\, d\nu \otimes \mathcal{L}^2 + \alpha_k,\\ &\int_\Omega \left|\boldsymbol{e}_{x'}(\boldsymbol{\varphi}_\varepsilon^k)\right|^2 dm_\varepsilon \otimes \mathcal{L}^2 \le \int_\Omega |\boldsymbol{e}_{x'}(\boldsymbol{\varphi}^\star)|^2\, dm \otimes \mathcal{L}^2 + \alpha_k.\end{aligned} \tag{6.28}$$

Let k_ε be the unique integer such that $\varepsilon_{k_\varepsilon+1} \le \varepsilon < \varepsilon_{k_\varepsilon}$ (notice that $k_\varepsilon \to \infty$). We set

$$\boldsymbol{\varphi}_\varepsilon = \boldsymbol{\varphi}_\varepsilon^{k_\varepsilon}. \tag{6.29}$$

By (3.2), (3.1), (6.24), (6.28) and (6.29), the sequence $(\boldsymbol{\varphi}_\varepsilon)$ strongly converges to $\boldsymbol{\varphi}$ in $L^1(\Omega;\mathbb{R}^3)$ and satisfies the assumptions (4.25) and (4.29) of Proposition 4.8. Therefore, the convergences (4.28) and (4.30) hold. We deduce that

$$\boldsymbol{\sigma}_\varepsilon(\boldsymbol{\varphi}_\varepsilon)\boldsymbol{e}_1 \xrightharpoonup{\nu_\varepsilon\otimes\mathcal{L}^2,\nu\otimes\mathcal{L}^2} \boldsymbol{\sigma}^\nu(\boldsymbol{\varphi})\boldsymbol{e}_1, \qquad \boldsymbol{e}_{x'}((\boldsymbol{\varphi}_\varepsilon)') \xrightharpoonup{m_\varepsilon\otimes\mathcal{L}^2,m\otimes\mathcal{L}^2} \boldsymbol{e}_{x'}((\boldsymbol{\varphi}^\star)').$$

On the other hand, (6.28) and (6.29) imply (since $k_\varepsilon \to \infty$)

$$\begin{aligned}
&\limsup_{\varepsilon\to 0}\int_\Omega |\boldsymbol{\sigma}_\varepsilon(\boldsymbol{\varphi}_\varepsilon)\boldsymbol{e}_1|^2\, d\nu_\varepsilon\otimes\mathcal{L}^2 \le \int_\Omega |\boldsymbol{\sigma}^\nu(\boldsymbol{\varphi})\boldsymbol{e}_1|^2\, d\nu\otimes\mathcal{L}^2,\\
&\limsup_{\varepsilon\to 0}\int_\Omega \left|\boldsymbol{e}_{x'}\left((\boldsymbol{\varphi}_\varepsilon)'\right)\right|^2 dm_\varepsilon\otimes\mathcal{L}^2 \le \int_\Omega |\boldsymbol{e}_{x'}((\boldsymbol{\varphi}^\star)')|^2\, dm\otimes\mathcal{L}^2,
\end{aligned}$$

yielding (6.9). Proposition 6.1 is proved provided we establish Proposition 6.3.

6.2. Proof of Proposition 6.3 . Let us prove that $\boldsymbol{\varphi}_\varepsilon^k$ belongs to $H^1(\Omega;\mathbb{R}^3)$. By (6.7), $\boldsymbol{\varphi}_\varepsilon^k$ belongs to $H^1(I_j^k\times\Omega';\mathbb{R}^3)$ for all $j\in\{1,\dots,n_k-1\}$, therefore it suffices to show that the traces of $\boldsymbol{\varphi}_\varepsilon^k$ coincide on each side of the common boundaries of $I_j^k\times\Omega'$ and $I_{j+1}^k\times\Omega'$, that is

$$(\boldsymbol{\varphi}_\varepsilon^k)^- = (\boldsymbol{\varphi}_\varepsilon^k)^+ \ \mathcal{H}^2\text{-a.e. on } \Sigma_{t_j^k} \quad \forall j\in\{1,\dots,n_k-1\}. \tag{6.30}$$

One easily deduce from formula (5.15) (applied to $\boldsymbol{v}^\delta=\boldsymbol{\varphi}$) that

$$\begin{aligned}
\varphi_1^+(t_j^k,x')-\varphi_1^+(t_{j-1}^k,x') = {} & \tfrac{1}{l+2}\int_{I_j^k}(\boldsymbol{\sigma}^\nu)_{11}(\boldsymbol{\varphi})(s_1,x')d\nu(s_1)\\
& -\sum_{\alpha=2}^3 \tfrac{l}{l+2}\int_{I_j^k}\frac{\partial\varphi_\alpha}{\partial x_\alpha}(s_1,x')ds_1.
\end{aligned}\tag{6.31}$$

On the other hand, by the properties of ϕ_ε^k and the definition of $\boldsymbol{\varphi}_\varepsilon^k$ (see (6.6), (6.7)), we have

$$\begin{aligned}
(\boldsymbol{\varphi}_\varepsilon^k)_1^-(t_j^k,x') = {} & \tfrac{1}{l+2}\int_{I_j^k}(\boldsymbol{\sigma}^\nu)_{11}(\boldsymbol{\varphi})(s_1,x')\, d\nu(s_1)\\
& -\sum_{\alpha=2}^3 \tfrac{l}{l+2}\int_{t_{j-1}^k}^{t_j^k}\frac{\partial\varphi_\alpha}{\partial x_\alpha}(s_1,x')ds_1 + \varphi_1^+(t_{j-1}^k,x').
\end{aligned}\tag{6.32}$$

We infer from (6.31) and (6.32) that $(\boldsymbol{\varphi}_\varepsilon^k)_1^-(t_j^k,x')) = \varphi_1^+(t_j^k,x')$. Since (6.6) and (6.7) imply $(\boldsymbol{\varphi}_\varepsilon^k)_1^+(t_{j-1}^k,x') = \varphi_1^+(t_{j-1}^k,x')$ for all $j\in\{1,\dots,n_k\}$, we deduce that (6.30) is satisfied by the first component of $\boldsymbol{\varphi}_\varepsilon^k$. Likewise, we deduce from the second equation in (5.15) that, for $\alpha\in\{2,3\}$,

$$\varphi_\alpha^+(t_j^k,x')-\varphi_\alpha^+(t_{j-1}^k,x') = \int_{I_j^k}(\boldsymbol{\sigma}^\nu)_{1\alpha}(\boldsymbol{\varphi})(s_1,x')\, d\nu(s_1) - \int_{I_j^k}\frac{\partial\varphi_1}{\partial x_\alpha}(s_1,x')\, ds_1,$$

and then from (6.7) that

$$(\boldsymbol{\varphi}^k_\varepsilon)^-_\alpha(t^k_j,x')=\int_{I^k_j}(\boldsymbol{\sigma}^\nu(\boldsymbol{\varphi}))_{1\alpha}(s_1,x')d\nu(s_1)-\int_{I^k_j}\frac{\partial\varphi_1}{\partial x_\alpha}(s_1,x')ds_1+\varphi^+_\alpha(t^k_{j-1},x'),$$

yielding $(\boldsymbol{\varphi}^k_\varepsilon)^-_\alpha(t^k_j,x'))=\varphi^+_\alpha(t^k_j,x')$. Noticing that (6.7) also implies that $(\boldsymbol{\varphi}^k_\varepsilon)^+_\alpha(t^k_{j-1},x')=\varphi^+_\alpha(t^k_{j-1},x')$ for all $j\in\{1,\dots,n_k\}$, we infer that $(\boldsymbol{\varphi}^k_\varepsilon)^-_\alpha(t^k_j,x'))=(\boldsymbol{\varphi}^k_\varepsilon)^+_\alpha(t^k_j,x'))$. Assertion (6.30) is proved and $\boldsymbol{\varphi}^k_\varepsilon$ belongs to $H^1(\Omega;\mathbb{R}^3)$.
The next lemma plays a crucial role in the proof of Proposition 6.3. In what follows, for all $x_1\in(0,L)$, we denote by j_{x_1} the unique integer satisfying

$$x_1\in\left(t^k_{j_{x_1}-1},t^k_{j_{x_1}}\right]. \tag{6.33}$$

Lemma 6.4. *We have*

$$\lim_{\varepsilon\to0}\nu_\varepsilon(I^k_j)=\nu(I^k_j)\ \textit{and}\ \lim_{\varepsilon\to0}m_\varepsilon(I^k_j)=m(I^k_j)\quad\forall k\in\mathbb{N},\ \forall j\in\{1,\dots,n_k\}. \tag{6.34}$$

For all $k\in\mathbb{N}$, the mapping $x_1\in(0,L]\to\nu(I^k_{j_{x_1}})$ defined by (6.4), (6.33) is Borel measurable and satisfies, for all $p\in(0,\infty)$,

$$\begin{aligned}&\lim_{\varepsilon\to0}\int\nu(I^k_{j_{x_1}})dm_\varepsilon(x_1)=\int\nu(I^k_{j_{x_1}})dm(x_1),\\&\lim_{k\to\infty}\int_0^L\nu(I^k_{j_{x_1}})^p\ d\mathcal{L}^1(x_1)=0,\quad\lim_{k\to\infty}\int_{[0,L]}\nu(I^k_{j_{x_1}})^p\ dm(x_1)=0.\end{aligned} \tag{6.35}$$

Proof. Since $\nu(\partial I^k_j)=m(\partial I^k_j)=0$ for all $k\in\mathbb{N}$, $j\in\{1,\dots,n_k\}$ (see (6.3)), the convergences (6.34) result from (3.3). By (6.3) and (6.33), we have

$$\nu(I^k_{j_{x_1}})=\sum_{j=1}^{n_k}\nu\left(I^k_j\right)\mathbb{1}_{I^k_j}(x_1), \tag{6.36}$$

hence the mapping $x_1\in(0,L]\to\nu(I^k_{j_{x_1}})$ is Borel-measurable and, by (6.34),

$$\begin{aligned}\lim_{\varepsilon\to0}\int\nu(I^k_{j_{x_1}})dm_\varepsilon(x_1)&=\lim_{\varepsilon\to0}\sum_{j=1}^{n_k}\nu(I^k_j)m_\varepsilon(I^k_j)=\sum_{j=1}^{n_k}\nu(I^k_j)m(I^k_j)\\&=\int\nu(I^k_{j_{x_1}})dm(x_1).\end{aligned}$$

The measure ν is bounded and the assumptions (6.3) imply that, for each fixed $x_1\in(0,L]$, the sequence of sets $(I^k_{j_{x_1}})_{k\in\mathbb{N}}$ is decreasing and satisfies $\bigcap_{k\in\mathbb{N}}\downarrow I^k_{j_{x_1}}=\{x_1\}$, therefore $\lim_{k\to\infty}\nu(I^k_{j_{x_1}})=\nu(\{x_1\})$. Applying the Dominated Convergence Theorem, noticing that, by (3.4), $\mathcal{L}^1(\mathcal{A}_\nu)=m(\mathcal{A}_\nu)=0$, we infer

$$\begin{aligned}&\lim_{k\to\infty}\int_0^L\nu(I^k_{j_{x_1}})^p\ d\mathcal{L}^1(x_1)=\int_{\mathcal{A}_\nu}\nu(\{x_1\})^pd\mathcal{L}^1(x_1)=0,\\&\lim_{k\to\infty}\int_{[0,L]}\nu(I^k_{j_{x_1}})^p\ dm(x_1)=\int_{\mathcal{A}_\nu}\nu(\{x_1\})^pdm(x_1)=0.\end{aligned}$$

□

Proof of (6.24)**.** By (4.3), (5.15), (5.19), (6.1), we have, forall $x_1 \in (0,L)$,

$$\int_{\Omega'} \left|\boldsymbol{\varphi}^+(t^k_{j_{x_1}-1}, x')\right|^2 \, dx' \leq C \int_{\Omega} |\boldsymbol{\sigma}^\nu(\boldsymbol{\varphi})|^2 \, d\nu \otimes \mathcal{L}^2 + C \int_{\Omega} \left|\frac{\partial \boldsymbol{\varphi}}{\partial x_\alpha}\right|^2 \, dx \leq C,$$

therefore, by (6.6), (6.7), and (6.33),

$$\begin{aligned}
&\sup_{x_1 \in (0,L)} \int_{\Omega'} |\boldsymbol{\varphi}^k_\varepsilon|^2(x_1, x')dx' \\
&\quad \leq C \left(\int_{\Omega} |\boldsymbol{\sigma}^\nu(\boldsymbol{\varphi})|^2 d\nu \otimes \mathcal{L}^2 + \int_{\Omega} |\frac{\partial \boldsymbol{\varphi}}{\partial x_\alpha}|^2 dx + \int_{\Omega'} |\boldsymbol{\varphi}^+(t^k_{j_{x_1}-1}, x')|^2 dx' \right) \leq C.
\end{aligned}$$

By integrating over $(0,L)$ with respect to m_ε, we obtain (6.24).

Proof of (6.25)**.** By (5.15), (5.19), (6.1), the following estimate holds for $x_1 \in I^k_j$ (or equivalently for $j = j_{x_1}$):

(6.37)

$$\begin{aligned}
\int_{\Omega'} |\boldsymbol{\varphi}^+(x_1, x') - \boldsymbol{\varphi}^+(t_j, x')|dx' &\leq C \int_{I^k_j \times \Omega'} |\boldsymbol{\sigma}^\nu(\boldsymbol{\varphi})| d\nu \otimes \mathcal{L}^2 + C \sum_{\alpha=2}^{3} \int_{I^k_j \times \Omega'} |\frac{\partial \boldsymbol{\varphi}}{\partial x_\alpha}| d\mathcal{L}^3 \\
&\leq C\nu(I^k_j)^{\frac{1}{2}} ||\boldsymbol{\sigma}^\nu(\boldsymbol{\varphi})||^{\frac{1}{2}}_{L^2_{\nu\otimes\mathcal{L}^2}} + C \left(\sup_{j \in \{1,\dots,n_k\}} \mathcal{L}^1(I^k_j) \right)^{\frac{1}{2}} \sum_{\alpha=2}^{3} ||\frac{\partial \boldsymbol{\varphi}}{\partial x_\alpha}||^{\frac{1}{2}}_{L^2(\Omega)} \\
&\leq C\nu(I^k_j)^{\frac{1}{2}} + C \left(\sup_{j \in \{1,\dots,n_k\}} \mathcal{L}^1(I^k_j) \right)^{\frac{1}{2}}.
\end{aligned}$$

By integration over $(0,L)$ with respect to $\mathcal{L}^1$, taking (6.3), (6.33), (6.35) into account, we infer

$$\lim_{k\to\infty} \int_{\Omega} |\boldsymbol{\varphi}^+(x_1, x') - \boldsymbol{\varphi}^+(t_{j_{x_1}}, x')|dx = 0. \tag{6.38}$$

By the same argument, we deduce from (6.6), (6.7) that

$$\lim_{k\to\infty} \int_{\Omega} |\boldsymbol{\varphi}^k_\varepsilon(x_1, x') - \boldsymbol{\varphi}^+(t_{j_{x_1}}, x')|dx = 0. \tag{6.39}$$

Assertion (6.25) results from (6.38) and (6.39).

Proof of (6.26)**.** Taking (3.2), (4.3), (6.6) and (6.7) into account, an elementary computation yields, for all $j \in \{1,\dots,n_k\}$ and for $\mathcal{L}^3$-a.e. $x \in I^k_j \times \Omega'$,

$$\begin{aligned}
\boldsymbol{\sigma}_\varepsilon(\boldsymbol{\varphi}^k_\varepsilon)(x)\boldsymbol{e}_1 &= \mu_\varepsilon \left(l\mathrm{tr}(\boldsymbol{e}(\boldsymbol{\varphi}^k_\varepsilon))\boldsymbol{I} + 2\boldsymbol{e}(\boldsymbol{\varphi}^k_\varepsilon) \right) \boldsymbol{e}_1 \\
&= \frac{1}{\nu_\varepsilon(I^k_j)} \int_{I^k_j} \boldsymbol{\sigma}^\nu(\boldsymbol{\varphi})(s_1, x')\boldsymbol{e}_1 d\nu(s_1) + \boldsymbol{r}^k_\varepsilon(x),
\end{aligned} \tag{6.40}$$

where for $\alpha \in \{2,3\}$,

$$
(6.41)\quad
\begin{aligned}
\frac{r^k_{\varepsilon 1}}{\mu_\varepsilon}(x) &:= l\sum_{\alpha=2}^{3}\Big(\frac{\partial\varphi^+_\alpha}{\partial x_\alpha}(t^k_{j-1},x') - \frac{\partial\varphi^+_\alpha}{\partial x_\alpha}(x_1,x')\Big)\\
&\quad + 2l\phi^k_\varepsilon(x_1)\sum_{\alpha=2}^{3}\int_{I^k_j}\frac{\partial(\boldsymbol{\sigma}^\nu(\boldsymbol{\varphi}))_{1\alpha}}{\partial x_\alpha}(s_1,x')d\nu(s_1) - l\sum_{\alpha=2}^{3}\int_{t^k_{j-1}}^{x_1}\frac{\partial^2\varphi_1}{\partial x_\alpha^2}(s_1,x')ds_1,\\
\frac{r^k_{\varepsilon\alpha}}{\mu_\varepsilon}(x) &:= \frac{1}{l+2}\phi^k_\varepsilon(x_1)\int_{I^k_j}\frac{\partial(\boldsymbol{\sigma}^\nu(\boldsymbol{\varphi}))_{11}}{\partial x_\alpha}(s_1,x')\,d\nu(s_1)\\
&\quad - \frac{l}{l+2}\sum_{\beta=2}^{3}\int_{t^k_{j-1}}^{x_1}\frac{\partial^2\varphi_\beta}{\partial x_\beta\partial x_\alpha}(s_1,x')ds_1 + \frac{\partial\varphi^+_1}{\partial x_\alpha}(t^k_{j-1},x') - \frac{\partial\varphi_1}{\partial x_\alpha}(x_1,x').
\end{aligned}
$$

We prove below that

$$
(6.42)\qquad \limsup_{k\to\infty}\limsup_{\varepsilon\to 0}\int_\Omega |\boldsymbol{r}^k_\varepsilon|^2 d\nu_\varepsilon\otimes\mathcal{L}^2 = 0.
$$

By (6.40), we have

$$
(6.43)\quad
\begin{aligned}
&\int_\Omega \Big|\boldsymbol{\sigma}_\varepsilon(\boldsymbol{\varphi}^k_\varepsilon)\boldsymbol{e}_1 - \boldsymbol{r}^k_\varepsilon\Big|^2 d\nu_\varepsilon\otimes\mathcal{L}^2\\
&\qquad = \sum_{j=1}^{n_k}\frac{\int_{I^k_j}\mu_\varepsilon^{-1}(x_1)dx_1}{(\nu_\varepsilon(I^k_j))^2}\int_{\Omega'}\left|\int_{I^k_j}\boldsymbol{\sigma}^\nu(\boldsymbol{\varphi})\boldsymbol{e}_1(s_1,x')d\nu(s_1)\right|^2 dx'\\
&\qquad \le \sum_{j=1}^{n_k}\frac{\nu(I^k_j)}{\nu_\varepsilon(I^k_j)}\int_{I^k_j\times\Omega'}|\boldsymbol{\sigma}^\nu(\boldsymbol{\varphi})\boldsymbol{e}_1|^2\ d\nu\otimes\mathcal{L}^2.
\end{aligned}
$$

Assertion (6.26) follows from (6.34), (6.42), (6.43).
Proof of (6.42). A computation analogous to (6.37) yields for $x_1\in I^k_j$, taking (5.19) into account,

$$
(6.44)\qquad \int_{\Omega'}\left|\frac{\partial\boldsymbol{\varphi}^+}{\partial x_\alpha}(x_1,x') - \frac{\partial\boldsymbol{\varphi}^+}{\partial x_\alpha}(t_j,x')\right|^2 dx' \le C\nu(I^k_j) + C\sup_{j\in\{1,\dots,n_k\}}\mathcal{L}^1(I^k_j).
$$

Similarly, by (5.4),

$$
(6.45)\qquad \int_{\Omega'}\left|\int_{I^k_j}\frac{\partial\boldsymbol{\sigma}^\nu}{\partial x_\alpha}(s_1,x')d\nu(s_1)\right|^2 dx' \le C\nu(I^k_j)\left|\left|\frac{\partial\boldsymbol{\sigma}^\nu}{\partial x_\alpha}\right|\right|^2_{L^2_{\nu\otimes\mathcal{L}^2}(\Omega)} \le C\nu(I^k_j),
$$

$$
(6.46)\quad
\begin{aligned}
\int_{\Omega'}\left|\int_{t^k_{j-1}}^{x_1}\frac{\partial^2\varphi_\beta}{\partial x_\beta\partial x_\alpha}(s_1,x')ds_1\right|^2 dx' &\le C\sup_{j\in\{1,\dots,n_k\}}\mathcal{L}^1(I^k_j)\left|\left|\frac{\partial^2\varphi_\beta}{\partial x_\beta\partial x_\alpha}\right|\right|^2_{L^2(\Omega)}\\
&\le C\sup_{j\in\{1,\dots,n_k\}}\mathcal{L}^1(I^k_j).
\end{aligned}
$$

Collecting (6.6), (6.41), (6.44), (6.45), (6.46), noticing that $\mu^2_\varepsilon\nu_\varepsilon = m_\varepsilon$, we infer

$$\int_\Omega |\boldsymbol{r}^k_\varepsilon|^2 d\nu_\varepsilon\otimes\mathcal{L}^2 \le C\int \nu(I^k_{j_{x_1}})dm_\varepsilon(x_1) + C\sup_{j\in\{1,\dots,n_k\}}\mathcal{L}^1(I^k_j)m_\varepsilon((0,L)). \tag{6.47}$$

Assertion (6.42) results from (6.3), (6.35), (6.47).
Proof of (6.27). By (6.7) we have, for $x_1\in I^k_j$,

$$\begin{aligned}
&\boldsymbol{e}_{x'}(\boldsymbol{\varphi}^k_\varepsilon)(x) = \boldsymbol{e}_{x'}(\boldsymbol{\varphi}^+)(t^k_{j-1},x') + \boldsymbol{R}^k_\varepsilon(x),\\
&\boldsymbol{R}^k_\varepsilon(x) := \phi^k_\varepsilon(x_1)\int_{I^k_j}\boldsymbol{e}_{x'}\left(\boldsymbol{\sigma}^\nu(\boldsymbol{\varphi})\boldsymbol{e}_1\right)(s_1,x')d\nu(s_1)\\
&\qquad\qquad - \sum_{\alpha,\beta=2}^3\int_{t^k_{j-1}}^{x_1}\tfrac{\partial^2\varphi_1}{\partial x_\alpha\partial x_\alpha}(s_1,x')ds_1\boldsymbol{e}_\alpha\odot\boldsymbol{e}_\beta.
\end{aligned} \tag{6.48}$$

We deduce from (6.6), (6.45), (6.46), (6.48), that $\int_\Omega\left|\boldsymbol{R}^k_\varepsilon\right|^2(x)dm_\varepsilon$ is bounded from above by the left-hand side of (6.47), hence, by (6.3), (6.35),

$$\lim_{k\to\infty}\sup_{\varepsilon>0}\int_\Omega\left|\boldsymbol{R}^k_\varepsilon\right|^2 dm_\varepsilon\otimes\mathcal{L}^2 = 0. \tag{6.49}$$

By (3.1) and (6.4) we have

$$\int_\Omega\left|\boldsymbol{e}_{x'}(\boldsymbol{\varphi}^+)\right|^2(t^k_{j_{x_1}-1},x')\,dm_\varepsilon\otimes\mathcal{L}^2 = \sum_{j=1}^{n_k} m_\varepsilon(I^k_j)\int_{\Omega'}\left|\boldsymbol{e}_{x'}(\boldsymbol{\varphi}^+)\right|^2(t^k_{j-1},x')\,dx',$$

yielding, by (6.34),

$$\lim_{\varepsilon\to 0}\int_\Omega\left|\boldsymbol{e}_{x'}(\boldsymbol{\varphi}^+)\right|^2(t^k_{j_{x_1}-1},x')dm_\varepsilon\otimes\mathcal{L}^2 = \int_\Omega\left|\boldsymbol{e}_{x'}(\boldsymbol{\varphi}^+)\right|^2(t^k_{j_{x_1}-1},x')dm\otimes\mathcal{L}^2. \tag{6.50}$$

By (6.3) and (6.33), for all $x_1\in(0,L)$, the sequence $(t^k_{j_{x_1-1}})_{k\in\mathbb{N}}$ converges to x_1 from below as $k\to\infty$. Therefore, by (5.14), for each $x\in\Omega$ the following holds

$$\lim_{k\to\infty}\left|\boldsymbol{e}_{x'}(\boldsymbol{\varphi}^+)\right|^2(t^k_{j_{x_1}-1},x') = \left|\boldsymbol{e}_{x'}(\boldsymbol{\varphi}^-)\right|^2(x). \tag{6.51}$$

On the other hand, by (5.15), $\left|\boldsymbol{e}_{x'}(\boldsymbol{\varphi}^+)\right|^2(t^k_{j_{x_1}-1},x')\le g(x)$, where

$$g(x) := \int_{(0,L)}\left|\boldsymbol{e}_{x'}(\boldsymbol{\sigma}^\nu(\boldsymbol{\varphi})\boldsymbol{e}_1)\right|^2(s_1,x')d\nu(s_1) + \sum_{\alpha,\beta=2}^3\int_0^L\left|\frac{\partial^2\varphi_1}{\partial x_\alpha\partial x_\beta}\right|^2(s_1,x')ds_1.$$

We deduce from (5.18) and (5.19) that $g\in L^1_{m\otimes\mathcal{L}^2}(\Omega)$, and then from (6.50), (6.51) and the Dominated Convergence Theorem, that

$$\lim_{k\to\infty}\int_\Omega\left|\boldsymbol{e}_{x'}(\boldsymbol{\varphi}^+)\right|^2(t^k_{j_{x_1}-1},x')\,dm\otimes\mathcal{L}^2 = \int_\Omega\left|\boldsymbol{e}_{x'}(\boldsymbol{\varphi}^-)\right|^2\,dm\otimes\mathcal{L}^2. \tag{6.52}$$

By (3.4) and (3.23) we have $|\boldsymbol{E}\boldsymbol{\varphi}|(\Sigma_{x_1})=0$ for m-a.e. $x_1\in(0,L)$, therefore Assertion (4.16) implies that $\boldsymbol{e}_{x'}(\boldsymbol{\varphi}^-)=\boldsymbol{e}_{x'}(\boldsymbol{\varphi}^\star)$ $m\otimes\mathcal{L}^2$-a.e.. Collecting (6.48), (6.49), (6.50), (6.52), and the last equation, the assertion (6.27) is proved.

6.3. Proof of Corollary 3.2. Choosing $\boldsymbol{\varphi} \in \mathcal{D}(\Omega \setminus \Sigma)$ in (6.14) (see (3.15)), taking (3.12) into account, we get $\int_{\Omega\setminus\Sigma} \boldsymbol{\sigma}(\boldsymbol{u}) : \boldsymbol{e}(\boldsymbol{\varphi})dx = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{\varphi} dx$ and infer, by the arbitrary choice of $\boldsymbol{\varphi}$, that $-\mathbf{div}\boldsymbol{a}\boldsymbol{e}(\boldsymbol{u}) = \boldsymbol{f}$ in $\Omega \setminus \Sigma$. Choosing $\boldsymbol{\varphi} \in BD_0^{\nu,m}(\Omega)$ such that $\boldsymbol{\varphi} \in C^\infty(U)$ for every connected component U of $\Omega \setminus \Sigma_\nu$, and integrating $\boldsymbol{a}\boldsymbol{e}(\boldsymbol{u}) : \boldsymbol{e}(\boldsymbol{\varphi})$ by parts over each connected component of $\Omega \setminus \Sigma$, taking the first line of (3.14) into account, we deduce

$$\sum_{t\in\mathcal{A}_m} \int_{\Sigma_t} \left((\boldsymbol{a}\boldsymbol{e}(\boldsymbol{u})\boldsymbol{e}_1)^- - (\boldsymbol{a}\boldsymbol{e}(\boldsymbol{u})\boldsymbol{e}_1)^+\right) \cdot \boldsymbol{\varphi} + m(\{t\})\boldsymbol{a}^{\|}\boldsymbol{e}_{x'}(\boldsymbol{u}^\star) : \boldsymbol{e}_{x'}(\boldsymbol{\varphi}^\star) d\mathcal{H}^2$$
$$+\sum_{t\in\mathcal{A}_\nu} \int_{\Sigma_t} (\boldsymbol{a}\boldsymbol{e}(\boldsymbol{u})\boldsymbol{e}_1)^- \cdot \boldsymbol{\varphi}^- - (\boldsymbol{a}\boldsymbol{e}(\boldsymbol{u})\boldsymbol{e}_1)^+ \cdot \boldsymbol{\varphi}^+ + (\boldsymbol{u}^+ - \boldsymbol{u}^-)\cdot \nu(\{t\})^{-1}\boldsymbol{A}(\boldsymbol{\varphi}^+ - \boldsymbol{\varphi}^-) d\mathcal{H}^2 = 0,$$

and obtain the transmission conditions stated in the second and third lines of (3.14). Conversely, any solution to (3.14) satisfies (3.6).

6.4. Sketch proof of Proposition 3.10 . Repeating the argument of the proof of Proposition 4.2, we establish the apriori estimates

$$\sup_{\varepsilon>0} \int_\Omega |\boldsymbol{u}_\varepsilon|^2 dm_\varepsilon \otimes \mathcal{L}^2 + \int_\Omega |\boldsymbol{u}_\varepsilon| dx + \int_\Omega \mu_\varepsilon \left|\boldsymbol{\nabla} \boldsymbol{u}_\varepsilon\right|^2 dx < \infty,$$

and deduce, up to a subsequence, the following convergences (analogous to (4.5))

$$\text{(6.53)} \quad \begin{aligned} &\boldsymbol{u}_\varepsilon \overset{\star}{\rightharpoonup} \boldsymbol{u} \quad \text{weakly* in } BV(\Omega;\mathbb{R}^n) \text{ for some } \boldsymbol{u} \in BV_0^{\nu,m}(\Omega), \\ &\mu_\varepsilon(\boldsymbol{C}\boldsymbol{\nabla}\boldsymbol{u}_\varepsilon)\boldsymbol{e}_1 \xrightharpoonup{\nu_\varepsilon\otimes\mathcal{L}^2, \nu\otimes\mathcal{L}^2} (\boldsymbol{C}\frac{\boldsymbol{D}\boldsymbol{u}}{\nu\otimes\mathcal{L}^{d-1}})\boldsymbol{e}_1, \quad \boldsymbol{\nabla}_{x'}\boldsymbol{u}_\varepsilon \xrightharpoonup{m_\varepsilon\otimes\mathcal{L}^2, m\otimes\mathcal{L}^2} \boldsymbol{\nabla}_{x'}\boldsymbol{u}^\star, \end{aligned}$$

where $BV_0^{\nu,m}(\Omega)$ and $\boldsymbol{\nabla}_{x'}\boldsymbol{v}$ are defined by (3.42) and (3.43). Fixing $\boldsymbol{v} \in BV_0^{\nu,m}(\Omega)$, $\delta > 0$, $k \in \mathbb{N}^*$, we set $\boldsymbol{\varphi} = \boldsymbol{v}^\delta$ and

$$\boldsymbol{\varphi}_\varepsilon^k(x) := \left(\int_{I^k_{j_{x_1}}} (\boldsymbol{T}^{-1}\boldsymbol{C}\,\frac{\boldsymbol{D}\boldsymbol{\varphi}}{\nu\otimes\mathcal{L}^{d-1}})\boldsymbol{e}_1(s_1,x')\ d\nu(s_1)\right)\phi_\varepsilon^k(x_1)$$
$$- \int_{t^k_{j_{x_1}-1}}^{x_1} (\boldsymbol{T}^{-1}\boldsymbol{C}\,\boldsymbol{\nabla}_{x'}\boldsymbol{\varphi})\,\boldsymbol{e}_1(s_1,x')ds_1 + \boldsymbol{\varphi}^+(t^k_{j_{x_1}-1},x').$$

Mimicking propositions 6.1 and 6.3, we exhibit a sequence $\boldsymbol{\varphi}_\varepsilon (= \boldsymbol{\varphi}_\varepsilon^{k_\varepsilon})$ satisfying

$$\text{(6.54)} \quad \begin{aligned} &\lim_{\varepsilon\to0} \int_\Omega |\boldsymbol{\varphi}_\varepsilon - \boldsymbol{\varphi}|\ dx = 0, \\ &\mu_\varepsilon(\boldsymbol{C}\,\boldsymbol{\nabla}\boldsymbol{\varphi}_\varepsilon)\boldsymbol{e}_1 \xrightarrow{\nu_\varepsilon\otimes\mathcal{L}^2, \nu\otimes\mathcal{L}^2} (\boldsymbol{C}\frac{\boldsymbol{D}\boldsymbol{\varphi}}{\nu\otimes\mathcal{L}^{d-1}})\boldsymbol{e}_1, \qquad \boldsymbol{\nabla}_{x'}\boldsymbol{\varphi}_\varepsilon \xrightarrow{m_\varepsilon\otimes\mathcal{L}^2, m\otimes\mathcal{L}^2} \boldsymbol{\nabla}_{x'}\boldsymbol{\varphi}^\star. \end{aligned}$$

Multiplying (3.37) by $\boldsymbol{\varphi}_\varepsilon$, integrating by parts, and applying the formula
(6.55)

$$\boldsymbol{C}\boldsymbol{\nabla}\boldsymbol{u}_\varepsilon : \boldsymbol{\nabla}\boldsymbol{\varphi}_\varepsilon = (\boldsymbol{T}^{-1}\boldsymbol{C}\,\boldsymbol{\nabla}\boldsymbol{u}_\varepsilon)\boldsymbol{e}_1 \cdot (\boldsymbol{C}\boldsymbol{\nabla}\boldsymbol{\varphi}_\varepsilon)\boldsymbol{e}_1 - (\boldsymbol{T}^{-1}\boldsymbol{C}\,\boldsymbol{\nabla}_{x'}\boldsymbol{u}_\varepsilon)\,\boldsymbol{e}_1 \cdot (\boldsymbol{C}\boldsymbol{\nabla}_{x'}\boldsymbol{\varphi}_\varepsilon)\,\boldsymbol{e}_1$$
$$+ \boldsymbol{C}\,\boldsymbol{\nabla}_{x'}\boldsymbol{u}_\varepsilon : \boldsymbol{\nabla}_{x'}\boldsymbol{\varphi}_\varepsilon,$$

proved below, we obtain

$$
\begin{aligned}
\int_\Omega \boldsymbol{f}\cdot\boldsymbol{\varphi}_\varepsilon dx &= \int_\Omega \mu_\varepsilon(\boldsymbol{T}^{-1}\boldsymbol{C}\,\nabla\boldsymbol{u}_\varepsilon)\boldsymbol{e}_1\cdot\mu_\varepsilon(\boldsymbol{C}\,\nabla\boldsymbol{\varphi}_\varepsilon)\boldsymbol{e}_1\; d\nu_\varepsilon\otimes\mathcal{L}^2\\
&\quad+\int_\Omega -(\boldsymbol{T}^{-1}\boldsymbol{C}\,\nabla_{x'}\boldsymbol{u}_\varepsilon)\boldsymbol{e}_1\cdot(\boldsymbol{C}\nabla_{x'}\boldsymbol{\varphi}_\varepsilon)\boldsymbol{e}_1+\boldsymbol{C}\,\nabla_{x'}\boldsymbol{u}_\varepsilon\cdot\nabla_{x'}\boldsymbol{\varphi}_\varepsilon\; dm_\varepsilon\otimes\mathcal{L}^2.
\end{aligned}
$$

Passing to the limit as $\varepsilon\to 0$ in accordance with (6.53) and (6.54), we find

$$
a(\boldsymbol{u},\boldsymbol{\varphi})=\int_\Omega \boldsymbol{u}\cdot\boldsymbol{\varphi}\; dx,
$$

where

$$
\begin{aligned}
a(\boldsymbol{u},\boldsymbol{\varphi}) &:= \int_\Omega (\boldsymbol{T}^{-1}\boldsymbol{C}\,\tfrac{\boldsymbol{D}\boldsymbol{u}}{\nu\otimes\mathcal{L}^{d-1}})\boldsymbol{e}_1\cdot(\boldsymbol{C}\,\tfrac{\boldsymbol{D}\boldsymbol{\varphi}}{\nu\otimes\mathcal{L}^{d-1}})\boldsymbol{e}_1\; d\nu\otimes\mathcal{L}^{d-1}\\
&\qquad-\int_\Omega (\boldsymbol{T}^{-1}\boldsymbol{C}\,\nabla_{x'}\boldsymbol{u}^\star)\boldsymbol{e}_1\cdot(\boldsymbol{C}\nabla_{x'}\boldsymbol{\varphi})^\star)\boldsymbol{e}_1+\boldsymbol{C}\,\nabla_{x'}\boldsymbol{u}^\star:\nabla_{x'}\boldsymbol{\varphi})^\star\; dm\otimes\mathcal{L}^{d-1}.
\end{aligned}
$$

An elementary computation shows that $a(\cdot,\cdot)$ is also given by (3.44). The rest of the proof is similar to that of Theorem 3.1.
Proof of (6.55). Noticing that $\boldsymbol{T}$ defined by (3.39) satisfies

$$
(\boldsymbol{T}\nabla\boldsymbol{v})\boldsymbol{e}_1=(\boldsymbol{C}\nabla\boldsymbol{v})\,\boldsymbol{e}_1-(\boldsymbol{C}\nabla_{x'}\boldsymbol{v})\,\boldsymbol{e}_1,
$$

and taking the invertibility of $\boldsymbol{T}$ and the symmetry of $\boldsymbol{T}^{-1}$ and $\boldsymbol{C}$ into account, we obtain

$$
\begin{aligned}
\boldsymbol{C}\nabla\boldsymbol{u}:\nabla\boldsymbol{v} &= (\boldsymbol{C}\nabla\boldsymbol{u})\boldsymbol{e}_1\cdot(\nabla\boldsymbol{v})\boldsymbol{e}_1+\boldsymbol{C}\nabla\boldsymbol{u}:\nabla_{x'}\boldsymbol{v}=(\boldsymbol{C}\nabla\boldsymbol{u})\boldsymbol{e}_1\cdot(\nabla\boldsymbol{v})\boldsymbol{e}_1+\nabla\boldsymbol{u}:\boldsymbol{C}\nabla_{x'}\boldsymbol{v}\\
&=(\boldsymbol{C}\nabla\boldsymbol{u})\boldsymbol{e}_1\cdot(\nabla\boldsymbol{v})\boldsymbol{e}_1+(\nabla\boldsymbol{u})\boldsymbol{e}_1\cdot(\boldsymbol{C}\nabla_{x'}\boldsymbol{v})\boldsymbol{e}_1+\nabla_{x'}\boldsymbol{u}:\boldsymbol{C}\nabla_{x'}\boldsymbol{v}\\
&=(\boldsymbol{C}\nabla\boldsymbol{u})\boldsymbol{e}_1\cdot\boldsymbol{T}^{-1}((\boldsymbol{C}\nabla\boldsymbol{v})\boldsymbol{e}_1-(\boldsymbol{C}\nabla_{x'}\boldsymbol{v})\boldsymbol{e}_1)\\
&\qquad\qquad+\boldsymbol{T}^{-1}((\boldsymbol{C}\nabla\boldsymbol{u})\boldsymbol{e}_1-(\boldsymbol{C}\nabla_{x'}\boldsymbol{u})\boldsymbol{e}_1)\cdot(\boldsymbol{C}\nabla_{x'}\boldsymbol{v})\boldsymbol{e}_1+\nabla_{x'}\boldsymbol{u}:\boldsymbol{C}\nabla_{x'}\boldsymbol{v}\\
&=(\boldsymbol{T}^{-1}\boldsymbol{C}\nabla\boldsymbol{u})\boldsymbol{e}_1\cdot(\boldsymbol{C}\nabla\boldsymbol{v})\boldsymbol{e}_1-(\boldsymbol{T}^{-1}\boldsymbol{C}\nabla_{x'}\boldsymbol{u})\boldsymbol{e}_1\cdot(\boldsymbol{C}\nabla_{x'}\boldsymbol{v})\boldsymbol{e}_1+\nabla_{x'}\boldsymbol{u}:\boldsymbol{C}\nabla_{x'}\boldsymbol{v}.
\end{aligned}
$$

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