Homogenization of Stratified Elastic Composites with High Contrast

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HOMOGENIZATION OF STRATIFIED ELASTIC COMPOSITES WITH HIGH CONTRAST

MICHEL BELLIEUD *

Abstract. We determine the asymptotic behavior of the solutions to the linear elastodynamic equations in a stratified media comprising an alternation of possibly very stiff layers with much softer ones, when the thickness of the layers tends to zero. The limit equations may depend on higher order terms, characterizing bending effects. A part of this work is set in the context of non-periodic homogenization and an extension to stochastic homogenization is presented.

Key words. homogenization, elasticity, non-local effects

AMS subject classifications. 35B27, 35B40, 35R60, 74B05, 74Q10

1. Introduction. In this paper we analyze the asymptotic behavior of the solution to the linear elastodynamic equations in a composite material wherein, at a microscopic scale, possibly very "stiff" layers alternate with a much "softer" medium. Stratified composite media have been intensively investigated over the last decades, especially in the context of diffusion equations [18, 27, 29, 30, 31, 32, 39, 52, 54]. As regards linear elasticity, layered elastic composites have been studied in [26, 28, 33, 38] under assumptions of uniform boundedness and uniform definite positiveness of the elasticity tensor guaranteeing that the effective equation is a standart linear elasticity equation. When these assumptions break down, as for instance in the so-called "high contrast case", the limit equilibrium equation may be of a quite different type: it may correspond, in theory, to the Euler equation associated to the minimization of any lower semi-continuous quadratic form on L^2 vanishing on rigid motions [20]. In particular, it may be non-local and depend on higher order derivatives of the displacement. Elastic media with high contrast have been studied under various geometrical assumptions. Composites with stiff grain-like inclusions have been investigated in [7, 8, 45], stiff fibered structures in [8, 12, 13, 46, 50], and stiff media with holes filled with a soft material in [22, 24, 47]. Our aim is to complement this body of work in the context of stratified media. Our approach is based on the two-scale convergence method [3, 5, 19, 23, 40, 41], which yields the convergence to an effective solution. It also yields a first order corrector result in L^2 (see Remark 3.14), but not the rigorous error estimates of higher order with respect to small parameters provided by the asymptotic expansions method [1, 2, 6, 15, 16, 21, 43, 44, 45, 48, 49].

For a given bounded smooth open subset Ω of $\mathbb{R}^3$, we consider a linear elastodynamic problem like (3.5). We assume that the Lamé coefficients take possibly large values in a subset B_ε of Ω and much smaller values elsewhere. The set B_ε consists of a non-periodic distribution of parallel disjoint homothetic layers of thickness r_ε, whose median planes are orthogonal to e_3 and separated by a minimal distance ε, where $\varepsilon, r_\varepsilon$ are positive reals converging to zero (see fig. 3.1). The effective volume fraction of the stiff phase B_ε is characterized by the parameter ϑ defined by (3.10). Both cases $\vartheta = 0$ and $0 < \vartheta < 1$ are investigated. The order of magnitude of the Lamé coefficients in the stiff phase is determined by the parameters k and κ defined by (3.8).

When the elasticity coefficients in the soft phase are of order 1 and the effective

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volume fraction of the stiff phase vanishes the limit behavior of the composite is governed, if $0 < k < +\infty$, by the equation

$$(1.1) \qquad (\rho + n\overline{\rho}_1)\frac{\partial^2 \boldsymbol{u}}{\partial t^2} - \mathbf{div}\boldsymbol{\sigma}(\boldsymbol{u}) - nk\mathbf{div}\boldsymbol{\sigma}_{x'}(\boldsymbol{u}') = (\rho + n\overline{\rho}_1)\boldsymbol{f} \qquad \text{in } \Omega\times(0,T),$$

where ρ denotes the mass density in the softer phase, and $\boldsymbol{u}'$, $\boldsymbol{\sigma}_{x'}$, $\boldsymbol{\sigma}$ and $\overline{\rho}_1$ are defined, respectively, by (2.1), (3.13), and (3.7). The function n characterizes the rescalled effective number of sections of stiff layers per unit length in the $\boldsymbol{e}_3$ direction and is obtained as the weak* limit in $L^\infty(\Omega)$ of the sequence (n_ε) defined by (3.14). When the order of magnitude of the elasticity coefficients in the stiff layers is larger, that is when $k = +\infty$, the functions u_1 and u_2 vanish on the set $\{n > 0\}$ and the behavior of u_3 is governed by the equation (3.19), (3.20) or (3.21), depending on the order of magnitude of κ. In the case $0 < \kappa < +\infty$, this equation involves the 4th partial derivatives of u_3 with respect to x_1, x_2:

$$(1.2) \qquad (\rho + n\overline{\rho}_1)\frac{\partial^2 u_3}{\partial t^2} - (\mathbf{div}\boldsymbol{\sigma}(\boldsymbol{u}))_3 + n\frac{\kappa}{3}\frac{l+1}{l+2}\sum_{\alpha,\beta=1}^{2}\frac{\partial^4 u_3}{\partial x_\alpha^2 \partial x_\beta^2}$$
$$= (\rho + n\overline{\rho}_1)f_3 \quad \text{in } \Omega\times(0,T),$$

revealing bending effects. The effective behavior on the set $\{n = 0\}$ is that of a homogeneous material without stiff layers. In Theorem 3.5, we extend these results to the stochastic case. The set $B_\varepsilon(\omega)$ then depends on a random element ω of some sample space $\mathfrak{O} \subset 2^{\mathbb{R}}$ equiped with a probability P satisfying (3.23). The limit problem as $\varepsilon \to 0$ is deduced from the above equations, P-almost surely, by substituting for n the conditional expectation $E_P^{\mathcal{F}}n_0(\omega)$ with respect to P given the σ-algebra $\mathcal{F}$ of the periodic sets, of the random variable n_0 defined by (3.24).

If the order of magnitude of the elasticity coefficients in the soft interlayers is strictly smaller than 1 and strictly larger than ε^2, the effective equations are deduced from (1.1), (1.2), formally, by removing the term $\mathbf{div}\boldsymbol{\sigma}(\boldsymbol{u})$ (see Theorem 3.7).

When the elastic moduli in the soft phase are of order ε^2, the effective behavior of the composite turns sensitive to the slightest geometrical perturbation (see Remark 3.20). The effective equation can not be expressed simply in terms of the function n as in the other cases. This characteristic renders the study of non-periodic homogenization a very difficult task: we only treat the case of an ε-periodic distribution of stiff layers. The homogenized problem then takes the form of a system of equations coupling some field $\boldsymbol{v}$, characterizing the effective displacement in the stiff layers, with the two-scale limit $\boldsymbol{u}_0 : \Omega\times(0,T) \times (-\frac{1}{2}, \frac{1}{2})^3 \to \mathbb{R}^3$ of the solution $(\boldsymbol{u}_\varepsilon)$ to (3.5) (see [3, 41]). This field $\boldsymbol{v}$ is obtained as the limit of the sequence $(\boldsymbol{u}_\varepsilon m_\varepsilon)$, where m_ε is the measure supported by the stiff layers defined by (3.39). If $0 < k < +\infty$, the effective behavior of the displacement in the stiff medium is governed by the equation

$$(1.3) \qquad \overline{\rho}_1\frac{\partial^2 \boldsymbol{v}}{\partial t^2} - k\mathbf{div}\boldsymbol{\sigma}_{x'}(\boldsymbol{v}') = \overline{\rho}_1\boldsymbol{f} + \boldsymbol{g}(\boldsymbol{u}_0) \qquad \text{in } \Omega \times (0,T),$$

associated with the boundary and initial conditions given in (3.45). This equation displays stretching vibrations with regard to the transversal components v_1, v_2 of $\boldsymbol{v}$. It is coupled with the soft phase through the field $\boldsymbol{g}(\boldsymbol{u}_0)$ which represents the sum of

the surface forces applied on each stiff layer by the adjacent soft medium. This field is defined by (3.42), in terms of the restriction of $\boldsymbol{u}_0$ to $\Omega \times (0,T) \times Y \setminus A$, which characterizes the effective displacement in the soft interstitial layers. The letters Y and A denote, respectively, the unit cell and the rescaled stiff layer (see (3.36), (3.37)). The effective displacement in the soft phase is governed by the equation

$$\rho \frac{\partial^2 \boldsymbol{u}_0}{\partial t^2} - \mathbf{div}_y(\boldsymbol{\sigma}_{0y}(\boldsymbol{u}_0)) = \rho \boldsymbol{f} \qquad \text{in } \ \Omega \times (0,T) \times Y \setminus A,$$

where $\boldsymbol{\sigma}_{0y}$ is defined by (3.42). This equation is coupled with the variable $\boldsymbol{v}$ by the relation (3.41) on $\Omega \times (0,T) \times A$. The weak limit of $(\boldsymbol{u}_\varepsilon)$ in $L^2(\Omega \times (0,T))$ is given by $\boldsymbol{u}(x,t) = \int_Y \boldsymbol{u}_0(x,t,y)dy$.

When the order of magnitude of the elasticity coefficients in the stiff layers is larger, the functions v_1 and v_2 vanish and the effective displacement in the stiff phase is governed by the equation of v_3 given by (3.46), (3.47) or (3.48), depending on the order of magnitude of κ. In the case $0 < \kappa < +\infty$, this equation,

$$\overline{\rho}_1 \frac{\partial^2 v_3}{\partial t^2} + \frac{\kappa}{3} \frac{l+1}{l+2} \sum_{\alpha,\beta=1}^{2} \frac{\partial^4 v_3}{\partial x_\alpha^2 \partial x_\beta^2} = \overline{\rho}_1 f_3 + (\boldsymbol{g}(\boldsymbol{u}_0))_3 \quad \text{in } \Omega \times (0,T),$$

involves the 4th partial derivatives of v_3 with respect to x_1, x_2, characterizing bending vibrations. Otherwise, the stiff layers display the behavior of a collection of unstretchable membranes if $(k,\kappa) = (+\infty, 0)$ and that of fixed bodies if $\kappa = \infty$.

Our results apply as well to equilibrium equations (see remarks 3.11, 3.19) and can be extended to multiphase composites (see remarks 3.10, 3.16, 3.17). Multiphase homogenized models have been studied in [8, 44, 45, 47, 48, 49].

As occurs in the case of fibers or grain-like inclusions embedded in a soft matrix (see [8, p.3]), the two-phase stratified composites modeled above by homogenization prove to be, in general, unsufficiently reinforced to "resist" to body forces. Accordingly, some of the components of the effective displacement may exhibit a quadratic behavior with respect to time, describing a motion of collapse. The whole displacement $\boldsymbol{u}$ does so, for instance, on the set $\widehat{\{n = 0\}}$ in (3.28)-(3.30). If $\kappa = 0$, a similar behavior is shown by u_3 in (3.28), (3.29), and by v_3 in (3.45), (3.46), if $\boldsymbol{f}$ is not parallel to the layers. In these cases, the sequence of the solutions to the associated equilibrium problems (see (3.32)) may fail to be bounded in L^2, and the effective equilibrium problem to be well-posed. Boundedness in L^2 is achieved by considering suitable multiphase media. n the fibered case, the choice of such media has been discussed in [8, Proposition 5.2]. We extend this discussion in remarks 3.11, 3.19, 6.3.

The paper is organised as follows: in Section 2 we specify the notations and in Section 3 we state our main results. In Section 4, we recall some classical results and introduce a non-periodic variant of the two-scale convergence for which we establish a compactness result. The effective equations are derived in Section 6 by employing apriori estimates demonstrated in Section 5, and a technical lemma proved in the appendix.

2. Notations. In this article, $\{\boldsymbol{e}_1, .., \boldsymbol{e}_N\}$ stands for the canonical basis of $\mathbb{R}^N$. Points in $\mathbb{R}^N$ or in $\mathbb{Z}^N$ and real-valued functions are represented by symbols beginning by a lightface lowercase (example $x, i, \det \boldsymbol{A}...$) and vectors and vector-valued functions by symbols beginning by a boldface lowercase (examples: $\boldsymbol{x}, \boldsymbol{i}, \boldsymbol{u}, \boldsymbol{f}, \boldsymbol{g}, \mathbf{div}\boldsymbol{\sigma}_\varepsilon,...$).

Matrices and matrix-valued functions are represented by symbols beginning by a boldface uppercase with the following exceptions: $\boldsymbol{\nabla u}$ (displacement gradient), $\boldsymbol{e}(\boldsymbol{u})$ (linearized strain tensor). We denote by u_i or $(\boldsymbol{u})_i$ the components of a vector $\boldsymbol{u}$ and by A_{ij} or $(\boldsymbol{A})_{ij}$ those of a matrix $\boldsymbol{A}$ (that is $\boldsymbol{u} = \sum_{i=1}^N u_i \boldsymbol{e}_i = \sum_{i=1}^N (\boldsymbol{u})_i \boldsymbol{e}_i$; $\boldsymbol{A} = \sum_{i,j=1}^N A_{ij} \boldsymbol{e}_i \otimes \boldsymbol{e}_j = \sum_{i,j=1}^N (\boldsymbol{A})_{ij} \boldsymbol{e}_i \otimes \boldsymbol{e}_j$). We do not employ the usual repeated index convention for summation. We denote by $\boldsymbol{A}:\boldsymbol{B} = \sum_{i,j=1}^N A_{ij} B_{ij}$ the inner product of two matrices, by ε_{ijk} the three-dimensional alternator, by $\boldsymbol{u} \wedge \boldsymbol{v} = \sum_{i,j,k=1}^3 \varepsilon_{ijk} u_j v_k \boldsymbol{e}_i$ the exterior product in $\mathbb{R}^3$, by $\mathbb{S}^M$ ($M \in \mathbb{N}$) the set of all real symmetric matrices of order M, by $\boldsymbol{I}_M$ the $M \times M$ identity matrix. The symbol $\sharp D$ denotes the cardinality of a finite set D. The letter C stands for different constants independent of ε whose precise values may vary. For any weakly differentiable vector field $\boldsymbol{\psi} : \Omega \subset \mathbb{R}^3 \to \mathbb{R}^3$, we set

$$\boldsymbol{\psi}' := \psi_1 \boldsymbol{e}_1 + \psi_2 \boldsymbol{e}_2; \quad \boldsymbol{e}_{x'}(\boldsymbol{\psi}) := \sum_{\alpha,\beta=1}^2 \frac{1}{2}\left(\frac{\partial \psi_\alpha}{\partial x_\beta} + \frac{\partial \psi_\beta}{\partial x_\alpha}\right) \boldsymbol{e}_\alpha \otimes \boldsymbol{e}_\beta \ (= \boldsymbol{e}_{x'}(\boldsymbol{\psi}'));$$

(2.1)

$$\boldsymbol{\sigma}_{x'}(\boldsymbol{\psi}) := \begin{pmatrix} 2\frac{\partial \psi_1}{\partial x_1} + \frac{2l}{l+2}\left(\frac{\partial \psi_1}{\partial x_1} + \frac{\partial \psi_2}{\partial x_2}\right) & \frac{\partial \psi_1}{\partial x_2} + \frac{\partial \psi_2}{\partial x_1} & 0 \\ \frac{\partial \psi_1}{\partial x_2} + \frac{\partial \psi_2}{\partial x_1} & 2\frac{\partial \psi_2}{\partial x_2} + \frac{2l}{l+2}\left(\frac{\partial \psi_1}{\partial x_1} + \frac{\partial \psi_2}{\partial x_2}\right) & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where the non-negative parameter l is defined in (3.6). We reproduce and modify here some notations from [8]: we denote by $C_\sharp^\infty(Y)$ (resp. $C_\sharp(Y)$) the set of Y-periodic functions from $C^\infty(\mathbb{R}^3)$ (resp. $C(\mathbb{R}^3)$), by $C_\sharp^\infty(Y \setminus B)$ the set of the restrictions of the elements of $C_\sharp^\infty(Y)$ to $Y \setminus B$, by $H_\sharp^1(Y)$ (resp. $H_\sharp^1(Y \setminus B)$) the completion of $C_\sharp^\infty(Y)$ (resp. $C_\sharp^\infty(Y \setminus B)$) with respect to the norm $w \to \left(\int_Y (|w|^2 + |\boldsymbol{\nabla} w|^2) dy\right)^{\frac{1}{2}}$ (resp. $w \to (\int_{Y\setminus B}(|w|^2 + |\boldsymbol{\nabla} w|^2)dy)^{\frac{1}{2}}$). For any subset Q of the unit cell Y, the symbol $Q_\sharp$ stands for the periodization on all $\mathbb{R}^3$ of Q, that is

$$(2.2) \qquad Q_\sharp := \bigcup_{z \in \mathbb{Z}^3} z + Q.$$

3. Setting of the problem and results. We consider a cylindrical domain $\Omega := \Omega' \times (0, L)$, where Ω' is a bounded smooth domain of $\mathbb{R}^2$. Given a small positive real ε, the non-periodic distribution B_ε of disjoint homothetical stiff layers B_ε^j will be described in terms of a finite subset ω_ε of $\mathbb{R}$

$$(3.1) \qquad \omega_\varepsilon := \left\{\omega_\varepsilon^j, j \in J_\varepsilon\right\}, \quad J_\varepsilon := \{1, .., \sharp\omega_\varepsilon\},$$

satisfying

$$(3.2) \qquad \omega_\varepsilon \subset (0, L), \quad \min_{j,j' \in J_\varepsilon, j \neq j'} |\omega_\varepsilon^j - \omega_\varepsilon^{j'}| = \varepsilon, \quad \mathrm{dist}(\omega_\varepsilon, \{0, L\}) > \frac{\varepsilon}{2},$$

and of a small parameters r_ε verifying

$$(3.3) \qquad \varepsilon > r_\varepsilon(1 + \delta) \qquad \text{for some } \delta > 0,$$

by setting (see Fig. 3.1)

$$(3.4) \qquad B_\varepsilon := \bigcup_{j \in J_\varepsilon} B_\varepsilon^j; \quad B_\varepsilon^j := \Omega' \times \left(\omega_\varepsilon^j + r_\varepsilon I \right); \quad I := \left(-\frac{1}{2}, \frac{1}{2} \right).$$
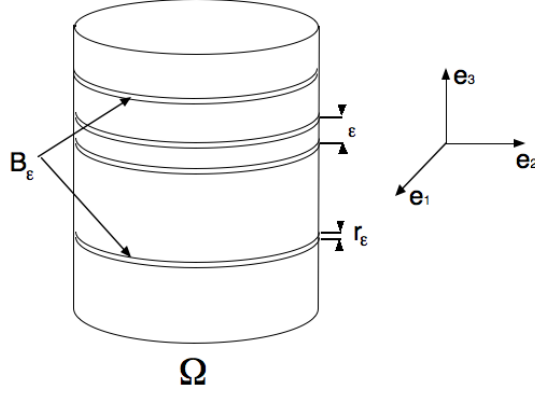

FIG. 3.1.

As in [8], we consider the vibration problem

$$(3.5) \quad (\mathcal{P}_\varepsilon) : \begin{cases} \rho_\varepsilon \dfrac{\partial^2 \boldsymbol{u}_\varepsilon}{\partial t^2} - \mathbf{div}(\boldsymbol{\sigma}_\varepsilon(\boldsymbol{u}_\varepsilon)) = \rho_\varepsilon \boldsymbol{f} \qquad \text{in } \Omega \times (0, T), \\[2mm] \boldsymbol{\sigma}_\varepsilon(\boldsymbol{u}_\varepsilon) = \lambda_\varepsilon \operatorname{tr}(\boldsymbol{e}(\boldsymbol{u}_\varepsilon)) \boldsymbol{I} + 2\mu_\varepsilon \boldsymbol{e}(\boldsymbol{u}_\varepsilon), \quad \boldsymbol{e}(\boldsymbol{u}_\varepsilon) = \dfrac{1}{2} (\boldsymbol{\nabla} \boldsymbol{u}_\varepsilon + \boldsymbol{\nabla}^T \boldsymbol{u}_\varepsilon), \\[2mm] \boldsymbol{u}_\varepsilon \in C([0, T]; H_0^1(\Omega, \mathbb{R}^3)) \cap C^1([0, T]; L^2(\Omega, \mathbb{R}^3)), \\[2mm] \boldsymbol{u}_\varepsilon(0) = \boldsymbol{a}_0, \quad \dfrac{\partial \boldsymbol{u}_\varepsilon}{\partial t}(0) = \boldsymbol{b}_0, \quad \boldsymbol{f} \in C(\overline{\Omega \times (0, T)}; \mathbb{R}^3), \\[2mm] (\boldsymbol{a}_0, \boldsymbol{b}_0) \in \left(H_0^1(\Omega, \mathbb{R}^3) \times L^2(\Omega, \mathbb{R}^3) \right) \cap C(\overline{\Omega}, \mathbb{R}^3)^2. \end{cases}$$

The Lamé coefficients μ_ε, λ_ε and the mass density ρ_ε are assumed to take constant values of possibly different orders of magnitude in the set of layers B_ε and in the set of interlayers $\Omega \setminus B_\varepsilon$. More precisely, we suppose that

$$(3.6) \quad \begin{aligned} &\mu_\varepsilon(x) = \mu_{1\varepsilon} \mathbb{1}_{B_\varepsilon}(x) + \mu_{0\varepsilon} \mathbb{1}_{\Omega \setminus B_\varepsilon}(x), \quad \lambda_\varepsilon(x) = \lambda_{1\varepsilon} \mathbb{1}_{B_\varepsilon}(x) + \lambda_{0\varepsilon} \mathbb{1}_{\Omega \setminus B_\varepsilon}(x), \\[2mm] &\mu_{1\varepsilon} \geq c > 0, \quad l_\varepsilon := \frac{\lambda_{1\varepsilon}}{\mu_{1\varepsilon}}, \quad \lim_{\varepsilon \to 0} l_\varepsilon = l \in [0, +\infty), \quad 0 \leq \lambda_{0\varepsilon} < C\mu_{0\varepsilon} \ll \mu_{\varepsilon 1}, \end{aligned}$$

and

$$(3.7) \qquad \rho_\varepsilon(x) = \rho \mathbb{1}_{\Omega \setminus B_\varepsilon} + \frac{\varepsilon}{r_\varepsilon} \overline{\rho}_1 \mathbb{1}_{B_\varepsilon}, \qquad \rho, \overline{\rho}_1 \in (0, +\infty).$$

We assume and set

$$(3.8) \qquad k := \lim_{\varepsilon \to 0} \frac{r_\varepsilon}{\varepsilon} \mu_{1\varepsilon} \in (0, +\infty], \qquad\qquad \kappa := \lim_{\varepsilon \to 0} \frac{r_\varepsilon^3}{\varepsilon} \mu_{1\varepsilon} \in [0, +\infty].$$

The weak* relative compactness in $L^\infty(0, T; L^2(\Omega; \mathbb{R}^3))$ of the sequence of the solutions to (3.5) is ensured by the following hypothesis:

$$(3.9) \qquad \sup_{\varepsilon > 0} \int_\Omega \Big(\rho_\varepsilon |\boldsymbol{b}_0|^2 + \boldsymbol{\sigma}_\varepsilon(\boldsymbol{a}_0) : \boldsymbol{e}(\boldsymbol{a}_0) \Big) dx + \int_{\Omega \times (0, T)} \rho_\varepsilon |\boldsymbol{f}|^2 dx dt < +\infty.$$

3.1. Case of interlayers with Lamé coefficients of order 1 . We consider the case of extremely thin layers of extremely large stiffness alternating with interlayers of elastic moduli of order 1. The effective volume fraction of the stiff layers is characterized by:

$$(3.10) \qquad\qquad\qquad\qquad \vartheta := \lim_{\varepsilon \to 0} \frac{r_\varepsilon}{\varepsilon}.$$

We assume in this subsection that

$$(3.11) \qquad\qquad\qquad\qquad \vartheta = 0,$$

$$(3.12) \qquad\qquad\qquad\qquad \mu_{0\varepsilon} = \mu > 0; \qquad \lambda_{0\varepsilon} = \lambda \geq 0.$$

We introduce the operator $\boldsymbol{\sigma} \colon H^1(\Omega; \mathbb{R}^3) \to L^2(\Omega; \mathbb{S}^3)$ and $n_\varepsilon \in L^\infty(\Omega)$ defined by

$$(3.13) \qquad \boldsymbol{\sigma}(\boldsymbol{\varphi}) := \lambda \operatorname{tr}(\boldsymbol{e}(\boldsymbol{\varphi})) \boldsymbol{I} + 2\mu \boldsymbol{e}(\boldsymbol{\varphi}) \quad \forall \boldsymbol{\varphi} \in H^1(\Omega; \mathbb{R}^3),$$

$$(3.14) \qquad n_\varepsilon(x) := \sum_{i \in Z_\varepsilon} \sharp \left(\omega_\varepsilon \cap \left(\varepsilon i - \frac{\varepsilon}{2}, \varepsilon i + \frac{\varepsilon}{2} \right] \right) \mathbb{1}_{\left(\varepsilon i - \frac{\varepsilon}{2}, \varepsilon i + \frac{\varepsilon}{2} \right)}(x_3),$$

$$(3.15) \qquad Z_\varepsilon := \left\{ i \in \mathbb{Z}, \ \left(\varepsilon i - \frac{\varepsilon}{2}, \varepsilon i + \frac{\varepsilon}{2} \right] \subset (0, L) \right\}.$$

Assumption (3.2) implies that $|n_\varepsilon|_{L^\infty(\Omega)} \leq 1$, therefore, up to a subsequence,

$$(3.16) \qquad\qquad n_\varepsilon \overset{\star}{\rightharpoonup} n \quad \text{weakly* in} \ L^\infty(\Omega) \quad \text{for some } n \in L^\infty(\Omega).$$

The scalar $\frac{1}{\varepsilon} n_\varepsilon(x)$ is an approximation at x of the local number of stiff layers per unit length in the $\boldsymbol{e}_3$ direction. For simplicity (see Remark 3.3), we assume that

$$(3.17) \quad \{n > 0\} \quad \text{is a finite union of open connected subsets of } \Omega \text{ if } k = +\infty.$$

Under these assumptions, we prove that the solution to (3.5) weakly* converges in $L^\infty(0, T; H_0^1(\Omega; \mathbb{R}^3))$ to the unique solution to $(\mathcal{P}_{(n,k,\kappa)}^{hom})$ defined, in terms of k, κ, n given by (3.8), (3.16), as follows: if $0 < k < +\infty$, we get (see (2.1))

$$(3.18) \quad \begin{matrix} (\mathcal{P}_{(n,k,0)}^{hom}) : \\ (0 < k < +\infty) \end{matrix} \begin{cases} (\rho + n\overline{\rho}_1) \dfrac{\partial^2 \boldsymbol{u}}{\partial t^2} - \mathbf{div}\boldsymbol{\sigma}(\boldsymbol{u}) \\ \qquad\qquad - nk\mathbf{div}\boldsymbol{\sigma}_{x'}(\boldsymbol{u}') = (\rho + n\overline{\rho}_1)\boldsymbol{f} \quad \text{in } \Omega \times (0, T), \\ \boldsymbol{u} \in C([0, T]; H_0^1(\Omega; \mathbb{R}^3)) \cap C^1([0, T]; L^2(\Omega; \mathbb{R}^3)), \\ \boldsymbol{u}(0) = \boldsymbol{a}_0, \ \dfrac{\partial \boldsymbol{u}}{\partial t}(0) = \boldsymbol{b}_0. \end{cases}$$

If $k = +\infty$ and $\kappa = 0$, the limit problem is deduced from (3.18), formally, by substituting $(0,0)$ for $(u_1(x), u_2(x))$ when $n(x) > 0$:

$$(3.19) \quad (\mathcal{P}^{hom}_{(n,+\infty,0)}) : \begin{cases} (\rho + n\bar{\rho}_1)\dfrac{\partial^2 u_3}{\partial t^2} - (\mathbf{div}\boldsymbol{\sigma}(\boldsymbol{u}))_3 = (\rho + n\bar{\rho}_1)f_3 \quad \text{in } \Omega \times (0,T), \\[2mm] \rho\dfrac{\partial^2 u_\alpha}{\partial t^2} - (\mathbf{div}\boldsymbol{\sigma}(\boldsymbol{u}))_\alpha = \rho f_\alpha \quad \text{in } \widehat{\{n=0\}} \times (0,T), \\[2mm] u_\alpha = 0 \quad \text{on } \partial\widehat{\{n=0\}} \times (0,T) \quad (\alpha \in \{1,2\}), \\[2mm] nu_1 = nu_2 = 0, \ \boldsymbol{u} \in C([0,T];H^1_0(\Omega;\mathbb{R}^3)) \cap C^1([0,T];L^2(\Omega;\mathbb{R}^3)), \\[2mm] \boldsymbol{u}(0) = (a_{01}\mathbb{1}_{\{n=0\}}, a_{02}\mathbb{1}_{\{n=0\}}, a_{03}), \\[2mm] \dfrac{\partial \boldsymbol{u}}{\partial t}(0) = (b_{01}\mathbb{1}_{\{n=0\}}, b_{02}\mathbb{1}_{\{n=0\}}, b_{03}). \end{cases}$$

The case $0 < \kappa < +\infty$ is characterized by the emergence of fourth order derivatives with respect to x_1, x_2 in the limit equations, revealing bending effects:

$$(3.20) \quad \begin{array}{c} (\mathcal{P}^{hom}_{(n,+\infty,\kappa)}) : \\ (0 < \kappa < +\infty) \end{array} \begin{cases} (\rho + n\bar{\rho}_1)\dfrac{\partial^2 u_3}{\partial t^2} - (\mathbf{div}\boldsymbol{\sigma}(\boldsymbol{u}))_3 + n\dfrac{\kappa}{3}\dfrac{l+1}{l+2}\displaystyle\sum_{\alpha,\beta=1}^{2}\dfrac{\partial^4 u_3}{\partial x_\alpha^2 \partial x_\beta^2} \\[4mm] \qquad\qquad\qquad\qquad = (\rho + n\bar{\rho}_1)f_3 \ \text{in } \Omega \times (0,T), \\[2mm] \rho\dfrac{\partial^2 u_\alpha}{\partial t^2} - (\mathbf{div}\boldsymbol{\sigma}(\boldsymbol{u}))_\alpha = \rho f_\alpha \quad \text{in } \widehat{\{n=0\}} \times (0,T), \\[2mm] u_\alpha = 0 \quad \text{on } \partial\widehat{\{n=0\}} \times (0,T) \quad (\alpha \in \{1,2\}), \\[2mm] nu_1 = nu_2 = 0, \ \boldsymbol{u} \in C([0,T];H^1_0(\Omega;\mathbb{R}^3)) \cap C^1([0,T];L^2(\Omega;\mathbb{R}^3)), \\[2mm] u_3 \in C([0,T]; L^2_n(0,L; H^2_0(\Omega'))), \\[2mm] \boldsymbol{u}(0) = (a_{01}\mathbb{1}_{\{n=0\}}, a_{02}\mathbb{1}_{\{n=0\}}, a_{03}), \\[2mm] \dfrac{\partial \boldsymbol{u}}{\partial t}(0) = (b_{01}\mathbb{1}_{\{n=0\}}, b_{02}\mathbb{1}_{\{n=0\}}, b_{03}). \end{cases}$$

If $\kappa = +\infty$, we get:

$$(3.21) \quad (\mathcal{P}^{hom}_{(n,+\infty,+\infty)}) : \begin{cases} \rho\dfrac{\partial^2 \boldsymbol{u}}{\partial t^2} - \mathbf{div}\boldsymbol{\sigma}(\boldsymbol{u}) = \rho\boldsymbol{f} \quad \text{in } \widehat{\{n=0\}} \times (0,T), \\[2mm] \boldsymbol{u} = 0 \quad \text{on } \partial\widehat{\{n=0\}} \times (0,T), \\[2mm] n\boldsymbol{u} = 0, \quad \boldsymbol{u} \in C([0,T]; H^1_0(\Omega;\mathbb{R}^3)) \cap C^1([0,T]; L^2(\Omega;\mathbb{R}^3)), \\[2mm] \boldsymbol{u}(0) = \boldsymbol{a}_0\mathbb{1}_{\{n=0\}}, \ \dfrac{\partial \boldsymbol{u}}{\partial t}(0) = \boldsymbol{b}_0\mathbb{1}_{\{n=0\}}. \end{cases}$$

THEOREM 3.1. *Assume (3.11), (3.12), (3.16), then the sequence $(\boldsymbol{u}_\varepsilon)$ of the solutions to (3.5) weakly* converges in $L^\infty(0,T; H^1_0(\Omega;\mathbb{R}^3))$ to the unique solution of the problem $(\mathcal{P}^{hom}_{(n,k,\kappa)})$ given by (3.18)-(3.21).*

REMARK 3.2. *(i) When stiff fibers [9, 13] (resp. grain-like inclusions [7]) embedded in a matrix of stiffness of order 1 are considered, the fibers (resp. the inclusions) disappear from the limit problem if $r_\varepsilon \ll \exp -\frac{1}{\varepsilon^2}$ (resp. $r_\varepsilon \ll \varepsilon^3$), where r_ε denotes*

the diameter of the sections of the fibers (resp. of the inclusions). This never occurs in the stratified case, whatever the choice of (r_ε). This is related to the fact that the harmonic capacity of a surface in Ω is always positive, whereas that of a line or a point are equal to zero.

(ii) Under (3.12), the case $\vartheta > 0$, $k < +\infty$ has been studied in [27], [38]. In the case $\vartheta > 0$, $k = +\infty$, we came up against technical complications (see Remark 6.1).

REMARK 3.3. *If (3.17) is not satisfied, the sequence $(\boldsymbol{u}_\varepsilon)$ of the solutions to (3.5) weakly* converges in $L^\infty(0, T; H_0^1(\Omega; \mathbb{R}^3))$ to the unique solution of the variational problem (4.23) with data deduced from (6.60), (6.61) by substituting $\widetilde{V}_{n,k,\kappa}$ and $\widetilde{a}_{n,k,\kappa}$, defined by (6.62), for $V_{n,k,\kappa}$ and $a_{n,k,\kappa}$. Assumption (3.17) ensures the equivalence between (3.19-3.21) and the last mentioned variational problem.*

3.2. Stochastic case. Fix $d > 0$ and set

$$\mathfrak{D} := \left\{ \omega \in 2^\mathbb{R}, \ \forall (\omega_1, \omega_2) \in \omega \times \omega, \ \omega_1 \neq \omega_2 \ \Rightarrow \ |\omega_1 - \omega_2| \geq d \right\},$$
$$(3.22) \qquad \omega_\varepsilon(\omega) := \varepsilon\omega \cap (\varepsilon, L - \varepsilon) \qquad \forall \omega \in \mathfrak{D},$$

where $2^\mathbb{R}$ denotes the set of all subsets of $\mathbb{R}$. Let $\mathcal{B}_\mathfrak{D}$ be the Borel σ-algebra generated by the Hausdorff distance on $\mathfrak{D}$ (see Remark 3.6), and P be a probability on $(\mathfrak{D}, \mathcal{B}_\mathfrak{D})$ satisfying

$$(3.23) \qquad P(A + z) = P(A) \quad \forall\, z \in \mathbb{Z}, \ \forall\, A \in \mathcal{B}_\mathfrak{D}.$$

We consider the random distribution of stiff homothetical layers $B_\varepsilon(\omega_\varepsilon(\omega))$ and the problem $(\mathcal{P}_\varepsilon(\omega))$ obtained by substituting $\omega_\varepsilon(\omega)$ for ω_ε in (3.4), (3.5). In what follows, $\mathcal{F}$ represents the σ-algebra of the Y-periodic elements of $\mathcal{B}_\mathfrak{D}$, $E_P^\mathcal{F} X$ the conditional expectation of a random variable X given $\mathcal{F}$ with respect to P, $n_\varepsilon(\omega)$ the element of $L^\infty(\Omega)$ defined by substituting $\omega_\varepsilon(\omega)$ for ω_ε in (3.14), and $n_0 : \mathfrak{D} \to \mathbb{N}$ the random variable given by

$$(3.24) \qquad n_0(\omega) := \sharp \left(\omega \cap \left[-\frac{1}{2}, \frac{1}{2} \right] \right) \qquad \forall \omega \in \mathfrak{D}.$$

The following theorem is proved in [10]:

THEOREM 3.4. *Under the assumptions stated above, there exists a sequence of reals (ε_k) converging to 0 and a P-negligible subset $\mathfrak{N}$ of $\mathfrak{D}$, such that for all $\omega \in \mathfrak{D} \setminus \mathfrak{N}$,*

$$(3.25) \qquad n_{\varepsilon_k}(\omega) \overset{\star}{\rightharpoonup} E_P^\mathcal{F} n_0(\omega) \quad weakly^* \ in \ L^\infty(\Omega).$$

The following result straightforwardly follows from theorems 3.1, 3.4:

THEOREM 3.5. *Assume (3.11), (3.12), and let (ε_k) and $\mathfrak{N}$ be the sequence and the P-negligible set given by Theorem 3.4. Then, for all $\omega \in \mathfrak{D} \setminus \mathfrak{N}$, the solution to $(\mathcal{P}_{\varepsilon_k}(\omega))$, weakly* converges in $L^\infty(0, T; H_0^1(\Omega; \mathbb{R}^3))$ to the unique solution to the problem $(\mathcal{P}_{(E_P^\mathcal{F} n_0(\omega)), k, \kappa}^{hom})$ defined by (3.18-3.21).*

REMARK 3.6. *The restriction of the Hausdorff distance $d_\mathcal{H}$ to $\mathfrak{D}$ is an extended metric on $\mathfrak{D}$, and the mapping $d_\mathfrak{D} : \mathfrak{D}^2 \to [0,1]$ defined by $d_\mathfrak{D}(\omega, \omega') := \min\{1, d_\mathcal{H}(\omega, \omega')\}$ is a finite metric on $\mathfrak{D}$ which turns $\mathfrak{D}$ into a complete metric space.*

3.3. Intermediate case . Under the assumptions

$$(3.26) \qquad \varepsilon^2 \ll \mu_{0\varepsilon} \ll 1, \quad 0 \le \lambda_{0\varepsilon} \le C\mu_{0\varepsilon},$$

$$(3.27) \qquad n_\varepsilon \to n \quad \text{strongly in} \quad L^2(\Omega),$$

and (3.17) (see Remark 3.9), we show that the solution to (3.5) weakly* converges in $L^\infty(0,T; L^2(\Omega; \mathbb{R}^3))$ to the unique solution to $(\mathcal{P}^{hom}_{(n,k,\kappa)})$ defined by

$$(3.28) \quad \begin{array}{c} (\mathcal{P}^{hom}_{(n,k,0)}): \\ (0<k<+\infty) \end{array} \left\{ \begin{array}{l} (\rho(1-\vartheta n)+n\bar{\rho}_1)\dfrac{\partial^2 \boldsymbol{u}}{\partial t^2} - nk\mathbf{div}\boldsymbol{\sigma}_{x'}(\boldsymbol{u}') \\ \qquad\qquad = (\rho(1-\vartheta n)+n\bar{\rho}_1)\boldsymbol{f} \quad \text{in } \Omega\times(0,T), \\ u_1, u_2 \in C([0,T]; L^2_n(0,L; H^1_0(\Omega'))), \\ \boldsymbol{u} \in C^1([0,T]; L^2(\Omega;\mathbb{R}^3)), \ \boldsymbol{u}(0)=\boldsymbol{a}_0, \ \dfrac{\partial \boldsymbol{u}}{\partial t}(0)=\boldsymbol{b}_0, \end{array} \right.$$

$$(3.29) \quad (\mathcal{P}^{hom}_{(n,+\infty,0)}): \left\{ \begin{array}{l} (\rho(1-\vartheta n)+n\bar{\rho}_1)\dfrac{\partial^2 u_3}{\partial t^2} = (\rho(1-\vartheta n)+n\bar{\rho}_1)f_3 \quad \text{in } \Omega\times(0,T), \\ \rho\dfrac{\partial^2 u_\alpha}{\partial t^2} = \rho f_\alpha \ \text{in } \widehat{\{n=0\}} \times (0,T), \ (\alpha \in \{1,2\}), \\ nu_1 = nu_2 = 0, \quad \boldsymbol{u} \in C^1([0,T]; L^2(\Omega;\mathbb{R}^3)), \\ \boldsymbol{u}(0) = (a_{01}\mathbb{1}_{\{n=0\}}, a_{02}\mathbb{1}_{\{n=0\}}, a_{03}), \\ \dfrac{\partial \boldsymbol{u}}{\partial t}(0) = (b_{01}\mathbb{1}_{\{n=0\}}, b_{02}\mathbb{1}_{\{n=0\}}, b_{03}), \end{array} \right.$$

$$(3.30) \quad \begin{array}{c} (\mathcal{P}^{hom}_{(n,+\infty,\kappa)}): \\ (0<\kappa<+\infty) \end{array} \left\{ \begin{array}{l} (\rho(1-\vartheta n)+n\bar{\rho}_1)\dfrac{\partial^2 u_3}{\partial t^2} + n\dfrac{\kappa}{3}\dfrac{l+1}{l+2}\displaystyle\sum_{\alpha,\beta=1}^{2}\dfrac{\partial^4 u_3}{\partial x_\alpha^2 \partial x_\beta^2} \\ \qquad\qquad = (\rho(1-\vartheta n)+n\bar{\rho}_1)f_3 \ \text{in } \Omega\times(0,T), \\ \rho\dfrac{\partial^2 u_\alpha}{\partial t^2} = \rho f_\alpha \ \text{in } \widehat{\{n=0\}} \times (0,T), \ (\alpha \in \{1,2\}), \\ nu_1 = nu_2 = 0, \\ u_3 \in C([0,T]; L^2_n(0,L; H^2_0(\Omega'))), \ \boldsymbol{u} \in C^1([0,T]; L^2(\Omega;\mathbb{R}^3)), \\ \boldsymbol{u}(0) = (a_{01}\mathbb{1}_{\{n=0\}}, a_{02}\mathbb{1}_{\{n=0\}}, a_{03}), \\ \dfrac{\partial \boldsymbol{u}}{\partial t}(0) = (b_{01}\mathbb{1}_{\{n=0\}}, b_{02}\mathbb{1}_{\{n=0\}}, b_{03}), \end{array} \right.$$

$$(3.31) \quad (\mathcal{P}^{hom}_{(n,+\infty,+\infty)}): \left\{ \begin{array}{l} \rho\dfrac{\partial^2 \boldsymbol{u}}{\partial t^2} = \rho\boldsymbol{f} \quad \text{in } \widehat{\{n=0\}} \times (0,T), \\ n\boldsymbol{u} = 0, \quad \boldsymbol{u} \in C^1([0,T]; L^2(\Omega;\mathbb{R}^3)), \\ \boldsymbol{u}(0) = \boldsymbol{a}_0\mathbb{1}_{\{n=0\}}, \ \dfrac{\partial \boldsymbol{u}}{\partial t}(0) = \boldsymbol{b}_0\mathbb{1}_{\{n=0\}}. \end{array} \right.$$

THEOREM 3.7. *Under (3.17), (3.26), (3.27), the solution to (3.5) weakly* converges in $L^\infty(0,T;L^2(\Omega;\mathbb{R}^3))$ to the unique solution to $(\mathcal{P}^{hom}_{(n,k,\kappa)})$ given by (3.28-3.31).*

REMARK 3.8. *(i) Problems (3.28-3.31) are formally deduced from (3.18-3.21) by removing the term "$\mathbf{div}\boldsymbol{\sigma}(\boldsymbol{u})$" (see (3.11)). This indicates that no strain energy is stored in the softer phase.*
(ii) Assumption (3.27), stronger than (3.16), precludes the application of Theorem 3.4 and the extension of Theorem 3.7 to the setting of stochastic homogenization.

REMARK 3.9. *If (3.17) is not satisfied, the sequence $(\boldsymbol{u}_\varepsilon)$ of the solutions to (3.5) weakly* converges in $L^\infty(0,T;L^2(\Omega;\mathbb{R}^3))$ to the unique solution of the variational problem (4.23), (6.60), (6.61). Under (3.17), this variational problem is equivalent to (3.29-3.31).*

REMARK 3.10. *Theorems 3.1, 3.7 can be extended to the case of a multiphase stratified elastic composite comprising several distributions of stiff layers of various stiffness and thickness (see Remark 6.2 for more details). Combining our results with those established in [8], [13], [14], one can also derive the effective equations governing the behavior of a composite comprising disjoint distributions of stiff layers alternating with softer interlayers in which fibers or grain-like inclusions are embedded (see also Remark 3.17).*

REMARK 3.11. *Under the assumptions of theorems 3.1, 3.7, the limit problem associated to the sequence of equilibrium problems*

$$(3.32) \qquad -\mathbf{div}(\boldsymbol{\sigma}_\varepsilon(\boldsymbol{u}_\varepsilon)) = \boldsymbol{f} \quad in \quad \Omega, \quad \boldsymbol{u}_\varepsilon \in H_0^1(\Omega,\mathbb{R}^3), \quad \boldsymbol{f} \in L^2(\Omega,\mathbb{R}^3),$$

is given by

$$(3.33) \qquad \boldsymbol{u} \in V \quad and \quad a(\boldsymbol{u},\boldsymbol{w}) = (\boldsymbol{f},\boldsymbol{w})_H, \quad \forall \boldsymbol{w} \in V,$$

where the Hilbert spaces V and H and the non-negative symmetric bilinear form $a(.,.)$ are specified in Remark 6.3. Under the assumptions of Theorem 3.7, this bilinear form fails to be coercive on $L^2(\Omega;\mathbb{R}^3)$ if $\kappa = 0$ and may fail to be so if the function n is not bounded from below by a positive constant. Remark 6.3 states conditions ensuring coerciveness in stratified multiphase media. Coerciveness can also be achieved by reinforcing the composite by stiff fibers embedded in the soft phase (see [8, Proposition 5.2]).

3.4. Case of soft interlayers with Lamé coefficients of order ε^2 . We assume that

$$(3.34) \qquad \mu_{0\varepsilon} = \varepsilon^2 \mu_0, \quad \lambda_{0\varepsilon} = \varepsilon^2 \lambda_0, \quad \mu_0 > 0, \quad \lambda_0 \geq 0,$$

and that the stiff layers are periodically distributed (see (3.15)):

$$(3.35) \qquad B_\varepsilon := \bigcup_{i \in Z_\varepsilon} B_\varepsilon^i; \quad B_\varepsilon^i := \Omega' \times (\varepsilon i + r_\varepsilon I).$$

Under these hypotheses, setting (see Remark 3.13)

$$(3.36) \quad Y := \left(-\frac{1}{2}, \frac{1}{2}\right)^3 ; \ B := \left(-\frac{1}{2}, \frac{1}{2}\right)^2 \times \left(-\frac{\vartheta}{2}, \frac{\vartheta}{2}\right) ; \ \Sigma := \left(-\frac{1}{2}, \frac{1}{2}\right)^2 \times \{0\},$$

$$(3.37) \quad A := B \quad \text{if} \quad \vartheta > 0, \qquad A := \Sigma \quad \text{if} \quad \vartheta = 0,$$

$$(3.38) \quad \vartheta > 0 \quad \text{if} \quad \kappa = 0,$$

and denoting by $\mathcal{M}(\overline{\Omega}; \mathbb{R}^3)$ the set of $\mathbb{R}^3$-valued Radon measures on $\overline{\Omega}$, we show that the solution $\boldsymbol{u}_\varepsilon$ to (3.5) two-scale converges to $\boldsymbol{u}_0 \in C([0,T]; L^2(\Omega, H^1_\sharp(Y; \mathbb{R}^3)))$ (see Section 4 for the definition of this convergence), and the sequence $(\boldsymbol{u}_\varepsilon m_\varepsilon)$, where m_ε is the measure defined by

$$(3.39) \qquad\qquad m_\varepsilon := \frac{\varepsilon}{r_\varepsilon} \mathbb{1}_{B_\varepsilon}(x) \mathcal{L}^3_{\lfloor \Omega},$$

weakly* converges in $L^\infty(0, T; \mathcal{M}(\overline{\Omega}; \mathbb{R}^3))$ to $\boldsymbol{v} \in C^1([0,T]; L^2(\Omega; \mathbb{R}^3))$, where $(\boldsymbol{u}_0, \boldsymbol{v})$ is the unique solution to the coupled system of equations (comparable in certain respects with [8, (2.17)])

$$(3.40) \qquad\qquad \begin{cases} (\mathcal{P}^{hom}_{soft}), \\ (\mathcal{P}^{hom}_{stiff}(k, \kappa)), \end{cases}$$

defined below in terms of k, κ, and ϑ given, respectively, by (3.8) and (3.10). The fields $\boldsymbol{u}_0$ and $\boldsymbol{v}$ are linked by the following relation in $\Omega \times (0, T) \times A$:

$$(3.41) \qquad\qquad \boldsymbol{v}(x,t) = \boldsymbol{u}_0(x,t,y) \qquad \text{in} \ \ \Omega \times (0,T) \times A.$$

We introduce the operators $\boldsymbol{e}_y, \boldsymbol{\sigma}_{0y} : H^1(Y; \mathbb{R}^3) \to L^2(Y; \mathbb{S}^3)$, $\boldsymbol{g} : \mathcal{H} \to \mathbb{R}^3$ defined by

$$(\boldsymbol{e}_y(\boldsymbol{w}))_{ij} = \frac{1}{2}\left(\frac{\partial w_i}{\partial y_j} + \frac{\partial w_j}{\partial y_i}\right), \ \ \boldsymbol{\sigma}_{0y}(\boldsymbol{w}) := \lambda_0 \operatorname{tr}(\boldsymbol{e}_y(\boldsymbol{w}))\boldsymbol{I} + 2\mu_0 \boldsymbol{e}_y(\boldsymbol{w}),$$

$$(3.42) \qquad \boldsymbol{g}(\boldsymbol{w}) := \begin{cases} -\displaystyle\int_{\partial(Y \backslash B) \cap \overline{B}} \boldsymbol{\sigma}_{0y}(\boldsymbol{w})\boldsymbol{\nu}_{Y \backslash B} d\mathcal{H}^2(y), & \text{if} \ \ A = B, \\ \displaystyle\int_{\Sigma} (\boldsymbol{\sigma}_{0y}(\boldsymbol{w}^+) - \boldsymbol{\sigma}_{0y}(\boldsymbol{w}^-)) \cdot \boldsymbol{e}_3 d\mathcal{H}^2(y) & \text{if} \ \ A = \Sigma, \end{cases}$$

where $\boldsymbol{\nu}_{Y \backslash B}$ stands for the outward normal to $\partial(Y \backslash B)$ and

$$(3.43) \qquad \mathcal{H} := \left\{ \boldsymbol{w} \in H^1(Y \backslash A; \mathbb{R}^3), \ \ \mathbf{div}(\boldsymbol{\sigma}_{0y}(\boldsymbol{w})) \in (H^1(Y \backslash A; \mathbb{R}^3))' \right\},$$

denoting by E' the topological dual of a Banach space E, and by $\boldsymbol{w}^+$ (resp. $\boldsymbol{w}^-$) the restriction of $\boldsymbol{w}$ to $\left(\frac{-1}{2}, \frac{1}{2}\right)^2 \times \left(0, \frac{1}{2}\right)$ (resp. $\left(\frac{-1}{2}, \frac{1}{2}\right)^2 \times \left(\frac{-1}{2}, 0\right)$). Problem $(\mathcal{P}^{hom}_{soft})$ in (3.40) is the equation of $\boldsymbol{u}_0$ in $\Omega \times (0, T) \times (Y \backslash A)$ coupled with $\boldsymbol{v}$ through (3.41) and given by (denoting by $\boldsymbol{\nu}$ the outward normal to ∂Y):

$$(3.44) \qquad (\mathcal{P}_{soft}^{hom}): \begin{cases} \rho \dfrac{\partial^2 \boldsymbol{u}_0}{\partial t^2} - \mathbf{div}_y(\boldsymbol{\sigma}_{0y}(\boldsymbol{u}_0)) = \rho \boldsymbol{f} \quad \text{in} \ \ \Omega \times (0,T) \times Y \setminus A, \\[2mm] (\boldsymbol{u}_0, \boldsymbol{v}) \quad \text{satisfies} \quad (3.41), \\[2mm] \boldsymbol{\sigma}_{0y}(\boldsymbol{u}_0)\boldsymbol{\nu}(y) = -\boldsymbol{\sigma}_{0y}(\boldsymbol{u}_0)\boldsymbol{\nu}(-y) \ \text{on} \ \ \Omega \times (0,T) \times \partial Y, \\[2mm] \boldsymbol{u}_0 \in C([0,T]; L^2(\Omega, H^1_\sharp(Y; \mathbb{R}^3))) \cap C^1([0,T]; L^2(\Omega \times Y; \mathbb{R}^3)), \\[2mm] \boldsymbol{u}_0(0)\mathbb{1}_{Y\setminus A} = \boldsymbol{a}_0 \mathbb{1}_{Y\setminus A}, \quad \dfrac{\partial \boldsymbol{u}_0}{\partial t}(0)\mathbb{1}_{Y\setminus A} = \boldsymbol{b}_0 \mathbb{1}_{Y\setminus A}. \end{cases}$$

Equation (3.44) governs the effective behavior of the displacement in the soft phase. Problem $(\mathcal{P}_{stiff}^{hom}(k,\kappa))$ in (3.40) is an equation of $\boldsymbol{v}$ in $\Omega \times (0,T)$ coupled with $(\mathcal{P}_{soft}^{hom})$ through the source term $\boldsymbol{g}(\boldsymbol{u}_0)$ defined by (3.42). This equation rules the effective behavior of the displacement in the stiff layers. Its form is determined by the order of magnitude of the coefficients k, κ. If $0 < k < +\infty$, we get (see (2.1))

$$(3.45) \qquad \begin{matrix} (\mathcal{P}_{stiff}^{hom}(k,0)): \\ (0 < k < +\infty) \end{matrix} \begin{cases} \overline{\rho}_1 \dfrac{\partial^2 \boldsymbol{v}}{\partial t^2} - k \mathbf{div}\boldsymbol{\sigma}_{x'}(\boldsymbol{v}') = \overline{\rho}_1 \boldsymbol{f} + \boldsymbol{g}(\boldsymbol{u}_0) \ \ \text{in} \ \Omega \times (0,T), \\[2mm] v_1, v_2 \in C([0,T]; L^2(0,L; H^1_0(\Omega'))) \cap C^1([0,T]; L^2(\Omega)), \\[2mm] \boldsymbol{v} \in C^1([0,T]; L^2(\Omega; \mathbb{R}^3)), \quad \boldsymbol{v}(0) = \boldsymbol{a}_0, \ \dfrac{\partial \boldsymbol{v}}{\partial t}(0) = \boldsymbol{b}_0. \end{cases}$$

If $(k,\kappa) = (+\infty, 0)$, we obtain

$$(3.46) \qquad (\mathcal{P}_{stiff}^{hom}(+\infty, 0)): \begin{cases} \overline{\rho}_1 \dfrac{\partial^2 v_3}{\partial t^2} = \overline{\rho}_1 f_3 + (\boldsymbol{g}(\boldsymbol{u}_0))_3 \quad \text{in} \ \ \Omega \times (0,T), \\[2mm] v_1 = v_2 = 0, \\[2mm] v_3 \in C^1([0,T], L^2(\Omega)), \ v_3(0) = a_{03}, \ \dfrac{\partial v_3}{\partial t}(0) = b_{03}. \end{cases}$$

If $0 < \kappa < +\infty$, the emergence of fourth derivatives of v_3 reveal bending effects:

$$(3.47) \qquad \begin{matrix} (\mathcal{P}_{stiff}^{hom}(+\infty, \kappa)): \\ (0 < \kappa < +\infty) \end{matrix} \begin{cases} \overline{\rho}_1 \dfrac{\partial^2 v_3}{\partial t^2} + \dfrac{\kappa}{3} \dfrac{l+1}{l+2} \displaystyle\sum_{\alpha,\beta=1}^{2} \dfrac{\partial^4 v_3}{\partial x_\alpha^2 \partial x_\beta^2} \\[4mm] \qquad\qquad = \overline{\rho}_1 f_3 + (\boldsymbol{g}(\boldsymbol{u}_0))_3 \quad \text{in} \ \Omega \times (0,T), \\[2mm] v_1 = v_2 = 0, \\[2mm] v_3 \in C([0,T]; L^2(0,L; H^2_0(\Omega'))) \cap C^1([0,T]; L^2(\Omega)), \\[2mm] v_3(0) = a_{03}, \ \dfrac{\partial v_3}{\partial t}(0) = b_{03}. \end{cases}$$

If $\kappa = +\infty$, the displacement in the stiff layers asymptotically vanishes:

$$(3.48) \qquad (\mathcal{P}_{stiff}^{hom}(+\infty, +\infty)): \quad \boldsymbol{v} = 0.$$

THEOREM 3.12. *Under (3.34), (3.35), the solution $\boldsymbol{u}_\varepsilon$ to (3.5) two-scale converges to $\boldsymbol{u}_0$ with respect to x and weakly* converges in $L^\infty(0,T; L^2(\Omega, \mathbb{R}^3))$ to $\boldsymbol{u} =$*

$\int_Y \boldsymbol{u}_0(.,y)dy$, and the sequence $(\boldsymbol{u}_\varepsilon m_\varepsilon)$, where m_ε is defined by (3.39), weakly* converges in $L^\infty(0,T;\mathcal{M}(\overline{\Omega},\mathbb{R}^3))$ to $\boldsymbol{v}\mathcal{L}^3_{\lfloor\Omega}$, where $(\boldsymbol{u}_0,\boldsymbol{v})$ is the unique solution to (3.40). Moreover, $\boldsymbol{u}_\varepsilon(\tau)$ two-scale converges to $\boldsymbol{u}_0(\tau)$ with respect to x, for each $\tau \in [0,T]$.

REMARK 3.13. *When $\vartheta = 0$, $k \in (0,+\infty]$, additional sub-regimes are likely to appear in the cases $r_\varepsilon = c\varepsilon^2$ and $\varepsilon^2 \ll r_\varepsilon \ll \varepsilon$, possibly attended by the emergence of bending effects. Similar phenomena have been studied by V. V. Zhikov [55] and V. V. Zhikov and S. E. Pastukhova [56] in the context of periodic singular structures.*

REMARK 3.14. *One can show (see [8, p.2548] for more details), that if $\boldsymbol{a}_0 = 0$ and if the fields $\boldsymbol{b}_0$, $\boldsymbol{f}$ are sufficiently regular, the following corrector result holds*

(3.49)
$$\lim_{\varepsilon\to 0}\left\|\boldsymbol{u}_\varepsilon - \boldsymbol{u}_0\left(x,t,\frac{x}{\varepsilon}\right)\right\|_{L^2(\Omega\times(0,T);\mathbb{R}^3)} = 0.$$

REMARK 3.15. *The effective problem (3.40) is non-local in space and time. Non-local effects [3], [5]-[19], [20, 21, 23, 34, 44], [47]-[50], and memory effects [1, 4, 37, 53] are typical of composite media with high contrast.*

REMARK 3.16 (Multiphase stratified elastic media). *In the same way as in [8, Section 4], we can extend Theorem 3.12 to the case of a multiphase medium whereby m ε-periodic disconnected families $B_\varepsilon^{[1]}$, ..., $B_\varepsilon^{[m]}$ of parallel layers are embedded in a soft matrix. The limit problem then takes the following form*

$$\begin{cases} (\mathcal{P}^{hom}_{soft}), \\ (\mathcal{P}^{hom\,[j]}_{stiff}), \ j \in \{1,...,m\}, \end{cases}$$

and can be written under the variational form (4.23) for some suitable choice of data H,V,a,h,ξ_0,ξ_1. Each family $B_\varepsilon^{[m]}$ is associated to some subset $A^{[j]}$ of Y like in (3.37). The system $(\mathcal{P}^{hom}_{soft})$ governs the effective displacement in the soft phase, and only differs from (3.44) by the relation (3.41) which is replaced by a series of relations on each set $\Omega\times(0,T)\times A^{[j]}$ between $\boldsymbol{u}_0$ and some auxiliary variable $\boldsymbol{v}^{[j]}$ characterizing the effective displacement in $B_\varepsilon^{[j]}$. Each problem $(\mathcal{P}^{hom\,[j]}_{stiff})$ consists of an equation of $\boldsymbol{v}^{[j]}$ of the same form as $(\mathcal{P}^{hom}_{stiff})$ in (3.40), coupled with $\boldsymbol{u}_0$ through the operator $\boldsymbol{g}^{[j]}$ deduced from (3.42) by replacing A by $A^{[j]}$.

REMARK 3.17. *Multiphase composites comprising, besides stiff layers, periodic distributions of fibers or grain-like inclusions, can also be considered. By way of illustration, let us examine the case of a composite consisting of an ε-periodic family $B_\varepsilon^{[1]}$ of parallel stiff layers of thickness $\vartheta\varepsilon$ $(\vartheta > 0)$ alternating with a softer medium in which an ε-periodic family $B_\varepsilon^{[2]}$ of parallel stiff fibers and an ε-periodic family $B_\varepsilon^{[3]}$ of stiff grain-like inclusions are embedded. The disconnected families $(B_\varepsilon^{[i]})$ are defined in terms of three connected subsets $B^{[1]}, B^{[2]}, B^{[3]}$ of Y with disjoint closure by setting $B_\varepsilon^{[i]} := \varepsilon\left(\bigcup_{j\in\mathbb{Z}^3} j + B^{[i]}\right) \cap \Omega$ (see figure 3.2). We assume that the Lamé coefficients take constant values of order at least 1 in each $B_\varepsilon^{[i]}$, and constant values $\varepsilon^2\lambda_0$, $\varepsilon^2\mu_0$ in $\Omega \setminus \bigcup_{i=1}^3 B_\varepsilon^{[i]}$. The limit problem then takes the form*

$$\begin{cases} (\mathcal{P}^{hom}_{soft}), \\ (\mathcal{P}^{hom\,[i]}_{stiff}), \ i \in \{1,2,3\}, \end{cases}$$

where, setting $B := \bigcup_{i=1}^3 B^{[i]}$, $\boldsymbol{y}_{B^{[i]}} := \frac{1}{|B^{[i]}|}\int_{B^{[i]}}\boldsymbol{y}dy$,

$$(3.50) \quad (\mathcal{P}_{soft}^{hom}): \begin{cases} \rho \dfrac{\partial^2 \boldsymbol{u}_0}{\partial t^2} - \mathbf{div}_y(\boldsymbol{\sigma}_{0y}(\boldsymbol{u}_0)) = \rho \boldsymbol{f} & in \ \Omega \times (0,T) \times (Y \setminus B), \\[2mm] \boldsymbol{u}_0(x,t,y) = \boldsymbol{v}^{[1]}(x,t) & in \ \Omega \times (0,T) \times B^{[1]}, \\[2mm] \boldsymbol{u}_0 = \boldsymbol{v}^{[i]} + \boldsymbol{r}^{[i]} \wedge (\boldsymbol{y} - \boldsymbol{y}_{B^{[i]}}) & in \ \Omega \times (0,T) \times B^{[i]}, \ (j \in \{2,3\}), \\[2mm] \boldsymbol{\sigma}_{0y}(\boldsymbol{u}_0).\boldsymbol{n}(y) = -\boldsymbol{\sigma}_{0y}(\boldsymbol{u}_0).\boldsymbol{n}(-y) & on \ \Omega \times (0,T) \times \partial Y, \\[2mm] \boldsymbol{u}_0 \in C([0,T]; L^2(\Omega, H^1_\sharp(Y; \mathbb{R}^3))) \cap C^1([0,T]; L^2(\Omega \times Y; \mathbb{R}^3)), \\[2mm] \boldsymbol{u}_0(0)1_{Y \setminus B} = \boldsymbol{a}_0 1_{Y \setminus B}, \quad \dfrac{\partial \boldsymbol{u}_0}{\partial t}(0)1_{Y \setminus B} = \boldsymbol{b}_0 1_{Y \setminus B}. \end{cases}$$

The problem $(\mathcal{P}_{stiff}^{hom \ [1]})$ in *(3.50)* is an equation of $\boldsymbol{v}^{[1]}$ characterizing the effective displacement in the stiff layers as described in Remark *3.16*. For $i \in \{2,3\}$, the prob-lem $(\mathcal{P}_{stiff}^{hom \ [i]})$ is a system of equations of $\boldsymbol{v}^{[i]}$, $\boldsymbol{r}^{[i]}$ defined in *[8, Section 4]*, governing the effective displacement in the fibers if $i = 2$, in the grain-like inclusions if $i = 3$. The couple $(\boldsymbol{v}^{[i]}, \frac{1}{\varepsilon}\boldsymbol{r}^{[i]})$ represents the reduction elements (on the principal axes of the fibers or at the geometrical center of gravity of the inclusions) of an helicoidal vector field locally approximating the displacement in the fibers or the grain-like inclusions.
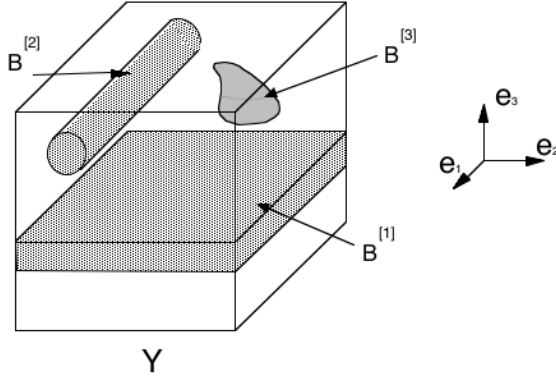

FIG. 3.2.

REMARK 3.18. *The solution to (3.44) is actually independent of y_1, y_2. We have $\boldsymbol{u}_0(x,y) = \check{\boldsymbol{u}}_0(x, y_3)$, where, setting $I := \left(-\frac{1}{2}, \frac{1}{2}\right)$, $\check{\boldsymbol{u}}_0(x, y_3)$ is the unique solution to*

$$(\mathcal{P}_{soft}^{hom}): \begin{cases} \rho \dfrac{\partial^2 \check{\boldsymbol{u}}_0}{\partial t^2} - \begin{pmatrix} \mu & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \lambda + 2\mu \end{pmatrix} \dfrac{\partial^2 \check{\boldsymbol{u}}_0}{\partial y_3^2} = \rho \boldsymbol{f} & in \ \Omega \times (0,T) \times I \setminus \left[-\dfrac{\vartheta}{2}, \dfrac{\vartheta}{2}\right], \\[3mm] \boldsymbol{v}(x,t) = \check{\boldsymbol{u}}_0(x,t,y_3) & in \ \Omega \times (0,T) \times \left[-\dfrac{\vartheta}{2}, \dfrac{\vartheta}{2}\right], \\[3mm] \dfrac{\partial \check{\boldsymbol{u}}_0}{\partial y_3}(., \dfrac{1}{2}) = \dfrac{\partial \check{\boldsymbol{u}}_0}{\partial y_3}(., -\dfrac{1}{2}), \\[3mm] \check{\boldsymbol{u}}_0 \in C([0,T]; L^2(\Omega, H^1_\sharp(I; \mathbb{R}^3))) \cap C^1([0,T]; L^2(\Omega \times I; \mathbb{R}^3)), \\[3mm] \check{\boldsymbol{u}}_0(0)1_{I \setminus [-\frac{\vartheta}{2}, \frac{\vartheta}{2}]} = \boldsymbol{a}_0 1_{I \setminus [-\frac{\vartheta}{2}, \frac{\vartheta}{2}]}, \quad \dfrac{\partial \check{\boldsymbol{u}}_0}{\partial t}(0)1_{I \setminus [-\frac{\vartheta}{2}, \frac{\vartheta}{2}]} = \boldsymbol{b}_0 1_{I \setminus [-\frac{\vartheta}{2}, \frac{\vartheta}{2}]}. \end{cases}$$

We retained the dependence on y_1, y_2 in (3.44) in order to extend our results to such multiphase media as described in Remark 3.17 and derive (3.50).

REMARK 3.19 (Equilibrium equations). *The conclusions of [8, Corollary 5.1] remain valid in the stratified case: if the solution to the equilibrium problem (3.32) two-scale converges to $\boldsymbol{u}_0 \in L^2(\Omega \times Y; \mathbb{R}^3)$, then $\boldsymbol{u}_0$ is a solution to the problem*

$$(3.51) \qquad \boldsymbol{u}_0 \in V \quad and \quad a(\boldsymbol{u}_0, \boldsymbol{w}_0) = (\boldsymbol{f}, \boldsymbol{w}_0)_H, \quad \forall \boldsymbol{w}_0 \in V,$$

where V, H, $a(.,.)$ are specified in Remark 3.16. The form $a(.,.)$ may fail to be coercive on $L^2(\Omega \times Y; \mathbb{R}^3)$, and problem (3.51) may then have no solution. The same conditions stated in Remark 6.3 for Theorem 3.7 also ensure the coercivity of $a(.,.)$ in the present context. Coerciveness is also achieved in the composite described in Remark 3.17 provided the Lamé coefficients of the fibers are at least of order $\frac{1}{\varepsilon^2}$ (see [8, Proposition 5.2]). Note in passing that one should substitute 1 for ρ_ε in [8, Formula 5.1], otherwise the proof of "(iii) $\Rightarrow$ (iv)" in [8, p. 2552] is false.

REMARK 3.20. *Under Assumption (3.34), the slightest perturbation of periodicity leads to a complete change of the form of the effective problem. For instance, if $m \in \mathbb{N}$ and $\omega_\varepsilon^j = \varepsilon \left(j + \frac{1}{2}\right)$ if j is a multiple of m, and $\omega_\varepsilon^j = \varepsilon j$ otherwise in (3.35), then the limit problem is a system of equations coupling $\boldsymbol{u}_0$ with m auxiliary variables $\boldsymbol{v}^{[1]}, ..., \boldsymbol{v}^{[m]}$ as described in Remark 3.16, which can not be expressed, as in theorems 3.1, 3.7, simply in terms of the function n defined by (3.16). The extension of Theorem 3.12 to the non-periodic case is far beyond the scope of this paper.*

4. Two-scale convergence and other analysis tools. In Section 4.1, we recall some properties of the two-scale convergence of G. Allaire [3] and G. Nguetseng [41] and reproduce some statements of [8] in a suitable form for the present context. In Section 4.2, we introduce a non-periodic notion of two-scale convergence with respect to a sequence of measures and establish a compactness result (Lemma 4.2). Two classical analysis results are recalled in Section 4.3.

4.1. Two-scale convergence. A sequence (f_ε) in $L^2(0, T; L^2(\Omega))$ is said to two-scale converge to $f_0 \in L^2(0, T; L^2(\Omega \times Y))$ with respect to x if, for all $\varphi_0 \in \mathcal{D}(\Omega \times (0, T), C_\sharp^\infty(Y))$,

$$(4.1) \qquad \lim_{\varepsilon \to 0} \int_{\Omega \times (0,T)} f_\varepsilon(x, t) \varphi_0\left(x, t, \frac{x}{\varepsilon}\right) dxdt = \int_{\Omega \times (0,T) \times Y} f_0 \varphi_0 \, dxdtdy,$$

$$\text{(notation: } f_\varepsilon \rightharpoonup f_0).$$

A sequence $(\varphi_\varepsilon) \subset L^2(0, T; L^2(\Omega))$ strongly two-scale converges to $\varphi_0 \in L^2(0, T; L^2(\Omega \times Y))$ with respect to x if

$$(4.2) \qquad \varphi_\varepsilon \rightharpoonup \varphi_0 \quad and \quad \lim_{\varepsilon \to 0} ||\varphi_\varepsilon||_{L^2(0,T;L^2(\Omega))} = ||\varphi_0||_{L^2(0,T;L^2(\Omega \times Y))},$$

$$\text{(notation: } \varphi_\varepsilon \longrightarrow\!\!\!\!\rightarrow \varphi_0).$$

The symbols $\rightharpoonup$ and $\longrightarrow\!\!\!\!\rightarrow$ will also denote the two-scale convergence and the strong two-scale convergence of sequences (f_ε) in $L^2(\Omega)$ independent of t, defined by formally considering them as constant in t. Any bounded sequence in $L^2(0, T; L^2(\Omega))$ has a

two-scale convergent subsequence [41]. An admissible sequence with respect to two-scale convergence is a sequence $(\varphi_\varepsilon) \subset L^2(0,T;L^2(\Omega))$ that two-scale converges to some $\varphi_0 \in L^2(0,T;L^2(\Omega \times Y))$ and such that, for every two-scale convergent sequence (f_ε),

$$(4.3) \qquad \lim_{\varepsilon \to 0} \int_{\Omega \times (0,T)} f_\varepsilon \varphi_\varepsilon dx dt = \int_{\Omega \times (0,T) \times Y} f_0 \varphi_0 dx dt dy.$$

A sequence (φ_ε) is admissible if and only if it strongly two-scale converges to some $\boldsymbol{\varphi_0}$ (see [8, p.2528]). For all $\psi_0 \in L^2(0,T;L^2(\Omega,C^\infty_\sharp(Y))) \cup L^2_\sharp(Y,C(\overline{\Omega \times (0,T)}))$, the sequence $(\psi_0(x,t,\frac{x}{\varepsilon}))_{\varepsilon > 0}$ strongly two-scale converges to ψ_0 (see [3], Lemma 5.2, Corollary 5.4). In particular, if Q is a Borel subset of Y, and $Q_\sharp$ is its periodization on $\mathbb{R}^3$ defined by (2.2), then the sequence $\left(\mathbb{1}_{Q_\sharp}\left(\frac{x}{\varepsilon}\right)\right)$ strongly two-scale converges to $\mathbb{1}_Q(y)$. Under (3.35), if $\vartheta > 0$, then $|\mathbb{1}_{\Omega \setminus B_\varepsilon} - \mathbb{1}_{(Y \setminus B)_\sharp}\left(\frac{x}{\varepsilon}\right)|_{L^2(\Omega)} \to 0$, therefore $\mathbb{1}_{\Omega \setminus B_\varepsilon} \longrightarrow\!\!\!\!\!\rightarrow \mathbb{1}_{Y \setminus B}$. If $\vartheta = 0$, $(\mathbb{1}_{\Omega \setminus B_\varepsilon})$ strongly converges to 1 in $L^2(\Omega)$, hence strongly two-scale converges to 1. We deduce that (see (3.37))

$$(4.4) \qquad \mathbb{1}_{\Omega \setminus B_\varepsilon} \longrightarrow\!\!\!\!\!\rightarrow \mathbb{1}_{Y \setminus A}, \quad \mathbb{1}_{B_\varepsilon} \longrightarrow\!\!\!\!\!\rightarrow \mathbb{1}_A.$$

The next Lemma is a straightforward variant of [8, Lemma 6.1].

LEMMA 4.1. *(i) Let (h_ε) be a bounded sequence in $L^\infty(\Omega \times (0,T) \times Y)$ such that $h_\varepsilon \longrightarrow\!\!\!\!\!\rightarrow h_0$. Then, for every sequence $(\chi_\varepsilon) \subset L^2(0,T;L^2(\Omega))$, the following implications hold:*

$$(4.5) \qquad \chi_\varepsilon \longrightarrow\!\!\!\!\!\rightarrow \chi_0 \quad \Longrightarrow \quad \chi_\varepsilon h_\varepsilon \longrightarrow\!\!\!\!\!\rightarrow \chi_0 h_0,$$

$$(4.6) \qquad \chi_\varepsilon \rightharpoonup\!\!\!\!\!\rightharpoonup \chi_0 \quad \Longrightarrow \quad \chi_\varepsilon h_\varepsilon \rightharpoonup\!\!\!\!\!\rightharpoonup \chi_0 h_0.$$

(ii) If (f_ε) is bounded in $L^\infty(0,T;L^2(\Omega))$, then (f_ε) two-scale converges, up to a subsequence, to some $f_0 \in L^\infty(0,T;L^2(\Omega \times Y))$. If in addition (f_ε) is bounded in $W^{1,\infty}(0,T;L^2(\Omega))$, then $f_0 \in W^{1,\infty}(0,T;L^2(\Omega \times Y))$ and $\left(\frac{\partial f_\varepsilon}{\partial t}\right)$ two-scale converges to $\frac{\partial f_0}{\partial t}$. Besides, if $f_\varepsilon(0) \rightharpoonup\!\!\!\!\!\rightharpoonup a_0$, then $a_0 = f_0(0)$ and $f_\varepsilon(\tau) \rightharpoonup\!\!\!\!\!\rightharpoonup f_0(\tau)$, $\forall \tau \in [0,T]$. Furthermore, if $\left(\frac{\partial f_\varepsilon}{\partial t}\right) \longrightarrow\!\!\!\!\!\rightarrow \frac{\partial f_0}{\partial t}$ and $f_\varepsilon(0) \longrightarrow\!\!\!\!\!\rightarrow a_0$, then $f_\varepsilon(\tau) \longrightarrow\!\!\!\!\!\rightarrow f_0(\tau)$, $\forall \tau \in [0,T]$.

4.2. Two-scale convergence with respect to (m_ε). One can easily check that the sequence (m_ε) defined by (3.39) is bounded in $\mathcal{M}(\overline{\Omega})$ and satisfies

$$(4.7) \qquad m_\varepsilon \overset{*}{\rightharpoonup} n\mathcal{L}^3_{\lfloor \Omega} \quad \text{weakly* in } \mathcal{M}(\overline{\Omega}),$$

where n is defined by (3.16). Notice that

$$(4.8) \qquad n = 1 \quad \text{under} \quad (3.35).$$

In what follows, the symbol $L^2_n(\Omega;\mathbb{R}^3)$ stands for the set of all Borel fields $\boldsymbol{w} : \Omega \to \mathbb{R}^3$ such that $\int_\Omega |\boldsymbol{w}|^2 n dx < +\infty$. Similarly, for any Hilbert space H, we denote by $L^2_n(0,L;H)$ the set of all Borel fields $\boldsymbol{w} : (0,L) \to H$ such that $\int_0^L |\boldsymbol{w}|^2_H n dx < +\infty$. We set (see (3.3), (3.4))

$$(4.9) \qquad y_\varepsilon(z) := \sum_{j \in J_\varepsilon} (z - \omega^j_\varepsilon) \mathbb{1}_{\omega^j_\varepsilon + r_\varepsilon(1+\delta)I}(z).$$

We say that a sequence (f_ε) in $L^2(0, T; L^2(\Omega))$ two-scale converges to $f_0 \in L^2(0, T; L_n^2(\Omega \times I))$ with respect to the sequence of measures (m_ε) if for each $\psi \in \mathcal{D}(\Omega \times (0, T); C_\sharp^\infty(I))$, the following holds

$$(4.10) \quad \lim_{\varepsilon \to 0} \int_{\Omega \times (0,T)} f_\varepsilon(x, t)\psi\left(x, t, \frac{y_\varepsilon(x_3)}{r_\varepsilon}\right) dm_\varepsilon dt = \int_{\Omega \times (0,T) \times I} f_0 \psi n \, dx \, dt \, dy_3.$$

(Notation: $f_\varepsilon \xrightarrow{m_\varepsilon} f_0$).

LEMMA 4.2. *Let (f_ε) be a sequence in $L^2(0, T; L^2(\Omega))$ satisfying*

$$(4.11) \quad \sup_{\varepsilon > 0} \int_{\Omega \times (0,T)} |f_\varepsilon|^2 dm_\varepsilon dt < +\infty.$$

Then (f_ε) two-scale converges with respect to (m_ε), up to a subsequence, to some $f_0 \in L_n^2(0, T; L^2(\Omega \times I))$. In addition, if

$$(4.12) \quad \sup_{\varepsilon > 0, \tau > 0} \int_\Omega |f_\varepsilon|^2(\tau) dm_\varepsilon < +\infty,$$

then $f_0 \in L^\infty(0, T; L_n^2(\Omega \times I))$.

   Proof. By Cauchy-Schwarz Inequality and by (4.11), we have

$$
(4.13) \quad
\begin{aligned}
&\left| \int_{\Omega \times (0,T)} f_\varepsilon(x, t)\psi\left(x, t, \frac{y_\varepsilon(x_3)}{r_\varepsilon}\right) dm_\varepsilon dt \right| \\
&\qquad \leq C \left(\int_{\Omega \times (0,T)} |f_\varepsilon|^2 dm_\varepsilon dt \right)^{\frac{1}{2}} \|\psi\|_{L^\infty(\Omega \times (0,T) \times I)} \\
&\qquad \leq C \|\psi\|_{L^\infty(\Omega \times (0,T) \times I)} \quad \forall \psi \in C(\overline{\Omega \times (0,T) \times I}).
\end{aligned}
$$

Hence, by the Riesz representation theorem, for each $\varepsilon > 0$ there exists a finite Radon measure $\theta_\varepsilon \in \mathcal{M}(\overline{\Omega \times (0,T) \times I})$ such that

$$(4.14) \quad \int \psi d\theta_\varepsilon = \int_{\Omega \times (0,T)} f_\varepsilon(x, t)\psi\left(x, t, \frac{y_\varepsilon(x_3)}{r_\varepsilon}\right) dm_\varepsilon dt \quad \forall \psi \in C(\overline{\Omega \times (0,T) \times I}).$$

By (4.13) and (4.14), the sequence (θ_ε) is bounded in $\mathcal{M}(\overline{\Omega \times (0,T) \times I})$, thus weakly* converges, up to a subsequence, to some $\theta \in \mathcal{M}(\overline{\Omega \times (0,T) \times I})$. By Cauchy-Schwarz inequality, we have

$$(4.15) \quad \left| \int \psi d\theta_\varepsilon \right| \leq \left(\int_{\Omega \times (0,T)} |f_\varepsilon|^2 dm_\varepsilon dt \right)^{\frac{1}{2}} \left(\int_{\Omega \times (0,T)} \left|\psi\left(x, t, \frac{y_\varepsilon(x_3)}{r_\varepsilon}\right)\right|^2 dm_\varepsilon dt \right)^{\frac{1}{2}}.$$

The proof of the next statement is similar to that of [3, Lemma 1.3]:

$$(4.16) \quad \lim_{\varepsilon \to 0} \int_\Omega \left|\varphi\left(x, \frac{y_\varepsilon(x_3)}{r_\varepsilon}\right)\right|^2 dm_\varepsilon = \int_{\Omega \times I} |\varphi|^2 n \, dx \, dy_3 \quad \forall \varphi \in C(\overline{\Omega \times I}).$$

We deduce from (4.11), (4.14), (4.15), (4.16), and from the weak* convergence in $\mathcal{M}(\overline{\Omega \times (0,T) \times I})$ of (θ_ε) to θ, that

$$\left| \int \psi d\theta \right| = \lim_{\varepsilon \to 0} \left| \int \psi d\theta_\varepsilon \right| \leq C \|\psi\|_{L^2_n(\Omega \times (0,T) \times I)} \quad \forall \psi \in C(\overline{\Omega \times (0,T) \times I}).$$

Thus, the linear form $\psi \to \int \psi d\theta$ is continuous on $C(\overline{\Omega \times (0,T) \times I})$ with respect to the strong topology of $L^2_n(\Omega \times (0,T) \times I)$. By a density argument, this linear form can be extended to a continuous linear form on $L^2_n(\Omega \times (0,T) \times I)$ which, by the Riesz representation theorem, takes the form $\psi \to \int_{\Omega \times (0,T) \times I} \psi f_0 n dx dt dy$ for some $f_0 \in L^2_n(\Omega \times (0,T) \times I)$. We infer that $\theta = n f_0 \mathcal{L}^5_{\lfloor \Omega \times (0,T) \times I}$, and then, taking (4.14) and the weak* convergence of (θ_ε) to θ into account, deduce (4.10). Under (4.12), by Fubini's Theorem and Cauchy-Schwarz inequality, we have

$$\int_{\Omega \times (0,T)} f_\varepsilon(x,t) \psi \left(x, t, \frac{y_\varepsilon(x_3)}{r_\varepsilon} \right) dm_\varepsilon dt \leq C \int_{(0,T)} \left(\int \left| \psi \left(x, \tau, \frac{y_\varepsilon(x_3)}{r_\varepsilon} \right) \right|^2 dm_\varepsilon \right)^{\frac{1}{2}} d\tau.$$

By passing to the limit as $\varepsilon \to 0$ in the last inequality, thanks to (4.16) and to the Dominated Convergence Theorem, we get $\int_{\Omega \times (0,T) \times I} f_0 \psi n dx dt dy_3 \leq C|\psi|_{L^1(0,T;L^2_n(\Omega \times I))}$ and deduce, by the arbitrary choice of ψ, that $f_0 \in L^\infty(0,T; L^2_n(\Omega \times I))$. □

4.3. Two classical results. For the reader's convenience, we reproduce below Lemma A2 of [11] (see also [17] for a more general version) and Theorem 6.2 of [8], which collects some abstract results proved in [25, 35, 36]. The lemma will be employed to identify the limit of the sequence $(\boldsymbol{u}_\varepsilon m_\varepsilon)$, where $\boldsymbol{u}_\varepsilon$ is the solution to (3.5). The theorem will be applied to check the existence, the uniqueness, and some regularity properties of the solution to Problem (3.5) and of the associated limit problems. In what follows, we denote by E' the continuous dual space of a Banach space E and by $\sigma(E', E)$ the weak* topology on E'.

LEMMA 4.3. *Let K be a compact subset of $\mathbb{R}^N$ and (θ_ε) a bounded sequence of positive Radon measures on K, weakly* converging in $\mathcal{M}(K)$ to some $\theta \in \mathcal{M}(K)$. Let (f_ε) be a sequence of θ_ε-measurable functions such that $\sup_\varepsilon \int |f_\varepsilon|^2 d\theta_\varepsilon < +\infty$. Then the sequence $(f_\varepsilon \theta_\varepsilon)$ is sequentially relatively compact in the weak* topology $\sigma(\mathcal{M}(K), C(K))$ and every cluster point is of the form $f\theta$, with $f \in L^2_\theta$. Moreover, if $f_\varepsilon \theta_\varepsilon \overset{\star}{\rightharpoonup} f\theta$, then*

$$(4.17) \qquad\qquad \liminf_{\varepsilon \to 0} \int |f_\varepsilon|^2 d\theta_\varepsilon \geq \int |f|^2 d\theta.$$

THEOREM 4.4. *Let V and H be separable Hilbert spaces such that $V \subset H = H' \subset V'$, with continuous and dense imbeddings. Let $\|.\|_V$, $|.|_H$, $((.,.))_V$, $(.,.)_H$ denote their respective norm and inner product. Let $a : V \times V \to \mathbb{R}$ be a continuous bilinear symmetric form on V. Let $A \in \mathcal{L}(V, V')$ be defined by $a(\xi, \widetilde{\xi}) = (A\xi, \widetilde{\xi})_{(V',V)}, \ \forall \, (\xi, \widetilde{\xi}) \in V^2$. Assume that*

$$(4.18) \qquad \exists (\lambda, \alpha) \in [0, +\infty) \times (0, +\infty), \quad a(\xi, \xi) + \lambda|\xi|^2_H \geq \alpha\|\xi\|^2_V, \ \forall \, \xi \in V.$$

Let $h \in L^2(0, T; H)$, $\xi_0 \in V$, $\xi_1 \in H$. Then there exists a unique solution ξ to

(4.19)
$$A\xi(t) + \frac{\partial^2 \xi}{\partial t^2}(t) = h(t), \quad \xi \in L^2(0, T; V),$$
$$\frac{\partial \xi}{\partial t} \in L^2(0, T; H), \quad \xi(0) = \xi_0, \quad \frac{\partial \xi}{\partial t}(0) = \xi_1.$$

Furthermore, we have

(4.20) $\quad \xi \in C([0, T]; V) \cap C^1([0, T]; H), \quad \dfrac{\partial \xi}{\partial t} \in L^2(0, T; V), \quad \dfrac{\partial^2 \xi}{\partial t^2} \in L^2(0, T; V').$

Besides, setting

(4.21) $\quad e(\tau) := \dfrac{1}{2}\left[\left(\dfrac{\partial \xi}{\partial t}(\tau), \dfrac{\partial \xi}{\partial t}(\tau)\right)_H + a(\xi(\tau), \xi(\tau))\right], \; \forall\, \tau \in [0, T],$

the following holds

(4.22) $\quad e(\tau) = e(0) + \displaystyle\int_0^\tau \left(h, \dfrac{\partial \xi}{\partial t}\right)_H dt, \qquad \forall\, \tau \in [0, T].$

Problem (4.19) is equivalent to

(4.23)
$$\int_0^T \left(a(\xi(t), \widetilde{\xi})\eta(t) + (\xi(t), \widetilde{\xi})_H \frac{\partial^2 \eta}{\partial t^2}(t)\right) dt + (\xi_0, \widetilde{\xi})_H \frac{\partial \eta}{\partial t}(0)$$
$$- (\xi_1, \widetilde{\xi})_H \eta(0) = \int_0^T (h, \widetilde{\xi})_H \eta(t) dt,$$
$$\forall\, \widetilde{\xi} \in V, \quad \forall\, \eta \in \mathcal{D}(]-\infty, T[); \quad \xi \in L^2(0, T; V), \frac{\partial \xi}{\partial t} \in L^2(0, T; H).$$

5. Asymptotic behavior of the solution to (3.5). In this section, we establish a series of estimates satisfied by the solution $\boldsymbol{u}_\varepsilon$ to (3.5) (see Proposition 5.2), and investigate in lemmas 5.3, 5.4 , and 5.5, the asymptotic behavior of sequences satisfying such estimates. These results are synthetized in Corollary 5.6. We start with a key inequality.

LEMMA 5.1. *We have*

(5.1) $\quad \displaystyle\int \left(\left|\dfrac{\varphi_1}{r_\varepsilon}\right|^2 + \left|\dfrac{\varphi_2}{r_\varepsilon}\right|^2 + |\varphi_3|^2\right) dm_\varepsilon \le \dfrac{C}{r_\varepsilon^2} \int |\boldsymbol{e}(\boldsymbol{\varphi})|^2 \, dm_\varepsilon \quad \forall \boldsymbol{\varphi} \in H_0^1(\Omega; \mathbb{R}^3).$

Proof. By (3.4) and (3.39), it is sufficient to show that for all $j \in J_\varepsilon$, and all $\boldsymbol{\varphi} \in H^1(B_\varepsilon^j; \mathbb{R}^3)$ such that $\boldsymbol{\varphi} = 0$ on $\partial B_\varepsilon^j \cap \partial\Omega$,

(5.2) $\quad \displaystyle\int_{B_\varepsilon^j} \left(\left|\dfrac{\varphi_1}{r_\varepsilon}\right|^2 + \left|\dfrac{\varphi_2}{r_\varepsilon}\right|^2 + |\varphi_3|^2\right) dx \le \dfrac{C}{r_\varepsilon^2} \int_{B_\varepsilon^j} |\boldsymbol{e}(\boldsymbol{\varphi})|^2 \, dx.$

By Korn's inequality, we have

$$\int_{\Omega' \times \left(-\frac{1}{2};\frac{1}{2}\right)} |\boldsymbol{\psi}|^2 dy \le C \int_{\Omega' \times \left(-\frac{1}{2};\frac{1}{2}\right)} |\boldsymbol{e}(\boldsymbol{\psi})|^2 dy \qquad \forall \boldsymbol{\psi} \in W,$$

$$W := \left\{ \boldsymbol{\psi} \in H^1 \left(\Omega' \times \left(-\frac{1}{2};\frac{1}{2}\right); \mathbb{R}^3 \right), \quad \boldsymbol{\psi} = 0 \text{ on } \partial\Omega' \times \left(-\frac{1}{2};\frac{1}{2}\right) \right\}.$$

By making the change of variable $y = (x_1, x_2, \frac{x_3 - \omega_\varepsilon^j}{r_\varepsilon})$, we get, for all $\boldsymbol{\psi} \in W$,

$$(5.3) \qquad \int_{B_\varepsilon^j} |\boldsymbol{\psi}|^2 \left(x_1, x_2, \frac{x_3 - \omega_\varepsilon^j}{r_\varepsilon} \right) dx \le C \int_{B_\varepsilon^j} |\boldsymbol{e}(\boldsymbol{\psi})|^2 \left(x_1, x_2, \frac{x_3 - \omega_\varepsilon^j}{r_\varepsilon} \right) dx.$$

Setting $\boldsymbol{\varphi}(x) = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \frac{1}{r_\varepsilon}\psi_3 \end{pmatrix} \left(x_1, x_2, \frac{x_3 - \omega_\varepsilon^j}{r_\varepsilon} \right)$, a straightforward computation yields

$$\int_{B_\varepsilon^j} \left(|\varphi_1|^2 + |\varphi_2|^2 + r_\varepsilon^2 |\varphi_3|^2 \right)(x)dx = \int_{B_\varepsilon^j} |\boldsymbol{\psi}|^2 \left(x_1, x_2, \frac{x_3 - \omega_\varepsilon^j}{r_\varepsilon} \right) dx$$

$$\le C \int_{B_\varepsilon^j} |\boldsymbol{e}(\boldsymbol{\psi})|^2 \left(x_1, x_2, \frac{x_3 - \omega_\varepsilon^j}{r_\varepsilon} \right) dx \le C \int_{B_\varepsilon^j} |\boldsymbol{e}(\boldsymbol{\varphi})|^2 (x)dx.$$

The inequality (5.2) is proved. $\square$

PROPOSITION 5.2. *There exists a unique solution $\boldsymbol{u}_\varepsilon$ to (3.5). Moreover,*

$$(5.4) \qquad \frac{\partial \boldsymbol{u}_\varepsilon}{\partial t} \in L^2(0, T; H_0^1(\Omega; \mathbb{R}^3)), \quad \frac{\partial^2 \boldsymbol{u}_\varepsilon}{\partial t^2} \in L^2(0, T; H^{-1}(\Omega; \mathbb{R}^3)).$$

Under (3.6), (3.9), there exists a constant $C > 0$ such that

$$\int_\Omega \mu_{0\varepsilon} |\boldsymbol{e}(\boldsymbol{u}_\varepsilon)|^2 (\tau)dx + \int_\Omega \left(\rho_\varepsilon \left| \frac{\partial \boldsymbol{u}_\varepsilon}{\partial t} \right|^2 + |\boldsymbol{u}_\varepsilon|^2 \right)(\tau)dx \le C \qquad \forall \tau \in [0, T],$$

$$(5.5) \qquad \int |\boldsymbol{e}(\boldsymbol{u}_\varepsilon)|^2 (\tau) + |\boldsymbol{u}_\varepsilon'|^2 (\tau)dm_\varepsilon \le C \left(\frac{r_\varepsilon}{\varepsilon} \mu_{1\varepsilon} \right)^{-1} \qquad \forall \tau \in [0, T],$$

$$\int |u_{\varepsilon 3}|^2 (\tau)dm_\varepsilon \le C \left(\frac{r_\varepsilon^3}{\varepsilon} \mu_{1\varepsilon} \right)^{-1}, \qquad \int |\boldsymbol{u}_\varepsilon|^2 (\tau)dm_\varepsilon \le C \quad \forall \tau \in [0, T].$$

Proof. Problem (3.5) is equivalent to (4.23), where $H := L^2(\Omega; \mathbb{R}^3)$, $(\boldsymbol{\xi}, \widetilde{\boldsymbol{\xi}})_H :=$ $\int_\Omega \rho_\varepsilon \boldsymbol{\xi} \cdot \widetilde{\boldsymbol{\xi}} dx$, $V := H_0^1(\Omega; \mathbb{R}^3)$ $(V' = H^{-1}(\Omega; \mathbb{R}^3))$, $a(\boldsymbol{\xi}, \widetilde{\boldsymbol{\xi}}) := \int_\Omega \boldsymbol{\sigma}_\varepsilon(\boldsymbol{\xi}) : \boldsymbol{e}(\widetilde{\boldsymbol{\xi}})dx$, and $(\boldsymbol{\xi}_0, \boldsymbol{\xi}_1, \boldsymbol{h}) = (\boldsymbol{a}_0, \boldsymbol{b}_0, \boldsymbol{f})$. By (3.7), $(H, (.,.)_H)$ is a Hilbert space and the assumptions of Theorem 4.4 are satisfied. Therefore, Problem (3.5) has a unique solution. Assertion (5.4) follows from (4.20). By (4.22) we have, for all $\tau \in [0, T]$,

$$\frac{1}{2} \int_\Omega \left(\rho_\varepsilon \left| \frac{\partial \boldsymbol{u}_\varepsilon}{\partial t} \right|^2 + \boldsymbol{\sigma}_\varepsilon(\boldsymbol{u}_\varepsilon) : \boldsymbol{e}(\boldsymbol{u}_\varepsilon) \right)(\tau)dx$$

$$= \frac{1}{2} \int_\Omega \left(\rho_\varepsilon |\boldsymbol{b}_0|^2 + \boldsymbol{\sigma}_\varepsilon(\boldsymbol{a}_0) : \boldsymbol{e}(\boldsymbol{a}_0) \right) dx + \int_{\Omega \times (0,\tau)} \rho_\varepsilon \boldsymbol{f} \cdot \frac{\partial \boldsymbol{u}_\varepsilon}{\partial t} dxdt.$$

We deduce from Cauchy-Schwarz Inequality that

$$\left| \int_{\Omega \times (0,\tau)} \rho_\varepsilon \boldsymbol{f} \cdot \frac{\partial \boldsymbol{u}_\varepsilon}{\partial t} dx dt \right| \leq \sqrt{\int_{\Omega \times (0,T)} \rho_\varepsilon |\boldsymbol{f}|^2 dx dt} \sqrt{\int_{\Omega \times (0,T)} \rho_\varepsilon \left| \frac{\partial \boldsymbol{u}_\varepsilon}{\partial t} \right|^2 dx dt}.$$

Taking (3.9) into account, we infer

(5.6)
$$\int_\Omega \left(\rho_\varepsilon \left| \frac{\partial \boldsymbol{u}_\varepsilon}{\partial t} \right|^2 dx + \boldsymbol{\sigma}_\varepsilon(\boldsymbol{u}_\varepsilon) : \boldsymbol{e}(\boldsymbol{u}_\varepsilon) \right)(\tau) dx$$
$$\leq C \left(1 + \sqrt{\int_{\Omega \times (0,T)} \rho_\varepsilon \left| \frac{\partial \boldsymbol{u}_\varepsilon}{\partial t} \right|^2 dx dt} \right) \qquad \forall \tau \in [0, T].$$

By integrating (5.6) with respect to τ over $(0, T)$, we deduce that $\int_{\Omega \times (0,T)} \rho_\varepsilon \left| \frac{\partial \boldsymbol{u}_\varepsilon}{\partial t} \right|^2 dx dt \leq C$ and then, coming back to (5.6), that

(5.7)
$$\int_\Omega \rho_\varepsilon \left| \frac{\partial \boldsymbol{u}_\varepsilon}{\partial t} \right|^2 (\tau) dx + \int_\Omega \boldsymbol{\sigma}_\varepsilon(\boldsymbol{u}_\varepsilon) : \boldsymbol{e}(\boldsymbol{u}_\varepsilon)(\tau) dx \leq C \qquad \forall \tau \in [0, T].$$

We infer from (3.5), (3.6), (3.39), and (5.7), that

(5.8)
$$\int_\Omega \rho_\varepsilon \left| \frac{\partial \boldsymbol{u}_\varepsilon}{\partial t} \right|^2 (\tau) + \mu_{0\varepsilon} |\boldsymbol{e}(\boldsymbol{u}_\varepsilon)|^2 (\tau) dx \leq C \qquad \forall \tau \in [0, T],$$
$$\int |\boldsymbol{e}(\boldsymbol{u}_\varepsilon)|^2 (\tau) dm_\varepsilon \leq C \left(\frac{r_\varepsilon}{\varepsilon} \mu_{1\varepsilon} \right)^{-1} \qquad \forall \tau \in [0, T].$$

By (3.7), (3.39), and (5.8), and by the continuity of $\boldsymbol{a}_0$ (see (3.5)), we have

(5.9)
$$\int_\Omega |\boldsymbol{u}_\varepsilon|^2 (\tau) dx + \int |\boldsymbol{u}_\varepsilon|^2 (\tau) dm_\varepsilon = \int_\Omega \left| \boldsymbol{a}_0 + \int_0^\tau \frac{\partial \boldsymbol{u}_\varepsilon}{\partial t}(t) dt \right|^2 d(\mathcal{L}^3 + m_\varepsilon)(x)$$
$$\leq C \left(1 + \int_{\Omega \times (0,T)} \rho_\varepsilon \left| \frac{\partial \boldsymbol{u}_\varepsilon}{\partial t} \right|^2 (t) dx dt \right) \leq C \qquad \forall \tau \in [0, T].$$

By (5.1) and (5.8), we have

$$\int \left| \frac{u_{\varepsilon 1}}{r_\varepsilon} \right|^2 + \left| \frac{u_{\varepsilon 2}}{r_\varepsilon} \right|^2 + |u_{\varepsilon 3}|^2 (\tau) dm_\varepsilon \leq \frac{C}{r_\varepsilon^2} \int |\boldsymbol{e}(\boldsymbol{u}_\varepsilon)|^2 (\tau) dm_\varepsilon \leq \frac{C\varepsilon}{r_\varepsilon^3 \mu_{1\varepsilon}} \qquad \forall \tau \in [0, T],$$

which, combined with (5.8), (5.9) completes the proof of (5.5). $\square$

LEMMA 5.3. *Let $(\boldsymbol{u}_\varepsilon)$ be a sequence in $W^{1,\infty}(0, T; H_0^1(\Omega; \mathbb{R}^3), L^2(\Omega; \mathbb{R}^3))$ satisfying*

(5.10)
$$\sup_{\varepsilon > 0, \tau \in (0,T)} \int |\boldsymbol{u}_\varepsilon|^2 + \left| \frac{\partial \boldsymbol{u}_\varepsilon}{\partial t} \right|^2 + |\boldsymbol{e}(\boldsymbol{u}_\varepsilon)|^2 (\tau) dm_\varepsilon < +\infty.$$

Then, up to a subsequence, the following holds

$$
\boldsymbol{u}_\varepsilon m_\varepsilon \overset{\star}{\rightharpoonup} n\boldsymbol{v}, \quad \frac{\partial \boldsymbol{u}_\varepsilon}{\partial t} m_\varepsilon \overset{\star}{\rightharpoonup} n\frac{\partial \boldsymbol{v}}{\partial t} \qquad \textit{weakly* in } \ L^\infty(0,T;\mathcal{M}(\overline{\Omega};\mathbb{R}^3)),
$$

(5.11) $\quad v_1, v_2 \in L^\infty(0,T; L_n^2(0,L; H_0^1(\Omega'))), \quad \boldsymbol{v} \in W^{1,\infty}(0,T; L_n^2(\Omega;\mathbb{R}^3)),$

$$
\boldsymbol{e}_{x'}(\boldsymbol{u}_\varepsilon')m_\varepsilon \overset{\star}{\rightharpoonup} n\boldsymbol{e}_{x'}(\boldsymbol{v}') \qquad \textit{weakly* in } \ L^\infty(0,T;\mathcal{M}(\overline{\Omega};\mathbb{S}^3)).
$$

Furthermore,

$$
nv_1 = nv_2 = 0 \qquad \textit{if} \qquad \liminf_{\varepsilon\to 0}\ \sup_{\tau\in(0,T)}\int |\boldsymbol{e}(\boldsymbol{u}_\varepsilon)|^2\,(\tau)dm_\varepsilon = 0,
$$

(5.12)

$$
n\boldsymbol{v} = 0 \qquad \textit{if} \qquad \liminf_{\varepsilon\to 0}\ \sup_{\tau\in(0,T)}\int \left|\frac{1}{r_\varepsilon}\boldsymbol{e}(\boldsymbol{u}_\varepsilon)\right|^2\,(\tau)dm_\varepsilon = 0.
$$

Moreover, if

$$
\sup_{\varepsilon>0,\tau\in(0,T)}\int_\Omega \mu_{0\varepsilon}\,|\boldsymbol{e}(\boldsymbol{u}_\varepsilon)|^2\,(\tau)dx < +\infty, \quad \varepsilon^2 \ll \mu_{\varepsilon 0}, \quad \textit{and}
$$

(5.13)

$\quad n_\varepsilon \to n$ *strongly in* $L^2(\Omega)$ *and* $\boldsymbol{u}_\varepsilon \overset{\star}{\rightharpoonup} \boldsymbol{u}$ *weakly* in $L^\infty(0,T; L^2(\Omega;\mathbb{R}^3))$,

or if

$$
\sup_{\varepsilon>0,\tau\in(0,T)}\int_\Omega |\boldsymbol{e}(\boldsymbol{u}_\varepsilon)|^2\,(\tau)dx < +\infty, \quad \textit{and}
$$

(5.14)

$$
\boldsymbol{u}_\varepsilon \to \boldsymbol{u} \qquad \textit{strongly in } L^2(0,T; L^2(\Omega;\mathbb{R}^3)),
$$

then $n\boldsymbol{u} = n\boldsymbol{v}$.

Proof. By applying Lemma 4.3 to $\theta_\varepsilon := \mathcal{L}^1_{\lfloor(0,T)} \otimes m_\varepsilon$, $K := \overline{(0,T)\times\Omega}$, $f_\varepsilon \in \{\boldsymbol{u}_\varepsilon, \frac{\partial \boldsymbol{u}_\varepsilon}{\partial t}, \boldsymbol{e}_{x'}(\boldsymbol{u}_\varepsilon')\}$, taking (4.7) and (5.10) into account, we obtain the convergences

$$
\boldsymbol{u}_\varepsilon m_\varepsilon \overset{\star}{\rightharpoonup} n\boldsymbol{v}, \quad \frac{\partial \boldsymbol{u}_\varepsilon}{\partial t} m_\varepsilon \overset{\star}{\rightharpoonup} n\boldsymbol{\gamma} \quad \textit{weakly* in } \ \mathcal{M}(\overline{(0,T)\times\Omega};\mathbb{R}^3)),
$$

(5.15)

$$
\boldsymbol{e}_{x'}(\boldsymbol{u}_\varepsilon')m_\varepsilon \overset{\star}{\rightharpoonup} n\boldsymbol{\Xi} \qquad \textit{weakly* in } \ \mathcal{M}(\overline{(0,T)\times\Omega};\mathbb{S}^3)),
$$

up to a subsequence, for some suitable $\boldsymbol{v},\boldsymbol{\gamma} \in L^2(0,T; L_n^2(\Omega;\mathbb{R}^3)), \boldsymbol{\Xi} \in L^2(0,T; L_n^2(\Omega;\mathbb{S}^3))$ such that $\Xi_{ij} = 0$ if $3 \in \{i,j\}$. By (5.10), the sequences $(\boldsymbol{u}_\varepsilon m_\varepsilon)$, $\left(\frac{\partial \boldsymbol{u}_\varepsilon}{\partial t} m_\varepsilon\right)$ and $(\boldsymbol{e}_{x'}(\boldsymbol{u}_\varepsilon')m_\varepsilon)$ are bounded in $L^\infty(0,T;\mathcal{M}(\overline{\Omega}))$, therefore the convergences (5.15) also hold with respect to the weak* topology of $L^\infty(0,T;\mathcal{M}(\overline{\Omega}))$. Let us fix $\boldsymbol{\Psi} \in C^\infty(\overline{\Omega\times(0,T)};\mathbb{S}^2)$. By integration by parts, we have

$$
\sum_{\alpha,\beta=1}^2 \int_{\Omega\times(0,T)}(\boldsymbol{e}_{x'}(\boldsymbol{u}_\varepsilon'))_{\alpha\beta}\Psi_{\alpha\beta}\,(x,t)\,dm_\varepsilon dt = -\int_{\Omega\times(0,T)}(u_{\varepsilon 1}\boldsymbol{e}_1 + u_{\varepsilon 2}\boldsymbol{e}_2)\cdot \mathbf{div}_{x'}\boldsymbol{\Psi}\,(x,t)\,dm_\varepsilon dt.
$$

By passing to the limit as $\varepsilon \to 0$, taking (5.15) into account, we obtain

$$(5.16) \qquad \sum_{\alpha,\beta=1}^{2} \int_{\Omega \times (0,T)} \Xi_{\alpha\beta} \Psi_{\alpha\beta} n\, dxdt = -\int_{\Omega \times (0,T)} (v_1 e_1 + v_2 e_2) \cdot \mathbf{div}_{x'} \mathbf{\Psi} n\, dxdt.$$

By making $\mathbf{\Psi}$ vary in $\mathcal{D}(\Omega \times (0,T); \mathbb{S}^2)$, we deduce that $n e_{x'}(\mathbf{v}')(= e_{x'}(n\mathbf{v}')) = n\mathbf{\Xi}$ in the sense of distributions on $\Omega \times (0,T)$, and then infer from Korn inequality in $H^1(\Omega')$ that $nv_1, nv_2 \in L^{\infty}(0,T; L^2(0,L; H^1(\Omega')))$, that is $v_1, v_2 \in L^{\infty}(0,T; L^2_n(0,L; H^1(\Omega')))$. By integrating (5.16) by parts for $\mathbf{\Psi} \in C^{\infty}(\overline{\Omega \times (0,T)}; \mathbb{S}^2)$, we get $\int_0^T \int_0^L \int_{\partial \Omega'} n\mathbf{v}' \cdot \mathbf{\Psi}\boldsymbol{\nu}\, d\mathcal{H}^1 dx_3 dt = 0$ and infer from the arbitrariness of $\mathbf{\Psi}$ that $v_1, v_2 \in L^{\infty}(0,T; L^2_n(0,L; H^1_0(\Omega')))$. We prove in a similar way that $\mathbf{v} \in W^{1,\infty}(0,T; L^2_n(\Omega; \mathbb{R}^3))$ and $\boldsymbol{\gamma} = \frac{\partial \mathbf{v}}{\partial t}$. Assertion (5.11) is proved. Assertion (5.12) is a consequence of the following inequalities (holding for $\alpha \in \{1,2\}$), deduced from (4.17), (5.1), (5.15)

$$\int_{\Omega \times (0,T)} |v_\alpha|^2 n\, dxdt \leq \liminf_{\varepsilon \to 0} \int_{\Omega \times (0,T)} |u_{\varepsilon\alpha}|^2 dm_\varepsilon(x)dt \leq \liminf_{\varepsilon \to 0} \int_{\Omega \times (0,T)} |e(\mathbf{u}_\varepsilon)|^2 dm_\varepsilon(x)dt,$$

$$\int_{\Omega \times (0,T)} |\mathbf{v}|^2 n\, dxdt \leq \liminf_{\varepsilon \to 0} \int_{\Omega \times (0,T)} |\mathbf{u}_\varepsilon|^2 dm_\varepsilon(x)dt \leq \liminf_{\varepsilon \to 0} \int_{\Omega \times (0,T)} \left|\frac{1}{r_\varepsilon} e(\mathbf{u}_\varepsilon)\right|^2 dm_\varepsilon(x)dt.$$

It remains to show that under (5.13) or (5.14), $n\mathbf{u} = n\mathbf{v}$. To that aim, we set

$$(5.17) \qquad \hat{\mathbf{v}}_\varepsilon(x,t) := \sum_{j \in J_\varepsilon} \mathbf{u}_\varepsilon(x_1, x_2, \omega_\varepsilon^j, t) \mathbb{1}_{\left(\omega_\varepsilon^j - \frac{r_\varepsilon}{2}, \omega_\varepsilon^j + \frac{r_\varepsilon}{2}\right]}(x_3).$$

By Fubini's Theorem, Jensen's inequality and Korn's inequality in $H^1_0(\Omega; \mathbb{R}^3)$, we have

$$\int |\mathbf{u}_\varepsilon - \hat{\mathbf{v}}_\varepsilon|^2(\tau) dm_\varepsilon = \frac{\varepsilon}{r_\varepsilon} \sum_{j \in J_\varepsilon} \int_{\Omega'} dx' \int_{\omega_\varepsilon^j - \frac{r_\varepsilon}{2}}^{\omega_\varepsilon^j + \frac{r_\varepsilon}{2}} \left|\mathbf{u}_\varepsilon(x', x_3, \tau) - \mathbf{u}_\varepsilon(x', \omega_\varepsilon^j, \tau)\right|^2 dx_3$$

$$(5.18) \qquad \leq \frac{\varepsilon}{r_\varepsilon} \sum_{j \in J_\varepsilon} \int_{\Omega'} dx' \int_{\omega_\varepsilon^j - \frac{r_\varepsilon}{2}}^{\omega_\varepsilon^j + \frac{r_\varepsilon}{2}} \left(\int_{\omega_\varepsilon^j - \frac{r_\varepsilon}{2}}^{\omega_\varepsilon^j + \frac{r_\varepsilon}{2}} \left|\frac{\partial \mathbf{u}_\varepsilon}{\partial x_3}\right| (x', s_3, \tau) ds_3\right)^2 dx_3$$

$$\leq \varepsilon r_\varepsilon \int_{\Omega} |\boldsymbol{\nabla} \mathbf{u}_\varepsilon|^2(\tau) dx \leq C\varepsilon r_\varepsilon \int_{\Omega} |e(\mathbf{u}_\varepsilon)|^2(\tau) dx.$$

Therefore, under (5.13) or (5.14), $\lim_{\varepsilon \to 0} \int |\mathbf{u}_\varepsilon - \hat{\mathbf{v}}_\varepsilon|^2(\tau) dm_\varepsilon = 0$. Hence, by (5.15),

$$(5.19) \qquad \hat{\mathbf{v}}_\varepsilon m_\varepsilon \overset{\star}{\rightharpoonup} n\mathbf{v} \quad \text{weakly* in } L^{\infty}(0,T; \mathcal{M}(\overline{\Omega}; \mathbb{R}^3)).$$

We define (see (3.1, 3.15))

$$(5.20) \qquad \overline{\mathbf{v}}_\varepsilon(x,t) := \sum_{i \in Z_\varepsilon} \left(\sum_{\left\{\omega_\varepsilon^j \in \omega_\varepsilon \cap \left(\varepsilon i - \frac{\varepsilon}{2}, \varepsilon i + \frac{\varepsilon}{2}\right]\right\}} \mathbf{u}_\varepsilon(x', \omega_\varepsilon^j, t) \right) \mathbb{1}_{\left(\varepsilon i - \frac{\varepsilon}{2}, \varepsilon i + \frac{\varepsilon}{2}\right]}(x_3).$$

Noticing that by (3.2) and (3.15),

$$(5.21) \qquad \omega_\varepsilon \subset \bigcup_{i \in Z_\varepsilon} \left(\varepsilon i - \frac{\varepsilon}{2}, \varepsilon i + \frac{\varepsilon}{2} \right],$$

we infer from (3.39) and (5.17) that

$$
\begin{aligned}
\int_\Omega |\overline{\boldsymbol{v}}_\varepsilon|^2(\tau) dx &\le C \sum_{i \in Z_\varepsilon} \sum_{\{\omega_\varepsilon^j \in \omega_\varepsilon \cap (\varepsilon i - \frac{\varepsilon}{2}, \varepsilon i + \frac{\varepsilon}{2}]\}} \varepsilon \int_{\Omega'} |\boldsymbol{u}_\varepsilon(x', \omega_\varepsilon^j, \tau)|^2 dx' \\
&= C \frac{\varepsilon}{r_\varepsilon} \sum_{j \in J_\varepsilon} r_\varepsilon \int_{\Omega'} |\boldsymbol{u}_\varepsilon(x', \omega_\varepsilon^j, \tau)|^2 dx' = C \int |\hat{\boldsymbol{v}}_\varepsilon|^2(\tau) dm_\varepsilon.
\end{aligned}
\tag{5.22}
$$

Therefore, by (5.10) and (5.18), under (5.13) or (5.14), the sequence $(\overline{\boldsymbol{v}}_\varepsilon)$ is bounded in $L^\infty(0, T; L^2(\Omega; \mathbb{R}^3))$. Thus the following convergence holds, up to a subsequence

$$(5.23) \qquad \overline{\boldsymbol{v}}_\varepsilon \overset{\star}{\rightharpoonup} \boldsymbol{w} \quad \text{weakly* in} \quad L^\infty(0, T; L^2(\Omega; \mathbb{R}^3)).$$

To identify $\boldsymbol{w}$, we fix $\boldsymbol{\varphi} \in \mathcal{D}(\Omega \times (0, T); \mathbb{R}^3)$ and set $\breve{\boldsymbol{\varphi}}_\varepsilon(x, t) := \sum_{i \in Z_\varepsilon} \boldsymbol{\varphi}(x', \varepsilon i, t)$ $\mathbb{1}_{(\varepsilon i - \frac{\varepsilon}{2}, \varepsilon i + \frac{\varepsilon}{2}]}(x_3)$. Noticing that $|\breve{\boldsymbol{\varphi}}_\varepsilon - \boldsymbol{\varphi}|_{L^\infty(\Omega \times (0, T); \mathbb{R}^3)} \le C\varepsilon$, we infer

$$(5.24) \qquad \int_{\Omega \times (0,T)} \boldsymbol{w} \cdot \boldsymbol{\varphi} \, dx dt = \lim_{\varepsilon \to 0} \int_{\Omega \times (0,T)} \overline{\boldsymbol{v}}_\varepsilon \cdot \breve{\boldsymbol{\varphi}}_\varepsilon \, dx dt.$$

On the other hand, by (3.39), (5.17) and (5.20), we have

$$
\begin{aligned}
\int_\Omega \overline{\boldsymbol{v}}_\varepsilon \cdot \breve{\boldsymbol{\varphi}}_\varepsilon(t) dx &= \sum_{i \in Z_\varepsilon} \sum_{\{\omega_\varepsilon^j \in \omega_\varepsilon \cap (\varepsilon i - \frac{\varepsilon}{2}, \varepsilon i + \frac{\varepsilon}{2}]\}} \int_{\Omega' \times (\varepsilon i - \frac{\varepsilon}{2}, \varepsilon i + \frac{\varepsilon}{2}]} \boldsymbol{u}_\varepsilon(x', \omega_\varepsilon^j, t) \cdot \boldsymbol{\varphi}(x', \varepsilon i, t) dx \\
&= \sum_{i \in Z_\varepsilon} \sum_{\{\omega_\varepsilon^j \in \omega_\varepsilon \cap (\varepsilon i - \frac{\varepsilon}{2}, \varepsilon i + \frac{\varepsilon}{2}]\}} \int_{\Omega' \times (\omega_\varepsilon^j + \frac{r_\varepsilon}{2}, \omega_\varepsilon^j + \frac{r_\varepsilon}{2})} \hat{\boldsymbol{v}}_\varepsilon(x, t) \cdot \boldsymbol{\varphi}(x', \varepsilon i, t) dm_\varepsilon.
\end{aligned}
\tag{5.25}
$$

For all $i \in Z_\varepsilon$ and all $\omega_\varepsilon^j \in \omega_\varepsilon \cap (\varepsilon i - \frac{\varepsilon}{2}, \varepsilon i + \frac{\varepsilon}{2}]$, we have $|\boldsymbol{\varphi}(x_1, x_2, \varepsilon i, t) - \boldsymbol{\varphi}(x, t)| \le C\varepsilon$ in $\Omega' \times (\omega_\varepsilon^j + \frac{r_\varepsilon}{2}, \omega_\varepsilon^j + \frac{r_\varepsilon}{2}) \times (0, T)$. Taking (5.21) into account, we deduce

$$
\left| \sum_{i \in Z_\varepsilon} \sum_{\{\omega_\varepsilon^j \in \omega_\varepsilon \cap (\varepsilon i - \frac{\varepsilon}{2}, \varepsilon i + \frac{\varepsilon}{2}]\}} \int_{\Omega' \times (\omega_\varepsilon^j + \frac{r_\varepsilon}{2}, \omega_\varepsilon^j + \frac{r_\varepsilon}{2})} \hat{\boldsymbol{v}}_\varepsilon(x, t) \cdot \boldsymbol{\varphi}(x', \varepsilon i, t) dm_\varepsilon - \int \hat{\boldsymbol{v}}_\varepsilon(t) \cdot \boldsymbol{\varphi}(t) dm_\varepsilon \right|
$$

$$
\le C\varepsilon \int |\hat{\boldsymbol{v}}_\varepsilon|(t) dm_\varepsilon.
$$

Therefore, by (5.19) and (5.25), the following holds

$$(5.26) \quad \lim_{\varepsilon \to 0} \int_{\Omega \times (0,T)} \overline{\boldsymbol{v}}_\varepsilon \cdot \breve{\boldsymbol{\varphi}}_\varepsilon \, dx dt = \lim_{\varepsilon \to 0} \int_{\Omega \times (0,T)} \hat{\boldsymbol{v}}_\varepsilon \cdot \boldsymbol{\varphi} \, dm_\varepsilon dt = \int_{\Omega \times (0,T)} n\boldsymbol{v} \cdot \boldsymbol{\varphi} \, dx dt.$$

Combining (5.24) and (5.26) we deduce, by the arbitrary choice of $\boldsymbol{\varphi}$,

$$(5.27) \qquad\qquad \boldsymbol{w} = n\boldsymbol{v}.$$

On the other hand, by (3.14), (5.20), (5.21), Jensen's inequality and Korn's inequality in $H_0^1(\Omega)$, we have

$$
\begin{aligned}
(5.28) \quad & \int_\Omega |n_\varepsilon \boldsymbol{u}_\varepsilon - \overline{\boldsymbol{v}}_\varepsilon|^2 (\tau) dx \\
& = \sum_{i \in Z_\varepsilon} \int_{\Omega' \times \left(\varepsilon i - \frac{\varepsilon}{2}, \varepsilon i + \frac{\varepsilon}{2}\right]} \left| \sum_{\left\{\omega_\varepsilon^j \in \omega_\varepsilon \cap \left(\varepsilon i - \frac{\varepsilon}{2}, \varepsilon i + \frac{\varepsilon}{2}\right]\right\}} \boldsymbol{u}_\varepsilon - \boldsymbol{u}_\varepsilon(x', \omega_\varepsilon^j, \tau) \right|^2 dx \\
& \le C \sum_{i \in Z_\varepsilon} \sum_{\left\{\omega_\varepsilon^j \in \omega_\varepsilon \cap \left(\varepsilon i - \frac{\varepsilon}{2}, \varepsilon i + \frac{\varepsilon}{2}\right]\right\}} \int_{\Omega' \times \left(\varepsilon i - \frac{\varepsilon}{2}, \varepsilon i + \frac{\varepsilon}{2}\right]} \left| \boldsymbol{u}_\varepsilon - \boldsymbol{u}_\varepsilon(x', \omega_\varepsilon^j, \tau) \right|^2 dx \\
& \le C\varepsilon^2 \sum_{i \in Z_\varepsilon} n_\varepsilon(\varepsilon i) \int_{\Omega' \times \left(\varepsilon i - \frac{\varepsilon}{2}, \varepsilon i + \frac{\varepsilon}{2}\right]} \left| \frac{\partial \boldsymbol{u}_\varepsilon}{\partial x_3} \right|^2 (\tau) dx \le C\varepsilon^2 \int_\Omega |\boldsymbol{e}(\boldsymbol{u}_\varepsilon)|^2 (\tau) dx.
\end{aligned}
$$

We infer that, under either (5.13) or (5.14), the sequence $\left(\int_{\Omega \times (0,T)} |n_\varepsilon \boldsymbol{u}_\varepsilon - \overline{\boldsymbol{v}}_\varepsilon|^2 (\tau) \right.$ $dxdt \big)$ converges to 0. Therefore, by (5.23) and (5.27),

$$(5.29) \qquad\qquad n_\varepsilon \boldsymbol{u}_\varepsilon \overset{\star}{\rightharpoonup} n\boldsymbol{v} \quad \text{weakly* in } L^\infty(0, T; L^2(\Omega; \mathbb{R}^3)).$$

Under (5.14), it easily follows from the strong convergence of $(\boldsymbol{u}_\varepsilon)$ to $\boldsymbol{u}$ in $L^2(0, T; L^2(\Omega; \mathbb{R}^3))$ and the weak* convergence of (n_ε) to n in $L^\infty(\Omega)$ (see (3.16)), that $(n_\varepsilon \boldsymbol{u}_\varepsilon)$ weakly converges to $n\boldsymbol{u}$ in $L^2(0, T; L^2(\Omega; \mathbb{R}^3))$, therefore $n\boldsymbol{u} = n\boldsymbol{v}$. The same conclusion holds under (5.13), because $(\boldsymbol{u}_\varepsilon)$ weakly* converges to $\boldsymbol{u}$ in $L^\infty(0, T; L^2(\Omega; \mathbb{R}^3))$ and (n_ε) is bounded in $L^\infty(\Omega)$ and strongly converges to n in $L^2(\Omega)$. The proof of Lemma 5.3 is concluded. $\square$

LEMMA 5.4. *Let* $(\boldsymbol{u}_\varepsilon)$ *be a sequence in* $L^\infty(0, T; H_0^1(\Omega; \mathbb{R}^3))$ *satisfying*

$$(5.30) \qquad\qquad \sup_{\varepsilon > 0, \tau \in (0,T)} \int \left| \frac{\partial \boldsymbol{u}_\varepsilon}{\partial t} \right|^2 + \left| \frac{1}{r_\varepsilon} \boldsymbol{e}(\boldsymbol{u}_\varepsilon) \right|^2 (\tau) dm_\varepsilon < +\infty.$$

Then, up to a subsequence, the convergences (5.11) *take place. Moreover,*

$$(5.31) \qquad\qquad v_3 \in L^\infty(0, T; L_n^2(0, L; H_0^2(\Omega'))).$$

Besides, up to a subsequence, the following convergences hold (see (4.10)*):*

$$
\begin{aligned}
(5.32) \quad & \frac{u_{\varepsilon\alpha}}{r_\varepsilon} \overset{m_\xi}{\rightharpoonup} \xi_\alpha(x, t) - \frac{\partial v_3}{\partial x_\alpha}(x, t) y_3 \qquad (\alpha \in \{1, 2\}), \\
& \left(\frac{1}{r_\varepsilon} \boldsymbol{e}(\boldsymbol{u}_\varepsilon) \right)_{\alpha\beta} \overset{m_\xi}{\rightharpoonup} \frac{1}{2} \left(\frac{\partial \xi_\alpha}{\partial x_\beta} + \frac{\partial \xi_\beta}{\partial x_\alpha} \right)(x, t) - \frac{\partial^2 v_3}{\partial x_\alpha \partial x_\beta}(x, t) y_3 \quad (\alpha, \beta \in \{1, 2\}), \\
& \xi_1, \xi_2 \in L^\infty(0, T; L_n^2(0, L; H_0^1(\Omega'))).
\end{aligned}
$$

Proof. By (5.1) and (5.30), we have

$$(5.33) \qquad \sup_{\varepsilon > 0, \tau \in (0,T)} \int \left(\left| \frac{u_{\varepsilon 1}}{r_\varepsilon} \right|^2 + \left| \frac{u_{\varepsilon 2}}{r_\varepsilon} \right|^2 + |u_{\varepsilon 3}|^2 \right) dm_\varepsilon < +\infty.$$

By (5.30) and (5.33), assumption (5.10) of Lemma 5.3 is verified, hence, up to a subsequence, the convergences (5.11) take place. By Lemma 4.2, (5.30) and (5.33), there exist $\boldsymbol{v} \in L^\infty(0,T; L_n^2(\Omega; \mathbb{R}^3))$, $\zeta_{01}, \zeta_{02}, \zeta_{03} \in L^\infty(0,T; L_n^2(\Omega \times I))$, and $\boldsymbol{\Xi}^b \in L^\infty(0,T; L_n^2(\Omega \times I; \mathbb{S}^3))$, such that (see (2.1))

$$(5.34) \qquad \begin{aligned} \boldsymbol{u} m_\varepsilon &\overset{\star}{\rightharpoonup} n\boldsymbol{v} \quad \text{weakly* in } L^\infty(0,T; \mathcal{M}(\overline{\Omega}; \mathbb{R}^3)); \qquad nv_1 = nv_2 = 0; \\ u_{\varepsilon 3} &\overset{m_\varepsilon}{\rightharpoonup} \zeta_{03}; \quad \frac{u_{\varepsilon \alpha}}{r_\varepsilon} \overset{m_\varepsilon}{\rightharpoonup} \zeta_{0\alpha} \quad (\alpha \in \{1,2\}); \qquad \frac{1}{r_\varepsilon} \boldsymbol{e}_{x'}(\boldsymbol{u}_\varepsilon) \overset{m_\varepsilon}{\rightharpoonup} \boldsymbol{\Xi}^b. \end{aligned}$$

We will use the following assertion, which will be proved later,

$$(5.35) \qquad n(x)\zeta_{03}(x,t,y_3) = n(x)v_3(x,t) \quad \text{a.e. in } \Omega \times (0,T) \times I.$$

Then, we fix a matrix field $\boldsymbol{\Psi}$ satisfying

$$(5.36) \qquad \boldsymbol{\Psi} \in C^\infty \left(\overline{\Omega \times (0,T)}; \mathcal{D}(I; \mathbb{S}^3) \right), \quad \Psi_{33} = 0.$$

Noticing that $x \to \boldsymbol{\Psi}\left(x, t, \frac{y_\varepsilon(x_3)}{r_\varepsilon} \right)$ vanishes on the complement of the support of m_ε, by integration by parts we get

$$(5.37) \qquad \begin{aligned} \int_{\Omega \times (0,T)} \boldsymbol{e}(\boldsymbol{u}_\varepsilon) : \boldsymbol{\Psi}\left(x, t, \frac{y_\varepsilon(x_3)}{r_\varepsilon} \right) dm_\varepsilon dt = &- \int_{\Omega \times (0,T)} \boldsymbol{u}_\varepsilon \cdot \mathbf{div}_x \boldsymbol{\Psi}\left(x, t, \frac{y_\varepsilon(x_3)}{r_\varepsilon} \right) dm_\varepsilon dt \\ &- \sum_{\alpha=1}^2 \int_{\Omega \times (0,T)} \frac{u_{\varepsilon \alpha}}{r_\varepsilon} \frac{\partial \Psi_{\alpha 3}}{\partial y_3} \left(x, t, \frac{y_\varepsilon(x_3)}{r_\varepsilon} \right) dm_\varepsilon dt. \end{aligned}$$

By passing to the limit as $\varepsilon \to 0$ in (5.37), taking (5.30), (5.34) and (5.35) into account, we infer

$$(5.38) \quad 0 = - \int_{\Omega \times (0,T) \times I} v_3 (\mathbf{div}_x \boldsymbol{\Psi})_3 \, n \, dx \, dt \, dy_3 - \sum_{\alpha=1}^2 \int_{\Omega \times (0,T) \times I} \zeta_{0\alpha} \frac{\partial \Psi_{\alpha 3}}{\partial y_3} n \, dx \, dt \, dy_3.$$

Fixing $\alpha \in \{1,2\}$, $\varphi \in C^\infty(\overline{\Omega \times (0,T)})$, $\psi \in \mathcal{D}(I)$, and selecting in (5.38) a field of the form $\boldsymbol{\Psi}(x,t,y_3) := \varphi(x,t)\psi(y_3)(\boldsymbol{e}_\alpha \otimes \boldsymbol{e}_3 + \boldsymbol{e}_3 \otimes \boldsymbol{e}_\alpha)$, we get

$$\begin{aligned} 0 = &- \int_{\Omega \times (0,T)} v_3(x,t) \frac{\partial \varphi}{\partial x_\alpha}(x,t) n \, dx \, dt \left(\int_I \psi(y_3) dy_3 \right) \\ &- \int_{\Omega \times (0,T)} \left(\int_I \zeta_{0\alpha}(x,t,y_3) \frac{\partial \psi}{\partial y_3}(y_3) dy_3 \right) \varphi(x,t) n \, dx \, dt. \end{aligned}$$

Choosing ψ such that $\left(\int_I \psi(y_3) dy_3\right) \neq 0$, and making φ vary in $C^\infty(\overline{\Omega \times (0,T)})$, we deduce that

(5.39) $$v_3 \in L^\infty(0,T; L^2_n(0,L; H^1_0(\Omega'))),$$

then, by integration by parts with respect to x_α, infer

$$0 = \int_{\Omega \times (0,T) \times I} \varphi(x,t)\psi(y_3)\frac{\partial v_3}{\partial x_\alpha}(x,t) n \, dx dt dy_3 - \int_{\Omega \times (0,T) \times I} \zeta_{0\alpha}(x,t,y_3)\frac{\partial \psi}{\partial y_3}(y_3)\varphi(x,t) n \, dx dt dy_3.$$

We deduce, from the arbitrary choice of φ and ψ, that

$$\zeta_{0\alpha} \in L^\infty\left(0,T; L^2_n\left(\Omega; H^1(I)\right)\right); \quad n\frac{\partial \zeta_{0\alpha}}{\partial y_3}(x,t,y_3) = -n\frac{\partial v_3}{\partial x_\alpha}(x,t) \quad \text{in } \Omega \times (0,T) \times I,$$

and then that

(5.40) $$n\zeta_{0\alpha}(x,t,y_3) = n\xi_\alpha(x,t) - n\frac{\partial v_3}{\partial x_\alpha}(x,t)y_3 \quad \text{in } \Omega \times (0,T) \times I,$$

for some suitable $\xi_\alpha \in L^\infty(0,T; L^2_n(\Omega))$. Next, we choose a matrix field $\boldsymbol{\Psi}$ satisfying (5.36) and $\Psi_{3k} = 0 \; \forall \, k \in \{1,2,3\}$. By multiplying (5.37) by $\frac{1}{r_\varepsilon}$, we get

$$\sum_{\alpha,\beta=1}^2 \int_{\Omega \times (0,T)} \frac{e_{\alpha\beta}(\boldsymbol{u}_\varepsilon)}{r_\varepsilon}\Psi_{\alpha\beta}\left(x,t,\frac{y_\varepsilon(x_3)}{r_\varepsilon}\right) dm_\varepsilon dt =$$

$$-\sum_{\alpha,\beta=1}^2 \int_{\Omega \times (0,T)} \frac{u_{\varepsilon\alpha}}{r_\varepsilon}\frac{\partial \Psi_{\alpha\beta}}{\partial x_\beta}\left(x,t,\frac{y_\varepsilon(x_3)}{r_\varepsilon}\right) dm_\varepsilon dt.$$

By passing to the limit as $\varepsilon \to 0$, thanks to (5.34) and (5.40), we find

$$\sum_{\alpha,\beta=1}^2 \int_{\Omega \times (0,T) \times I} \Xi^b_{\alpha\beta}\Psi_{\alpha\beta} n \, dx dt dy$$

$$= -\sum_{\alpha,\beta=1}^2 \int_{\Omega \times (0,T) \times I} \left(\xi_\alpha(x,t) - \frac{\partial v_3}{\partial x_\alpha}(x,t)y_3\right)\frac{\partial \Psi_{\alpha\beta}}{\partial x_\beta}(x,t,y_3) n \, dx dt dy_3.$$

By the arbitrary choice of the functions $\Psi_{\alpha\beta}(= \Psi_{\beta\alpha})$ in $C^\infty\left(\overline{\Omega \times (0,T)}; \mathcal{D}(I)\right)$ and (5.39), we deduce that for $\alpha, \beta \in \{1,2\}$, the following holds

$$\xi_\alpha \in L^\infty(0,T; L^2_n(0,L; H^1_0(\Omega'))), \quad v_3 \in L^\infty(0,T; L^2_n(0,L; H^2_0(\Omega'))),$$

$$n\Xi^b_{\alpha\beta}(x,t,y_3) = \frac{1}{2}n\left(\frac{\partial \xi_\alpha}{\partial x_\beta} + \frac{\partial \xi_\beta}{\partial x_\alpha}\right)(x,t) - n\frac{\partial^2 v_3}{\partial x_\alpha \partial x_\beta}(x,t)y_3 \text{ in } \Omega \times (0,T) \times I.$$

The proof of Lemma 5.4 is concluded.
Proof of (5.35). By (5.17) and (5.30), we have

$$\int |u_{\varepsilon 3} - \hat{v}_{\varepsilon 3}|^2(\tau) dm_\varepsilon = \frac{\varepsilon}{r_\varepsilon} \sum_{j \in J_\varepsilon} \int_{\Omega'} dx' \int_{\omega_\varepsilon^j - \frac{r_\varepsilon}{2}}^{\omega_\varepsilon^j + \frac{r_\varepsilon}{2}} |u_{\varepsilon 3}(x, \tau) - u_{\varepsilon 3}(x', \omega_\varepsilon^j, \tau)|^2 dx_3$$

(5.41)

$$\leq \varepsilon r_\varepsilon \sum_{j \in J_\varepsilon} \int_{\Omega'} dx' \int_{\omega_\varepsilon^j - \frac{r_\varepsilon}{2}}^{\omega_\varepsilon^j + \frac{r_\varepsilon}{2}} \left| \frac{\partial u_{\varepsilon 3}}{\partial x_3} \right|^2 (x', x_3, \tau) dx_3 \leq C r_\varepsilon^2 \int |e(\boldsymbol{u}_\varepsilon)|^2(\tau) dm_\varepsilon \leq C r_\varepsilon^4.$$

We easily deduce from (5.34) and (5.41) that

$$(5.42) \qquad \hat{v}_{\varepsilon 3} m_\varepsilon \overset{\star}{\rightharpoonup} n v_3 \quad \text{weakly* in } L^\infty(0, T; \mathcal{M}(\overline{\Omega})); \qquad \hat{v}_{\varepsilon 3} \overset{m_\varepsilon}{\rightharpoonup} \zeta_{03}.$$

Fixing $\psi \in \mathcal{D}(\Omega \times (0, T) \times I)$, we set (see (4.9))

$$(5.43) \qquad \widetilde{\psi}_\varepsilon \left(x, t, \frac{y_\varepsilon(x_3)}{r_\varepsilon} \right) := \sum_{j \in J_\varepsilon} \psi \left(x', \omega_\varepsilon^j, t, \frac{y_\varepsilon(x_3)}{r_\varepsilon} \right) \mathbb{1}_{\left(\omega_\varepsilon^j - \frac{r_\varepsilon}{2}, \omega_\varepsilon^j + \frac{r_\varepsilon}{2} \right)}(x_3).$$

We have

$$(5.44) \qquad \left| \widetilde{\psi}_\varepsilon \left(x, t, \frac{y_\varepsilon(x_3)}{r_\varepsilon} \right) - \psi \left(x, t, \frac{y_\varepsilon(x_3)}{r_\varepsilon} \right) \right| \leq C r_\varepsilon \quad \text{in } B_\varepsilon.$$

By making the change of variables $y = \frac{x_3 - \omega_\varepsilon^j}{r_\varepsilon}$, we get $\int_{\omega_\varepsilon^j - \frac{r_\varepsilon}{2}}^{\omega_\varepsilon^j + \frac{r_\varepsilon}{2}} \psi \left(x', \omega_\varepsilon^j, t, \frac{y_\varepsilon(x_3)}{r_\varepsilon} \right) dx_3 = r_\varepsilon \int_I \psi \left(x', \omega_\varepsilon^j, t, y_3 \right) dy_3$. We infer

$$\int_{\Omega \times (0, T)} \hat{v}_{\varepsilon 3} \widetilde{\psi}_\varepsilon \left(x, t, \frac{y_\varepsilon(x_3)}{r_\varepsilon} \right) dm_\varepsilon dt$$

$$= \frac{\varepsilon}{r_\varepsilon} \sum_{j \in J_\varepsilon} \int_{\Omega' \times (0, T)} dx' dt \int_{\omega_\varepsilon^j - \frac{r_\varepsilon}{2}}^{\omega_\varepsilon^j + \frac{r_\varepsilon}{2}} u_{\varepsilon 3}(x', \omega_\varepsilon^j, t) \psi \left(x', \omega_\varepsilon^j, t, \frac{y_\varepsilon(x_3)}{r_\varepsilon} \right) dx_3$$

(5.45)

$$= \frac{\varepsilon}{r_\varepsilon} \sum_{j \in J_\varepsilon} \int_{\Omega' \times (0, T)} r_\varepsilon u_{\varepsilon 3}(x', \omega_\varepsilon^j, t) \left(\int_I \psi \left(x', \omega_\varepsilon^j, t, y_3 \right) dy_3 \right) dx' dt$$

$$= \frac{\varepsilon}{r_\varepsilon} \sum_{j \in J_\varepsilon} \int_{\Omega' \times (0, T)} dx' dt \int_{\omega_\varepsilon^j - \frac{r_\varepsilon}{2}}^{\omega_\varepsilon^j + \frac{r_\varepsilon}{2}} u_{\varepsilon 3}(x', \omega_\varepsilon^j, t) \left(\int_I \psi \left(x', \omega_\varepsilon^j, t, y_3 \right) dy_3 \right) dx_3$$

$$= \int_{\Omega \times (0, T)} \hat{v}_{\varepsilon 3}(x, t) \widehat{\psi}_\varepsilon(x, t) dm_\varepsilon dt,$$

where $\widehat{\psi}_\varepsilon(x, t) := \sum_{j \in J_\varepsilon} \left(\int_I \psi \left(x', \omega_\varepsilon^j, t, y_3 \right) dy_3 \right) \mathbb{1}_{\left(\omega_\varepsilon^j - \frac{r_\varepsilon}{2}, \omega_\varepsilon^j + \frac{r_\varepsilon}{2} \right)}(x_3)$. Noticing that $\left| \widehat{\psi}_\varepsilon(x, t) - \left(\int_I \psi(x, t, y_3) dy_3 \right) \right| \leq C r_\varepsilon$ in B_ε, we deduce successively from (5.42), (5.45), (5.44), and again (5.42) that

$$\int_{\Omega\times(0,T)\times I} v_3(x,t)\Big(\int_I \psi(x,t,y_3)\,dy_3\Big)n\,dx\,dt = \lim_{\varepsilon\to 0}\int_{\Omega\times(0,T)} \hat{v}_{\varepsilon 3}(x,t)\Big(\int_I \psi(x,t,y_3)\,dy_3\Big)dm_\varepsilon dt$$

(5.46)
$$= \lim_{\varepsilon\to 0}\int_{\Omega\times(0,T)} \hat{v}_{\varepsilon 3}(x,t)\widehat{\psi}_\varepsilon(x,t)dm_\varepsilon dt = \lim_{\varepsilon\to 0}\int_{\Omega\times(0,T)} \hat{v}_{\varepsilon 3}(x,t)\widetilde{\psi}_\varepsilon\left(x,t,\frac{y_\varepsilon(x_3)}{r_\varepsilon}\right)dm_\varepsilon dt$$

$$= \lim_{\varepsilon\to 0}\int_{\Omega\times(0,T)} \hat{v}_{\varepsilon 3}(x,t)\psi\left(x,t,\frac{y_\varepsilon(x_3)}{r_\varepsilon}\right)dm_\varepsilon dt = \int_{\Omega\times(0,T)\times I}\zeta_{03}(x,t,y_3)\psi(x,t,y_3)n\,dx\,dt\,dy_3.$$

By the arbitrary choice of ψ, assertion (5.35) is proved. □

The next Lemma is specific to the periodic case. Given a sequence $(\boldsymbol{u}_\varepsilon)$ satisfying (5.10) and (5.47) (and possibly (5.30)), we establish some relations satisfied by its two-scale limit $\boldsymbol{u}_0$ and by the field $\boldsymbol{v}$ introduced in Lemma 5.3.

LEMMA 5.5. *Assume that B_ε is the ε-periodic set defined by (3.35) and let $(\boldsymbol{u}_\varepsilon)$ be a sequence in $L^\infty(0,T;H_0^1(\Omega;\mathbb{R}^3))$ satisfying (5.10) and*

(5.47)
$$\sup_{\tau\in[0,T],\ \varepsilon>0}\int_\Omega |\boldsymbol{u}_\varepsilon|^2 + \left|\frac{\partial\boldsymbol{u}_\varepsilon}{\partial t}\right|^2 + \varepsilon^2|\boldsymbol{e}(\boldsymbol{u}_\varepsilon)|^2(\tau)dx < +\infty, \quad \boldsymbol{u}_\varepsilon(0) = \boldsymbol{a}_0.$$

Then, up to a subsequence, the convergences (5.11) take place with $n=1$. Moreover

(5.48)
$$\boldsymbol{u}_\varepsilon \rightharpoonup \boldsymbol{u}_0 \text{ and } \varepsilon\boldsymbol{e}(\boldsymbol{u}_\varepsilon)\rightharpoonup \boldsymbol{e}_y(\boldsymbol{u}_0) \text{ in accordance with (4.1)},$$
$$\boldsymbol{u}_0 \in L^\infty(0,T;L^2(\Omega;H_\sharp^1(Y;\mathbb{R}^3)))\cap W^{1,\infty}(0,T;L^2(\Omega\times Y;\mathbb{R}^3)),$$
$$\frac{\partial\boldsymbol{u}_\varepsilon}{\partial t} \rightharpoonup \frac{\partial\boldsymbol{u}_0}{\partial t}, \qquad \boldsymbol{u}_\varepsilon(\tau) \rightharpoonup \boldsymbol{u}_0(\tau) \ \forall\ \tau\in[0,T].$$

Furthermore,
(i) If $\vartheta > 0$, then

(5.49)
$$\boldsymbol{u}_0(x,t,y) = \boldsymbol{v}(x,t) \quad in \ \Omega\times(0,T)\times B.$$

(ii) If $\vartheta = 0$ and if the estimate (5.30) is satisfied, then

(5.50)
$$\boldsymbol{u}_0(x,t,y) = \boldsymbol{v}(x,t) \quad on \ \Omega\times(0,T)\times\Sigma.$$

Proof. The convergences (5.11) are deduced from Lemma 5.3. Under (5.47), by Lemma 4.1 (ii), the sequence $(\boldsymbol{u}_\varepsilon)$ (resp. $(\varepsilon\boldsymbol{e}(\boldsymbol{u}_\varepsilon))$) two-scale converges, up to a subsequence, to some $\boldsymbol{u}_0 \in L^\infty(0,T;L^2(\Omega\times Y;\mathbb{R}^3))$ (resp. $\boldsymbol{\Xi}^m \in L^\infty(0,T;L^2(\Omega\times Y;\mathbb{S}^3)))$. Choosing $\boldsymbol{\Psi} \in \mathcal{D}(\Omega\times(0,T);C_\sharp^\infty(Y;\mathbb{S}^3))$ and passing to the limit as $\varepsilon\to 0$ in the equation

(5.51)
$$\int_{\Omega\times(0,T)} \varepsilon\boldsymbol{e}(\boldsymbol{u}_\varepsilon) : \boldsymbol{\Psi}\left(x,t,\frac{x}{\varepsilon}\right)dx\,dt =$$
$$-\varepsilon\int_{\Omega\times(0,T)} \boldsymbol{u}_\varepsilon\cdot\mathbf{div}_x\boldsymbol{\Psi}\left(x,t,\frac{x}{\varepsilon}\right)dx\,dt - \int_{\Omega\times(0,T)} \boldsymbol{u}_\varepsilon\cdot\mathbf{div}_y\boldsymbol{\Psi}\left(x,t,\frac{x}{\varepsilon}\right)dx\,dt,$$

we infer $\int_{\Omega\times(0,T)\times Y} \boldsymbol{\Xi}^m : \boldsymbol{\Psi} dxdtdy = -\int_{\Omega\times(0,T)\times Y} \boldsymbol{u}_0 \cdot \mathbf{div}_y \boldsymbol{\Psi} dxdtdy$ and deduce, by the arbitrary choice of $\boldsymbol{\Psi}$, that $\boldsymbol{u}_0 \in L^\infty(0,T; L^2(\Omega; H^1_\sharp(Y;\mathbb{R}^3)))$ and $\boldsymbol{e}_y(\boldsymbol{u}_0) = \boldsymbol{\Xi}^m$. By Lemma 4.1 (ii), $\boldsymbol{u}_0 \in W^{1,\infty}(0,T; L^2(\Omega\times Y;\mathbb{R}^3))$ and the convergences of the last line of (5.48) hold. Assertion (5.48) is proved.

If $\vartheta > 0$, by (5.10) and (5.47), the sequence $(\varepsilon\boldsymbol{e}(\boldsymbol{u}_\varepsilon)\mathbb{1}_{B_\varepsilon})$ strongly converges to 0 in $L^2(\Omega\times(0,T);\mathbb{R}^3)$. On the other hand, by (4.4), (5.48) and Lemma 4.1, $(\varepsilon\boldsymbol{e}(\boldsymbol{u}_\varepsilon)\mathbb{1}_{B_\varepsilon})$ two-scale converges to $\boldsymbol{e}_y(\boldsymbol{u}_0)\mathbb{1}_B$, hence $\boldsymbol{e}_y(\boldsymbol{u}_0) = 0$ in $\Omega \times (0,T) \times B$ and, for a. e. $(x,t) \in \Omega\times(0,T)$, the restriction of $\boldsymbol{u}_0(x,t,.)$ to B is a rigid displacement. Since $\boldsymbol{u}_0$ is Y-periodic, we deduce that

$$(5.52) \qquad\qquad \boldsymbol{u}_0 = \boldsymbol{a} \ \text{ in } \Omega\times(0,T)\times B,$$

for some $\boldsymbol{a} \in L^\infty(0,T; L^2(\Omega;\mathbb{R}^3))$. By (4.4), (5.48) and Lemma 4.1 (i), the sequence $(\boldsymbol{u}_\varepsilon\mathbb{1}_{B_\varepsilon})$ two-scale converges to $\boldsymbol{u}_0(x,t,y)\mathbb{1}_B(y)$. Fixing $\boldsymbol{\varphi}\in\mathcal{D}(\Omega\times(0,T);\mathbb{R}^3)$, taking (5.11), (5.47) and (5.52) into account, and noticing that $\frac{\varepsilon}{r_\varepsilon} \to \frac{1}{|B|}$, we deduce

$$\int_{\Omega\times(0,T)} \boldsymbol{v}\cdot\boldsymbol{\varphi} dxdt = \lim_{\varepsilon\to 0}\int_{\Omega\times(0,T)} \boldsymbol{u}_\varepsilon\cdot\boldsymbol{\varphi} dm_\varepsilon dt = \lim_{\varepsilon\to 0} \frac{\varepsilon}{r_\varepsilon}\int_{\Omega\times(0,T)} \boldsymbol{u}_\varepsilon\cdot\boldsymbol{\varphi}\mathbb{1}_{B_\varepsilon}(x)dxdt$$

$$= \frac{1}{|B|}\int_{\Omega\times(0,T)\times B} \boldsymbol{u}_0\cdot\boldsymbol{\varphi}(x,t)1_B(y)dxdtdy = \int_{\Omega\times(0,T)} \boldsymbol{a}\cdot\boldsymbol{\varphi} dxdt,$$

and infer, from the arbitrary choice of $\boldsymbol{\varphi}$, that $\boldsymbol{v} = \boldsymbol{a}$. Assertion (5.49) is proved.

Let us assume now that $\vartheta = 0$ (i.e. that $r_\varepsilon \ll \varepsilon$). Since the stiff layers are periodicaly distributed, by (3.35) the field $\overline{\boldsymbol{v}}_\varepsilon$ defined by (5.20) takes the form

$$(5.53) \qquad \overline{\boldsymbol{v}}_\varepsilon(x,t) := \sum_{i\in Z_\varepsilon} \boldsymbol{u}_\varepsilon(x_1,x_2,\varepsilon i,t)\mathbb{1}_{\left(\varepsilon i-\frac{\varepsilon}{2},\varepsilon i+\frac{\varepsilon}{2}\right]}(x_3),$$

and coincides in B_ε with the field $\hat{\boldsymbol{v}}_\varepsilon$ given by (5.17). Therefore, by (5.18),

$$(5.54) \qquad \int |\overline{\boldsymbol{v}}_\varepsilon - \boldsymbol{u}_\varepsilon|^2(\tau)dm_\varepsilon \leq C\frac{r_\varepsilon}{\varepsilon}\int_\Omega \varepsilon^2|\boldsymbol{e}(\boldsymbol{u}_\varepsilon)|^2(\tau)dx \quad \forall\tau\in[0,T].$$

Since $r_\varepsilon \ll \varepsilon$, we deduce from (5.11), (5.47) and (5.54) that $\int |\overline{\boldsymbol{v}}_\varepsilon|^2(\tau)dm_\varepsilon \leq C$. On the other hand, taking (3.39), (3.35) and (5.53) into account, it is easy to check that $\int |\overline{\boldsymbol{v}}_\varepsilon|^2(\tau)dm_\varepsilon = \int_\Omega |\overline{\boldsymbol{v}}_\varepsilon|^2(\tau)dx$, therefore the sequence $(\overline{\boldsymbol{v}}_\varepsilon)$ is bounded in $L^\infty(0,T; L^2(\Omega;\mathbb{R}^3))$. It then follows from Lemma 4.1 (ii) that

$$(5.55) \qquad\qquad\qquad \overline{\boldsymbol{v}}_\varepsilon \rightharpoonup\!\!\!\rightharpoonup \overline{\boldsymbol{v}}_0,$$

up to a subsequence, for some $\overline{\boldsymbol{v}}_0 \in L^\infty(0,T; L^2(\Omega\times Y;\mathbb{R}^3))$. We establish below that

$$(5.56) \qquad \frac{\partial\overline{\boldsymbol{v}}_0}{\partial y_3} = 0 \ \text{ a.e. in } \Omega\times(0,T)\times Y, \qquad \boldsymbol{u}_0 = \overline{\boldsymbol{v}}_0 \ \text{ on } \ \Omega\times(0,T)\times\Sigma,$$

and that (see (2.1))

$$\boldsymbol{v}'(x,t) = \overline{\boldsymbol{v}}_0'(x,t,y) \quad \text{a.e. in } \Omega \times (0,T) \times Y,$$

(5.57)

$$v_3(x,t) = \int_{\left(-\frac{1}{2},\frac{1}{2}\right)^2} \overline{v}_{03}(x,t,s_1,s_2,y_3)ds_1ds_2.$$

The next equation (proved below)

(5.58)
$$\frac{\partial \overline{v}_{03}}{\partial y_\alpha} = 0 \quad \forall \alpha \in \{1,2\}, \quad \text{if (5.30) holds true,}$$

joined with (5.57), yields (5.50). It remains to prove (5.56), (5.57) and (5.58).
Proof of (5.56). Let us fix $\boldsymbol{\psi} \in \mathcal{D}(\Omega \times (0,T); \mathcal{D}_\sharp(Y; \mathbb{R}^3))$. By (5.53) we have

$$\int_{\Omega \times (0,T)} \overline{\boldsymbol{v}}_\varepsilon \cdot \frac{\partial \boldsymbol{\psi}}{\partial y_3}\left(x,t,\frac{x}{\varepsilon}\right) dxdt$$

$$= \sum_{i \in Z_\varepsilon} \int_{\Omega' \times (0,T)} \boldsymbol{u}_\varepsilon(x_1,x_2,\varepsilon i,t) \cdot \left(\int_{\left(\varepsilon i - \frac{\varepsilon}{2}, \varepsilon i + \frac{\varepsilon}{2}\right)} \frac{\partial \boldsymbol{\psi}}{\partial y_3}\left(x,t,\frac{x'}{\varepsilon},\frac{x_3}{\varepsilon}\right) dx_3\right) dx'dt.$$

Since $\boldsymbol{\psi}(x,t,.) \in \mathcal{D}_\sharp(Y; \mathbb{R}^3)$, the following holds

$$\int_{\left(\varepsilon i - \frac{\varepsilon}{2}, \varepsilon i + \frac{\varepsilon}{2}\right)} \frac{\partial \boldsymbol{\psi}}{\partial y_3}\left(x,t,\frac{x'}{\varepsilon},\frac{x_3}{\varepsilon}\right) dx_3 = \frac{1}{\varepsilon}\int_{\left(-\frac{1}{2},\frac{1}{2}\right)} \frac{\partial \boldsymbol{\psi}}{\partial y_3}\left(x,t,\frac{x'}{\varepsilon},y_3\right) dy_3 = 0,$$

therefore $\int_{\Omega \times (0,T)} \overline{\boldsymbol{v}}_\varepsilon \cdot \frac{\partial \boldsymbol{\psi}}{\partial y_3}\left(x,t,\frac{x}{\varepsilon}\right) dxdt = 0$. By passing to the limit as $\varepsilon \to 0$, we infer $\int_{\Omega \times (0,T) \times Y} \overline{\boldsymbol{v}}_0 \cdot \frac{\partial \boldsymbol{\psi}}{\partial y_3}(x,t,y) \, dxdtdy = 0$, and deduce from the arbitrary choice of $\boldsymbol{\psi}$ that

(5.59)
$$\frac{\partial \overline{v}_0}{\partial y_3} = 0 \quad \text{in} \quad \Omega \times (0,T) \times Y.$$

We set $Y^+ := \left(-\frac{1}{2},\frac{1}{2}\right)^2 \times \left(0,\frac{1}{2}\right)$ and fix $\boldsymbol{\Psi} \in \mathcal{D}(\Omega \times (0,T); \mathcal{D}_\sharp(Y; \mathbb{S}^3))$. Then, for each $i \in Z_\varepsilon$ (defined by (3.15)), the field $\boldsymbol{\Psi}\left(x,t,\frac{x}{\varepsilon}\right)$ vanishes on $\partial\left(\Omega' \times \left(\varepsilon i - \frac{\varepsilon}{2}, \varepsilon i + \frac{\varepsilon}{2}\right)\right) \times (0,T)$, and, for ε small enough, the support of $\boldsymbol{\Psi}\left(x,t,\frac{x}{\varepsilon}\right)$ is included in $\bigcup_{i \in Z_\varepsilon} \Omega' \times \left[\varepsilon i - \frac{\varepsilon}{2}, \varepsilon i + \frac{\varepsilon}{2}\right] \times (0,T)$. Hence, by integration by parts, we get

$$\int_{\Omega \times (0,T)} \varepsilon \boldsymbol{e}(\boldsymbol{u}_\varepsilon) : \boldsymbol{\Psi}\left(x,t,\frac{x}{\varepsilon}\right) \mathbb{1}_{Y_\sharp^+}\left(\frac{x}{\varepsilon}\right) dxdt = \sum_{i \in Z_\varepsilon} \int_{\Omega' \times \left(\varepsilon i, \varepsilon i + \frac{\varepsilon}{2}\right) \times (0,T)} \varepsilon \boldsymbol{e}(\boldsymbol{u}_\varepsilon) : \boldsymbol{\Psi}\left(x,t,\frac{x}{\varepsilon}\right) dxdt$$

(5.60)
$$= -\sum_{i \in Z_\varepsilon} \varepsilon \int_{\Omega' \times \{\varepsilon i\} \times (0,T)} \boldsymbol{u}_\varepsilon \cdot \boldsymbol{\Psi}\left(x,t,\frac{x'}{\varepsilon},0\right) \boldsymbol{e}_3 d\mathcal{H}^2(x)dt$$

$$-\int_{\Omega \times (0,T)} \varepsilon \boldsymbol{u}_\varepsilon \cdot \mathbf{div}_x \boldsymbol{\Psi}\left(x,t,\frac{x}{\varepsilon}\right) \mathbb{1}_{Y_\sharp^+}\left(\frac{x}{\varepsilon}\right) + \boldsymbol{u}_\varepsilon \cdot \mathbf{div}_y \boldsymbol{\Psi}\left(x,t,\frac{x}{\varepsilon}\right) \mathbb{1}_{Y_\sharp^+}\left(\frac{x}{\varepsilon}\right) dxdt.$$

We set $\overline{\boldsymbol{\Psi}}_\varepsilon\left(x,t,\frac{x'}{\varepsilon},0\right) := \sum_{i \in Z_\varepsilon} \boldsymbol{\Psi}\left(x_1,x_2,\varepsilon i,t,\frac{x'}{\varepsilon},0\right) \mathbb{1}_{\left(\varepsilon i - \frac{\varepsilon}{2}, \varepsilon i + \frac{\varepsilon}{2}\right)}(x_3)$. Notice that

$$\left| \overline{\boldsymbol{\Psi}}_\varepsilon \left(x, t, \frac{x'}{\varepsilon}, 0 \right) - \boldsymbol{\Psi} \left(x, t, \frac{x'}{\varepsilon}, 0 \right) \right| \leq C\varepsilon,$$

(5.61)

$$\left| \frac{\partial}{\partial y_\alpha} \overline{\boldsymbol{\Psi}}_\varepsilon \left(x, t, \frac{x'}{\varepsilon}, 0 \right) - \frac{\partial}{\partial y_\alpha} \boldsymbol{\Psi} \left(x, t, \frac{x'}{\varepsilon}, 0 \right) \right| \leq C\varepsilon \quad \text{for} \quad \alpha \in \{1, 2\}.$$

By the definitions $\overline{\boldsymbol{\Psi}}_\varepsilon$ and $\overline{\boldsymbol{v}}_\varepsilon$ (see (5.53)), there holds

(5.62) $$-\sum_{i \in Z_\varepsilon} \varepsilon \int_{\Omega' \times \{\varepsilon i\} \times (0,T)} \boldsymbol{u}_\varepsilon \cdot \boldsymbol{\Psi}(x, t, \frac{x'}{\varepsilon}, 0) \boldsymbol{e}_3 d\mathcal{H}^2(x) dt = -\int_{\Omega \times (0,T)} \overline{\boldsymbol{v}}_\varepsilon \cdot \overline{\boldsymbol{\Psi}}_\varepsilon \left(x, t, \frac{x'}{\varepsilon}, 0 \right) \boldsymbol{e}_3 dx dt.$$

Taking (5.55) and (5.61) into account, and noticing that by (5.59) we have $\overline{\boldsymbol{v}}_0(x, t, y) = \overline{\boldsymbol{v}}_0(x, t, y', 0)$ in $\Omega \times (0,T) \times Y$, we obtain

(5.63)
$$\lim_{\varepsilon \to 0} -\int_{\Omega \times (0,T)} \overline{\boldsymbol{v}}_\varepsilon \cdot \overline{\boldsymbol{\Psi}}_\varepsilon \left(x, t, \frac{x'}{\varepsilon}, 0 \right) \boldsymbol{e}_3 dx dt = -\int_{\Omega \times (0,T) \times Y} \overline{\boldsymbol{v}}_0 \cdot \boldsymbol{\Psi}(x, t, y', 0) \boldsymbol{e}_3 dx dt dy$$

$$= -\int_{\Omega \times (0,T) \times \Sigma} \overline{\boldsymbol{v}}_0 \cdot \boldsymbol{\Psi} \boldsymbol{e}_3 dx dt d\mathcal{H}^2(y).$$

By passing to the limit as $\varepsilon \to 0$ in (5.60), applying Lemma 4.1 (i) with $h_\varepsilon = \mathbb{1}_{Y^+} \left(\frac{x}{\varepsilon} \right)$ and taking (5.48), (5.62) and (5.63) into account, we get

$$\int_{\Omega \times (0,T) \times Y^+} \boldsymbol{e}_y(\boldsymbol{u}_0) : \boldsymbol{\Psi} dx dt dy = -\int_{\Omega \times (0,T) \times \Sigma} \overline{\boldsymbol{v}}_0 \cdot \boldsymbol{\Psi} \boldsymbol{e}_3 dx dt d\mathcal{H}^2(y) - \int_{\Omega \times (0,T) \times Y^+} \boldsymbol{u}_0 \cdot \mathbf{div}_y \boldsymbol{\Psi} dx dt dy.$$

By integration by parts, we have

$$-\int_{\Omega \times (0,T) \times Y^+} \boldsymbol{u}_0 \cdot \mathbf{div}_y \boldsymbol{\Psi} dx dt dy = \int_{\Omega \times (0,T) \times \Sigma} \boldsymbol{u}_0 \cdot \boldsymbol{\Psi} \boldsymbol{e}_3 dx dt d\mathcal{H}^2(y) + \int_{\Omega \times (0,T) \times Y^+} \boldsymbol{e}_y(\boldsymbol{u}_0) : \boldsymbol{\Psi} dx dt dy.$$

Joining the last two equations, we infer that $\int_{\Omega \times (0,T) \times \Sigma} \boldsymbol{u}_0 \cdot \boldsymbol{\Psi} \boldsymbol{e}_3 dx dt d\mathcal{H}^2(y) = \int_{\Omega \times (0,T) \times \Sigma} \overline{\boldsymbol{v}}_0 \cdot \boldsymbol{\Psi} \boldsymbol{e}_3 dx dt d\mathcal{H}^2(y)$. By the arbitrary choice of $\boldsymbol{\Psi}$ (and by (5.59)), we deduce that (5.56) holds.

Proof of (5.57). Let us fix $\boldsymbol{\psi} \in \mathcal{D}(\Omega \times (0,T); \mathbb{R}^3)$ and set

(5.64) $$\overline{\boldsymbol{\psi}}_\varepsilon(x, t) := \sum_{i \in Z_\varepsilon} \boldsymbol{\psi}(x_1, x_2, \varepsilon i, t) \mathbb{1}_{\left(\varepsilon i - \frac{\varepsilon}{2}, \varepsilon i + \frac{\varepsilon}{2} \right]}(x_3).$$

By (3.39) and (5.53) we have

(5.65)
$$\int_{\Omega \times (0,T)} \overline{\boldsymbol{v}}_\varepsilon \cdot \overline{\boldsymbol{\psi}}_\varepsilon dm_\varepsilon dt = \frac{\varepsilon}{r_\varepsilon} \sum_{i \in Z_\varepsilon} \int_{\Omega' \times \left(\varepsilon i - \frac{r_\varepsilon}{2}, \varepsilon i + \frac{r_\varepsilon}{2} \right) \times (0,T)} \boldsymbol{u}_\varepsilon(x', \varepsilon i, t) \cdot \boldsymbol{\psi}(x', \varepsilon i, t) dx dt$$

$$= \sum_{i \in Z_\varepsilon} \int_{\Omega' \times \left(\varepsilon i - \frac{\varepsilon}{2}, \varepsilon i + \frac{\varepsilon}{2} \right) \times (0,T)} \boldsymbol{u}_\varepsilon(x', \varepsilon i, t) \cdot \boldsymbol{\psi}(x', \varepsilon i, t) dx dt = \int_{\Omega \times (0,T)} \overline{\boldsymbol{v}}_\varepsilon \cdot \overline{\boldsymbol{\psi}}_\varepsilon dx dt.$$

We infer from (5.55) and from the estimate

(5.66)
$$|\boldsymbol{\psi} - \overline{\boldsymbol{\psi}}_\varepsilon|_{L^\infty(\Omega\times(0,T);\mathbb{R}^3)} \leq C\varepsilon,$$

that

(5.67)
$$\lim_{\varepsilon\to0} \int_{\Omega\times(0,T)} \overline{\boldsymbol{v}}_\varepsilon \cdot \overline{\boldsymbol{\psi}}_\varepsilon dxdt = \int_{\Omega\times(0,T)} \left(\int_Y \overline{\boldsymbol{v}}_0(x,t,y)dy\right) \boldsymbol{\psi}(x,t)dxdt.$$

By (5.54) and (5.66), the following holds

(5.68)
$$\lim_{\varepsilon\to0} \left| \int_{\Omega\times(0,T)} \boldsymbol{u}_\varepsilon \cdot \boldsymbol{\psi} dm_\varepsilon dt - \int_{\Omega\times(0,T)} \overline{\boldsymbol{v}}_\varepsilon \cdot \overline{\boldsymbol{\psi}}_\varepsilon dm_\varepsilon dt \right| = 0.$$

The weak* convergence of $(\boldsymbol{u}_\varepsilon m_\varepsilon)$ to $\boldsymbol{v}$ and (5.65), (5.67), (5.68), imply

$$\int_{\Omega\times(0,T)} \boldsymbol{v} \cdot \boldsymbol{\psi} dxdt = \lim_{\varepsilon\to0} \int_{\Omega\times(0,T)} \boldsymbol{u}_\varepsilon \cdot \boldsymbol{\psi} dm_\varepsilon dt = \int_{\Omega\times(0,T)} \left(\int_Y \overline{\boldsymbol{v}}_0(x,t,y)dy\right) \cdot \boldsymbol{\psi}(x,t)dxdt,$$

yielding, by the arbitrary choice of $\boldsymbol{\psi}$,

(5.69)
$$\boldsymbol{v}(x,t) = \int_Y \overline{\boldsymbol{v}}_0(x,t,y)dy \quad \text{in} \quad \Omega\times(0,T).$$

By (5.59) and (5.69), the proof of (5.57) is achieved provided that we establish that

(5.70)
$$\frac{\partial \overline{v}_{0\alpha}}{\partial y_\beta} = 0 \quad \forall \alpha, \beta \in \{1,2\}.$$

To that aim, let us fix $\boldsymbol{\Psi} \in \mathcal{D}\left(\Omega\times(0,T); C_\sharp^\infty\left(\left(-\frac{1}{2},\frac{1}{2}\right)^2;\mathbb{S}^3\right)\right)$. Since $\boldsymbol{u}_\varepsilon$ vanishes on $\partial\Omega \times (0,T)$, by integrating by parts with respect to x_1 and x_2, we get (see (2.1))

$$\int_{\Omega\times(0,T)} \boldsymbol{e}_{x'}(\boldsymbol{u}_\varepsilon') : \boldsymbol{\Psi}\left(x,t,\frac{x'}{\varepsilon}\right) dm_\varepsilon dt = -\int_{\Omega\times(0,T)} \boldsymbol{u}_\varepsilon' \cdot \mathbf{div}_{x'}' \boldsymbol{\Psi}\left(x,t,\frac{x'}{\varepsilon}\right) dm_\varepsilon dt$$
$$- \frac{1}{\varepsilon} \int_{\Omega\times(0,T)} \boldsymbol{u}_\varepsilon' \cdot \mathbf{div}_{y'}' \boldsymbol{\Psi}\left(x,t,\frac{x'}{\varepsilon}\right) dm_\varepsilon dt.$$

By (5.10), the left-hand side and the first term on the right-hand side of the above equation are bounded, therefore

(5.71)
$$\lim_{\varepsilon\to0} \int_{\Omega\times(0,T)} \boldsymbol{u}_\varepsilon' \cdot \mathbf{div}_{y'}' \boldsymbol{\Psi}\left(x,t,\frac{x'}{\varepsilon}\right) dm_\varepsilon dt = 0.$$

On the other hand, by (5.54) and (5.61), there holds

(5.72) $\lim_{\varepsilon\to0} \left| \int_{\Omega\times(0,T)} \boldsymbol{u}_\varepsilon' \cdot \mathbf{div}_{y'}' \boldsymbol{\Psi}\left(x,t,\frac{x'}{\varepsilon}\right) dm_\varepsilon dt - \int_{\Omega\times(0,T)} \overline{\boldsymbol{v}}_\varepsilon' \cdot \mathbf{div}_{y'}' \overline{\boldsymbol{\Psi}}_\varepsilon\left(x,t,\frac{x'}{\varepsilon}\right) dm_\varepsilon dt \right| = 0.$

A computation analogous to (5.65) yields

$$(5.73) \quad \int_{\Omega \times (0,T)} \overline{\boldsymbol{v}}'_\varepsilon \cdot \mathbf{div}'_{y'} \overline{\boldsymbol{\Psi}}_\varepsilon \left(x, t, \frac{x'}{\varepsilon}\right) dm_\varepsilon dt = \int_{\Omega \times (0,T)} \overline{\boldsymbol{v}}'_\varepsilon \cdot \mathbf{div}'_{y'} \overline{\boldsymbol{\Psi}}_\varepsilon \left(x, t, \frac{x'}{\varepsilon}\right) dx dt.$$

By (5.61) and by the two-scale convergence of $\overline{\boldsymbol{v}}_\varepsilon$ to $\overline{\boldsymbol{v}}_0$ (see (5.55)), there holds

$$(5.74) \quad \lim_{\varepsilon \to 0} \int_{\Omega \times (0,T)} \overline{\boldsymbol{v}}'_\varepsilon \cdot \mathbf{div}'_{y'} \overline{\boldsymbol{\Psi}}_\varepsilon \left(x, t, \frac{x'}{\varepsilon}\right) dx dt = \int_{\Omega \times (0,T) \times Y} \overline{\boldsymbol{v}}'_0 \cdot \mathbf{div}'_{y'} \boldsymbol{\Psi} \left(x, t, y'\right) dx dt dy.$$

Combining (5.71)-(5.74), we get

$$(5.75) \quad \int_{\Omega \times (0,T) \times Y} \overline{\boldsymbol{v}}'_0 \cdot \mathbf{div}'_{y'} \boldsymbol{\Psi} \left(x, t, y'\right) dx dt dy = 0,$$

hence $\boldsymbol{e}_{y'}(\overline{\boldsymbol{v}}'_0) = 0$, in the sense of distributions. We deduce that $y' \to \overline{\boldsymbol{v}}'_0(x, t, y', y_3)$ is a rigid displacement. By integrating (5.75) by parts, we infer

$$\int_{\Omega \times (0,T) \times \partial\left(-\frac{1}{2}, \frac{1}{2}\right)^2} \overline{\boldsymbol{v}}'_0(x, t, y) \cdot \boldsymbol{\Psi} \left(x, t, y'\right) \boldsymbol{\nu} dx dt d\mathcal{H}^1(y') = 0,$$

and deduce from the arbitrary choice of $\boldsymbol{\Psi} \in \mathcal{D}\left(\Omega \times (0,T); C_\sharp^\infty \left(\left(-\frac{1}{2}, \frac{1}{2}\right)^2; \mathbb{S}^3\right)\right)$, that $\overline{\boldsymbol{v}}'_0 \in L^2\left(\Omega \times (0,T), H_\sharp^1\left(\left(-\frac{1}{2}, \frac{1}{2}\right)^2; \mathbb{R}^3\right)\right)$. The periodicity of $\overline{\boldsymbol{v}}'_0$ with respect to y' and the fact that $y' \to \overline{\boldsymbol{v}}'_0(x, t, y', y_3)$ is a rigid displacement imply that $y' \to \overline{\boldsymbol{v}}'_0(x, t, y')$ is a constant field. Assertion (5.70) is proved. The proof of (5.57) is achieved.

Proof of (5.58). We assume (5.30), fix $\psi \in \mathcal{D}\left(\Omega \times (0,T); \mathcal{D}_\sharp\left(\left(-\frac{1}{2}, \frac{1}{2}\right)^2\right)\right)$, $\eta \in \mathcal{D}(I)$, and $\alpha \in \{1, 2\}$. Noticing that the mapping $x \to \psi\left(x, t, \frac{x'}{\varepsilon}\right) \eta\left(\frac{y_\varepsilon(x_3)}{r_\varepsilon}\right)$ is compactly supported in B_ε, by integration by parts we obtain

$$(5.76) \quad \begin{aligned} &\int_{\Omega \times (0,T)} \left(\frac{\partial u_{\varepsilon\alpha}}{\partial x_3} + \frac{\partial u_{\varepsilon 3}}{\partial x_\alpha}\right) \psi\left(x, t, \frac{x'}{\varepsilon}\right) \eta\left(\frac{y_\varepsilon(x_3)}{r_\varepsilon}\right) dm_\varepsilon dt \\ &= -\int_{\Omega \times (0,T)} \left(u_{\varepsilon\alpha} \frac{\partial \psi}{\partial x_3}\left(x, t, \frac{x'}{\varepsilon}\right) + u_{\varepsilon 3} \frac{\partial \psi}{\partial x_\alpha}\left(x, t, \frac{x'}{\varepsilon}\right)\right) \eta\left(\frac{y_\varepsilon(x_3)}{r_\varepsilon}\right) dm_\varepsilon dt \\ &\quad - \int_{\Omega \times (0,T)} \frac{u_{\varepsilon\alpha}}{r_\varepsilon} \psi\left(x, t, \frac{x'}{\varepsilon}\right) \frac{\partial \eta}{\partial x_3}\left(\frac{y_\varepsilon(x_3)}{r_\varepsilon}\right) dm_\varepsilon dt \\ &\quad - \frac{1}{\varepsilon} \int_{\Omega \times (0,T)} u_{\varepsilon 3} \frac{\partial \psi}{\partial y_\alpha}\left(x, t, \frac{x'}{\varepsilon}\right) \eta\left(\frac{y_\varepsilon(x_3)}{r_\varepsilon}\right) dm_\varepsilon dt. \end{aligned}$$

By (5.1) we have

$$\int \left|\frac{\partial u_{\varepsilon\alpha}}{\partial x_3} + \frac{\partial u_{\varepsilon 3}}{\partial x_\alpha}\right|^2 + \left|\frac{u_{\varepsilon\alpha}}{r_\varepsilon}\right|^2 + |u_{\varepsilon 3}|^2(\tau) dm_\varepsilon \leq C \int \frac{1}{r_\varepsilon^2} |e(\boldsymbol{u}_\varepsilon)|^2(\tau) dm_\varepsilon,$$

hence, by (5.30), all terms of the three first lines of (5.76) are bounded. We infer

$$\lim_{\varepsilon \to 0} \int_{\Omega \times (0,T)} u_{\varepsilon 3} \frac{\partial \psi}{\partial y_\alpha}\left(x, t, \frac{x'}{\varepsilon}\right) \eta\left(\frac{y_\varepsilon(x_3)}{r_\varepsilon}\right) dm_\varepsilon dt = 0,$$

and then deduce from (5.54) and from an estimate analogous to (5.61) that

$$(5.77) \qquad \lim_{\varepsilon \to 0} \int_{\Omega \times (0,T)} \overline{v}_{\varepsilon 3} \frac{\partial \overline{\psi}_{\varepsilon}}{\partial y_{\alpha}} \left(x, t, \frac{x'}{\varepsilon} \right) \eta \left(\frac{y_{\varepsilon}(x_3)}{r_{\varepsilon}} \right) dm_{\varepsilon} dt = 0,$$

where $\overline{\psi}_{\varepsilon}$ is defined by (5.64). Taking (5.53) and (5.64) into account, and noticing that $\frac{1}{r_{\varepsilon}} \int_{\left(\varepsilon i - \frac{r_{\varepsilon}}{2}, \varepsilon i + \frac{r_{\varepsilon}}{2} \right)} \eta \left(\frac{y_{\varepsilon}(x_3)}{r_{\varepsilon}} \right) dx_3 = \int_I \eta(y) dy$, we get

$$(5.78) \qquad \begin{aligned} &\int_{\Omega \times (0,T)} \overline{v}_{\varepsilon 3} \frac{\partial \overline{\psi}_{\varepsilon}}{\partial y_{\alpha}} \left(x, t, \frac{x'}{\varepsilon} \right) \eta \left(\frac{y_{\varepsilon}(x_3)}{r_{\varepsilon}} \right) dm_{\varepsilon} dt \\ &= \frac{\varepsilon}{r_{\varepsilon}} \sum_{i \in Z_{\varepsilon}} \int_{\Omega' \times \left(\varepsilon i - \frac{r_{\varepsilon}}{2}, \varepsilon i + \frac{r_{\varepsilon}}{2} \right) \times (0,T)} u_{\varepsilon 3}(x', \varepsilon i, t) \frac{\partial \psi}{\partial y_{\alpha}} \left(x', \varepsilon i, t, \frac{x'}{\varepsilon} \right) \eta \left(\frac{y_{\varepsilon}(x_3)}{r_{\varepsilon}} \right) dx' dx_3 dt \\ &= \varepsilon \sum_{i \in Z_{\varepsilon}} \int_{\Omega' \times (0,T)} u_{\varepsilon 3}(x', \varepsilon i, t) \frac{\partial \psi}{\partial y_{\alpha}} \left(x', \varepsilon i, t, \frac{x'}{\varepsilon} \right) dx' dt \left(\int_I \eta(y) dy \right) \\ &= \left(\int_{\Omega \times (0,T)} \overline{v}_{\varepsilon 3} \frac{\partial \overline{\psi}_{\varepsilon}}{\partial y_{\alpha}} \left(x, t, \frac{x'}{\varepsilon} \right) dx dt \right) \left(\int_I \eta(y) dy \right). \end{aligned}$$

On the other hand, by (5.55) and an estimate analogous to (5.61), there holds

$$(5.79) \qquad \lim_{\varepsilon \to 0} \int_{\Omega \times (0,T)} \overline{v}_{\varepsilon 3} \frac{\partial \overline{\psi}_{\varepsilon}}{\partial y_{\alpha}} \left(x, t, \frac{x'}{\varepsilon} \right) dx dt = \int_{\Omega \times (0,T) \times Y} \boldsymbol{v}_{03} \frac{\partial \psi}{\partial y_{\alpha}} dx dt dy.$$

Joining (5.77), (5.78), (5.79), and choosing η such that $\int_I \eta dy_3 \neq 0$, we infer that $\int_{\Omega \times (0,T) \times Y} \boldsymbol{v}_{03} \frac{\partial \psi}{\partial y_{\alpha}} dx dt dy = 0$. By the arbitraryness of ψ, Assertion (5.58) is proved. $\square$

In the next Corollary, we derive from Proposition 5.2 and lemmas 5.3, 5.4, 5.5, a series of convergences and identification relations for various sequences associated with the solution to (3.5).

COROLLARY 5.6. *Let $\boldsymbol{u}_{\varepsilon}$ be the solution to* (3.5).
(i) Up to a subsequence, the convergences (5.11) *hold and*

$$(5.80) \qquad n v_1 = n v_2 = 0 \quad if \quad k = +\infty, \qquad n \boldsymbol{v} = 0 \quad if \quad \kappa = +\infty.$$

Under (5.13) *or* (5.14), $n \boldsymbol{u} = n \boldsymbol{v}$.
(ii) If $\kappa > 0$, the relation (5.31) *and convergences* (5.32) *hold.*
(iii) In the periodic case, that is under (3.34) *and* (3.35), *the convergences and relations* (5.48) *hold. If $\vartheta > 0$ (resp. $\vartheta = 0$), the relations* (5.49) *(resp.* (5.50)) *are verified.*

Proof. Noticing that by (3.8) and (5.5), the estimate (5.10) holds, Assertion (i) follows from Lemma 5.3 (Assertion (5.80) is a consequence of (3.8), (5.5), and (5.12)). If $\kappa > 0$, by (3.8) and (5.5) the estimate (5.30) holds, and Assertion (ii) follows from Lemma 5.4. In the periodic case, by (3.34), (3.35), and (5.5), $\boldsymbol{u}_{\varepsilon}$ satisfies (5.10) and (5.47) and by (3.8), (3.38) and (5.5) it satisfies (5.30) if $\vartheta = 0$. Hence (iii) results from Lemma 5.5. $\square$

6. Proof of theorems 3.1, 3.7, 3.12.

In the spirit of Tartar [53, 54], we will multiply (3.5) by an appropriate test field ϕ_ε, integrate by parts, and, passing to the limit as $\varepsilon \to 0$ by means of the convergences derived in Corollary 5.6, obtain a variational problem equivalent to the announced limit problem, and also to (4.23) for some suitable H, V, a, h, ξ_0, ξ_1. Theorem 4.4 will yield existence, uniqueness, and regularity of the effective displacement. Uniqueness implies that the convergences obtained in Corollary 5.6 for subsequences hold for the complete sequences.

6.1. Proof of Theorem 3.12.   We set

(6.1)
$$H := \left\{ \begin{array}{l} (\boldsymbol{w}_0, \boldsymbol{\psi}) \in L^2(\Omega \times Y; \mathbb{R}^3) \times L^2(\Omega; \mathbb{R}^3), \\ \boldsymbol{w}_0 = \boldsymbol{\psi} \ \text{in } \Omega \times B \end{array} \right\},$$
$$((\boldsymbol{w}_0, \boldsymbol{\psi}), (\widetilde{\boldsymbol{w}}_0, \widetilde{\boldsymbol{\psi}}))_H := \int_{\Omega \times Y} \rho \boldsymbol{w}_0 \cdot \widetilde{\boldsymbol{w}}_0 dxdy, \quad \right\} \quad \text{if } \vartheta > 0,$$
$$H := L^2(\Omega \times Y; \mathbb{R}^3) \times L^2(\Omega; \mathbb{R}^3),$$
$$((\boldsymbol{w}_0, \boldsymbol{\psi}), (\widetilde{\boldsymbol{w}}_0, \widetilde{\boldsymbol{\psi}}))_H := \int_{\Omega \times Y} \rho \boldsymbol{w}_0 \cdot \widetilde{\boldsymbol{w}}_0 dxdy + \int_\Omega \bar{\rho}_1 \boldsymbol{\psi} \cdot \widetilde{\boldsymbol{\psi}} dx, \right\} \quad \text{if } \vartheta = 0.$$

We easily deduce from the positiveness of ρ and $\bar{\rho}_1$ (see (3.7)) that H is a Hilbert space. We fix a couple $(\boldsymbol{w}_0, \boldsymbol{\psi}) \in L^2(0,T; H)$ satisfying (see (3.37))

(6.2) $\quad \boldsymbol{w}_0 \in C^\infty([0,T]; \mathcal{D}(\Omega; C_\sharp^\infty(Y; \mathbb{R}^3))), \quad \boldsymbol{\psi} \in C^\infty([0,T]; \mathcal{D}(\Omega; \mathbb{R}^3)),$

(6.3) $\quad \boldsymbol{w}_0(T) = \dfrac{\partial \boldsymbol{w}_0}{\partial t}(T) = \boldsymbol{\psi}(T) = \dfrac{\partial \boldsymbol{\psi}}{\partial t}(T) = 0,$

(6.4) $\quad \psi_1 = \psi_2 = 0 \ \text{ if } \ k = +\infty, \qquad \boldsymbol{\psi} = 0 \ \text{ if } \ \kappa = +\infty,$

(6.5) $\quad \boldsymbol{w}_0(.,y) = \boldsymbol{\psi}(.) \ \text{ in } \Omega \times (0,T) \times A.$

We choose a sequence (α_ε) of positive reals such that

(6.6) $\quad\quad \varepsilon \ll \alpha_\varepsilon \ll 1 \quad \text{and} \quad \left[\dfrac{r_\varepsilon}{\varepsilon} \ll \alpha_\varepsilon \ll 1 \quad \text{if } \vartheta = 0 \right],$

and set

(6.7) $$C_\varepsilon := \{ x \in \Omega, \ \text{dist}(x, B_\varepsilon) < \alpha_\varepsilon r_\varepsilon \}.$$

It is usefull to notice that

(6.8) $$\mathcal{L}^3(C_\varepsilon \setminus B_\varepsilon) \le C \dfrac{\alpha_\varepsilon r_\varepsilon}{\varepsilon},$$

and that, by (6.5), the following estimate holds for $m \in \{1, 2\}$:

(6.9)
$$\left| \boldsymbol{\psi}(x,t) - \boldsymbol{w}_0\left(x,t,\dfrac{x}{\varepsilon}\right) \right| + \left| \dfrac{\partial^m \boldsymbol{\psi}}{\partial t^m}(x,t) - \dfrac{\partial^m \boldsymbol{w}_0}{\partial t^m}\left(x,t,\dfrac{x}{\varepsilon}\right) \right| \le C\beta_\varepsilon \ \text{ in } C_\varepsilon \times (0,T),$$
$$\beta_\varepsilon = \alpha_\varepsilon \quad \text{if } \vartheta > 0, \quad \beta_\varepsilon = \dfrac{r_\varepsilon}{\varepsilon} \quad \text{if } \vartheta = 0.$$

By (6.7), we can fix a sequence (η_ε) in $C^\infty(\overline{\Omega})$ satisfying

$$(6.10) \quad 0 \le \eta_\varepsilon \le 1, \qquad \eta_\varepsilon = 1 \quad \text{in} \quad B_\varepsilon, \qquad \eta_\varepsilon = 0 \quad \text{in} \quad \Omega \setminus C_\varepsilon, \qquad |\boldsymbol{\nabla}\eta_\varepsilon| < \frac{C}{r_\varepsilon \alpha_\varepsilon}.$$

The sequence of test fields $(\boldsymbol{\phi}_\varepsilon)$ mentioned above will be defined by

$$(6.11) \qquad \boldsymbol{\phi}_\varepsilon(x,t) := \eta_\varepsilon(x)\,\widehat{\boldsymbol{\psi}}_\varepsilon(x,t) + (1 - \eta_\varepsilon(x))\boldsymbol{w}_0\left(x,t,\frac{x}{\varepsilon}\right),$$

where $\widehat{\boldsymbol{\psi}}_\varepsilon$ is described in Section 7. As $\boldsymbol{\phi}_\varepsilon(x,t) = \boldsymbol{w}_0\left(x,t,\frac{x}{\varepsilon}\right)$ in $\Omega \setminus C_\varepsilon \times (0,T)$, we deduce from (6.5), (6.9), (6.10), (6.11), and (7.4) that the following estimates hold in $\Omega \times (0,T)$ for $m \in \{1,2\}$:

$$(6.12) \quad \left|\boldsymbol{\phi}_\varepsilon(x,t) - \boldsymbol{w}_0\left(x,t,\frac{x}{\varepsilon}\right)\right| + \left|\frac{\partial^m \boldsymbol{\phi}_\varepsilon}{\partial t^m}(x,t) - \frac{\partial^m \boldsymbol{w}_0}{\partial t^m}\left(x,t,\frac{x}{\varepsilon}\right)\right| \le C\beta_{1\varepsilon},$$
$$\beta_{1\varepsilon} = r_\varepsilon + \alpha_\varepsilon \quad \text{if } \vartheta > 0, \quad \beta_{1\varepsilon} = \frac{r_\varepsilon}{\varepsilon} \quad \text{if} \quad \vartheta = 0.$$

It is also interesting to notice that by (6.10), (6.11), and (7.4),

$$(6.13) \quad |\boldsymbol{\phi}_\varepsilon(x,t) - \boldsymbol{\psi}(x,t)| + \left|\frac{\partial^m \boldsymbol{\phi}_\varepsilon}{\partial t^m}(x,t) - \frac{\partial^m \boldsymbol{\psi}}{\partial t^m}(x,t)\right| \le Cr_\varepsilon \quad \text{in} \quad B_\varepsilon \times (0,T).$$

By (3.5) and (3.6) we have $|\boldsymbol{\sigma}_\varepsilon(\boldsymbol{\phi}_\varepsilon)| \le C\mu_{0\varepsilon}|\boldsymbol{\nabla}\boldsymbol{\phi}_\varepsilon|$ in $(\Omega \setminus B_\varepsilon) \times (0,T)$, therefore by (6.9), (6.10), (6.12), (7.4), the next estimates are satisfied in $(C_\varepsilon \setminus B_\varepsilon) \times (0,T)$

$$|\boldsymbol{\sigma}_\varepsilon(\boldsymbol{\phi}_\varepsilon)| \le C\mu_{0\varepsilon}\left(|\boldsymbol{\nabla}\eta_\varepsilon|\left|\widehat{\boldsymbol{\psi}}_\varepsilon(x,t) - \boldsymbol{w}_0\left(x,t,\frac{x}{\varepsilon}\right)\right| + \left|\boldsymbol{\nabla}\widehat{\boldsymbol{\psi}}_\varepsilon(x,t)\right| + \left|\boldsymbol{\nabla}\left(\boldsymbol{w}_0\left(x,t,\frac{x}{\varepsilon}\right)\right)\right|\right)$$
$$(6.14) \qquad \le C\mu_{0\varepsilon}\left(\frac{1}{\alpha_\varepsilon r_\varepsilon}(r_\varepsilon + \beta_\varepsilon) + \frac{C}{\varepsilon}\right) \le C\mu_{0\varepsilon}\beta_{2\varepsilon},$$
$$\beta_{2\varepsilon} = \frac{1}{\alpha_\varepsilon} + \frac{1}{\varepsilon} \quad \text{if } \vartheta > 0, \quad \beta_{2\varepsilon} = \frac{1}{\alpha_\varepsilon \varepsilon} \quad \text{if } \vartheta = 0.$$

Applying (4.5) to $\chi_\varepsilon = \rho_\varepsilon \mathbb{1}_{\Omega \setminus B_\varepsilon}$, $h_0 \in \{\boldsymbol{w}_0, \frac{\partial^2 \boldsymbol{w}_0}{\partial t^2}, \boldsymbol{w}_0(0), \frac{\partial^2 \boldsymbol{w}_0}{\partial t^2}(0)\}$, we deduce from (3.7), (4.4), (6.6), and (6.12), that the following convergences hold for $m \in \{1,2\}$

$$(6.15) \quad \begin{aligned} &\rho_\varepsilon \boldsymbol{\phi}_\varepsilon \mathbb{1}_{\Omega \setminus B_\varepsilon} \longrightarrow \rho \mathbb{1}_{Y \setminus A}(y)\boldsymbol{w}_0, && \rho_\varepsilon \frac{\partial^m \boldsymbol{\phi}_\varepsilon}{\partial t^m}\mathbb{1}_{\Omega \setminus B_\varepsilon} \longrightarrow \rho \mathbb{1}_{Y \setminus A}(y)\frac{\partial^m \boldsymbol{w}_0}{\partial t^m}, \\ &\rho_\varepsilon \boldsymbol{\phi}_\varepsilon(0)\mathbb{1}_{\Omega \setminus B_\varepsilon} \longrightarrow \rho \mathbb{1}_{Y \setminus A}(y)\boldsymbol{w}_0(0), && \rho_\varepsilon \frac{\partial \boldsymbol{\phi}_\varepsilon}{\partial t}(0)\mathbb{1}_{\Omega \setminus B_\varepsilon} \longrightarrow \rho \mathbb{1}_{Y \setminus A}(y)\frac{\partial \boldsymbol{w}_0}{\partial t}(0). \end{aligned}$$

By multiplying (3.5) by $\boldsymbol{\phi}_\varepsilon$, after integrations by parts we obtain (see (6.3))

$$(6.16) \quad \begin{aligned} \int_{\Omega \times (0,T)} \rho_\varepsilon \boldsymbol{u}_\varepsilon \cdot \frac{\partial^2 \boldsymbol{\phi}_\varepsilon}{\partial t^2}\,dxdt &+ \int_\Omega \rho_\varepsilon \boldsymbol{a}_0 \cdot \frac{\partial \boldsymbol{\phi}_\varepsilon}{\partial t}(0)dx - \int_\Omega \rho_\varepsilon \boldsymbol{b}_0 \cdot \boldsymbol{\phi}_\varepsilon(0)dx \\ &+ \int_{\Omega \times (0,T)} \boldsymbol{e}(\boldsymbol{u}_\varepsilon) : \boldsymbol{\sigma}_\varepsilon(\boldsymbol{\phi}_\varepsilon)dxdt = \int_{\Omega \times (0,T)} \rho_\varepsilon \boldsymbol{f} \cdot \boldsymbol{\phi}_\varepsilon dxdt. \end{aligned}$$

By (3.7) and (3.39), we have

$$
(6.17) \quad
\int_{\Omega \times (0,T)} \rho_\varepsilon \boldsymbol{u}_\varepsilon \cdot \frac{\partial^2 \boldsymbol{\phi}_\varepsilon}{\partial t^2} dx dt = \int_{\Omega \times (0,T)} \rho \mathbb{1}_{\Omega \setminus B_\varepsilon} \boldsymbol{u}_\varepsilon \cdot \frac{\partial^2 \boldsymbol{\phi}_\varepsilon}{\partial t^2} dx dt
$$
$$
+ \int_{\Omega \times (0,T)} \overline{\rho}_1 \boldsymbol{u}_\varepsilon \cdot \frac{\partial^2 \boldsymbol{\phi}_\varepsilon}{\partial t^2} dm_\varepsilon(x) dt.
$$

We deduce from (5.48) (see Corollary 5.6), and (6.15) that

$$
\lim_{\varepsilon \to 0} \int_{\Omega \times (0,T)} \rho_\varepsilon \mathbb{1}_{\Omega \setminus B_\varepsilon} \boldsymbol{u}_\varepsilon \cdot \frac{\partial^2 \boldsymbol{\phi}_\varepsilon}{\partial t^2} dx dt = \int_{\Omega \times (0,T) \times Y} \rho \mathbb{1}_{Y \setminus A}(y) \boldsymbol{u}_0 \cdot \frac{\partial^2 \boldsymbol{w}_0}{\partial t^2} dx dt dy.
$$

By (5.11), and (6.13), we have

$$
\lim_{\varepsilon \to 0} \int_{\Omega \times (0,T)} \overline{\rho}_1 \boldsymbol{u}_\varepsilon \cdot \frac{\partial^2 \boldsymbol{\phi}_\varepsilon}{\partial t^2} dm_\varepsilon(x) dt = \int_{\Omega \times (0,T)} \overline{\rho}_1 \boldsymbol{v} \cdot \frac{\partial^2 \boldsymbol{\psi}}{\partial t^2} dx dt.
$$

The last three equations imply

$$
(6.18) \quad \lim_{\varepsilon \to 0} \int_{\Omega \times (0,T)} \rho_\varepsilon \boldsymbol{u}_\varepsilon \cdot \frac{\partial^2 \boldsymbol{\phi}_\varepsilon}{\partial t^2} dx dt = \int_{\Omega \times (0,T) \times Y \setminus A} \rho \boldsymbol{u}_0 \cdot \frac{\partial^2 \boldsymbol{w}_0}{\partial t^2} dx dt dy + \int_{\Omega \times (0,T)} \overline{\rho}_1 \boldsymbol{v} \cdot \frac{\partial^2 \boldsymbol{\psi}}{\partial t^2} dx dt.
$$

As, by (3.5), $\boldsymbol{a}_0$, $\boldsymbol{b}_0$, and $\boldsymbol{f}$ are continuous, we obtain by the same argument

$$
\lim_{\varepsilon \to 0} \int_{\Omega} \rho_\varepsilon \boldsymbol{a}_0 \cdot \frac{\partial \boldsymbol{\phi}_\varepsilon}{\partial t}(0) dx = \int_{\Omega \times Y \setminus A} \rho \boldsymbol{a}_0 \cdot \frac{\partial \boldsymbol{w}_0}{\partial t}(0) dx dy + \int_{\Omega} \overline{\rho}_1 \boldsymbol{a}_0 \cdot \frac{\partial^2 \boldsymbol{\psi}}{\partial t^2}(0) dx,
$$

$$
(6.19) \quad \lim_{\varepsilon \to 0} \int_{\Omega} \rho_\varepsilon \boldsymbol{b}_0 \cdot \boldsymbol{\phi}_\varepsilon(0) dx = \int_{\Omega \times Y \setminus A} \rho \boldsymbol{b}_0 \cdot \boldsymbol{w}_0(0) dx dy + \int_{\Omega} \overline{\rho}_1 \boldsymbol{b}_0 \cdot \boldsymbol{\psi}(0) dx,
$$

$$
\lim_{\varepsilon \to 0} \int_{\Omega \times (0,T)} \rho_\varepsilon \boldsymbol{f} \cdot \boldsymbol{\phi}_\varepsilon dx dt = \int_{\Omega \times (0,T) \times Y \setminus A} \rho \boldsymbol{f} \cdot \boldsymbol{w}_0 dx dt dy + \int_{\Omega \times (0,T)} \overline{\rho}_1 \boldsymbol{f} \cdot \frac{\partial^2 \boldsymbol{\psi}}{\partial t^2} dx dt.
$$

We split the 4th term on the left-hand side of (6.16) into the sum of three terms:

$$
(6.20) \quad
\int_{\Omega \times (0,T)} \boldsymbol{e}(\boldsymbol{u}_\varepsilon) : \boldsymbol{\sigma}_\varepsilon(\boldsymbol{\phi}_\varepsilon) dx dt = I_{1\varepsilon} + I_{2\varepsilon} + I_{3\varepsilon}; \quad I_{1\varepsilon} := \int_{\Omega \setminus C_\varepsilon \times (0,T)} \varepsilon \boldsymbol{e}(\boldsymbol{u}_\varepsilon) : \frac{1}{\varepsilon} \boldsymbol{\sigma}_\varepsilon(\boldsymbol{\phi}_\varepsilon) dx dt,
$$
$$
I_{2\varepsilon} := \int_{C_\varepsilon \setminus B_\varepsilon \times (0,T)} \varepsilon \boldsymbol{e}(\boldsymbol{u}_\varepsilon) : \frac{1}{\varepsilon} \boldsymbol{\sigma}_\varepsilon(\boldsymbol{\phi}_\varepsilon) dx dt, \quad I_{3\varepsilon} := \int_{B_\varepsilon \times (0,T)} \boldsymbol{e}(\boldsymbol{u}_\varepsilon) : \boldsymbol{\sigma}_\varepsilon(\boldsymbol{\phi}_\varepsilon) dx dt.
$$

By (6.10) and (6.11), we have $\boldsymbol{\phi}_\varepsilon \mathbb{1}_{\Omega \setminus C_\varepsilon} = \boldsymbol{w}_0 \left(x, t, \frac{x}{\varepsilon} \right) \mathbb{1}_{\Omega \setminus C_\varepsilon}$. Taking (3.5), (3.6), and (3.34) into account, we deduce that

$$
(6.21) \quad \left| \frac{1}{\varepsilon} \boldsymbol{\sigma}_\varepsilon (\boldsymbol{\phi}_\varepsilon) - \boldsymbol{\sigma}_{0y}(\boldsymbol{w}_0) \left(x, t, \frac{x}{\varepsilon} \right) \right| \mathbb{1}_{\Omega \setminus C_\varepsilon} \leq C \varepsilon,
$$

where the operator $\boldsymbol{\sigma}_{0y}$ is defined by (3.42). The following convergence

(6.22)
$$\mathbb{1}_{\Omega \setminus C_\varepsilon} \longrightarrow\!\!\!\!\!\rightarrow \mathbb{1}_{Y \setminus A},$$

follows from (4.4) and from the strong convergence of $\mathbb{1}_{C_\varepsilon \setminus B_\varepsilon}$ to 0 in $L^2(\Omega)$, which results from (6.6) and (6.8). By applying Assertion (4.5) of Lemma 4.1 to $h_0 := \boldsymbol{\sigma}_{0y}(\boldsymbol{w}_0)$ and $\chi_\varepsilon := \mathbb{1}_{\Omega \setminus C_\varepsilon}$, taking (6.21) into account, we infer

(6.23)
$$\frac{1}{\varepsilon} \boldsymbol{\sigma}_\varepsilon (\boldsymbol{\phi}_\varepsilon) \, \mathbb{1}_{\Omega \setminus C_\varepsilon} \longrightarrow\!\!\!\!\!\rightarrow \boldsymbol{\sigma}_{0y}(\boldsymbol{w}_0) \mathbb{1}_{Y \setminus A}(y).$$

We deduce from (5.48), (6.20), and (6.23) that

(6.24)
$$\lim_{\varepsilon \to 0} I_{1\varepsilon} = \int_{\Omega \times (0,T) \times Y \setminus A} \boldsymbol{e}_y(\boldsymbol{u}_0) : \boldsymbol{\sigma}_{0y}(\boldsymbol{w}_0) dx dt dy.$$

By (3.34), (6.8) and (6.14), we have $\int_{C_\varepsilon \setminus B_\varepsilon \times (0,T)} \left| \frac{1}{\varepsilon} \boldsymbol{\sigma}_\varepsilon(\boldsymbol{\phi}_\varepsilon) \right|^2 dx dt \leq C \left(\alpha_\varepsilon + \frac{\varepsilon^2}{\alpha_\varepsilon} \right)$ if $\vartheta > 0$, and $\int_{C_\varepsilon \setminus B_\varepsilon \times (0,T)} \left| \frac{1}{\varepsilon} \boldsymbol{\sigma}_\varepsilon(\boldsymbol{\phi}_\varepsilon) \right|^2 dx dt \leq C \frac{r_\varepsilon}{\alpha_\varepsilon \varepsilon}$ if $\vartheta = 0$. Therefore, by (6.6), the sequence $\left(\frac{1}{\varepsilon} \boldsymbol{\sigma}_\varepsilon(\boldsymbol{\phi}_\varepsilon) \mathbb{1}_{C_\varepsilon \setminus B_\varepsilon} \right)$ strongly converges to 0 in $L^2(\Omega \times (0,T); \mathbb{S}^3)$. Accordingly, we infer from (5.47) and (6.20) that

(6.25)
$$\lim_{\varepsilon \to 0} I_{2\varepsilon} = 0.$$

Finally, the limit of the sequence $(I_{3\varepsilon})$ defined by (6.20) is computed in Lemma 7.1 in terms of k and κ. Passing to the limit as $\varepsilon \to 0$ in (6.16), collecting (6.4), (6.18), (6.19), (6.20), (6.24), (6.25), and (7.5), we obtain the variational formulation given, according to the order of magnitude of k and κ, by (6.26), (6.39), (6.41), or (6.32). We distinguish 4 cases:

Case $0 < k < +\infty$. We find

(6.26)
$$\int_{\Omega \times (0,T) \times Y \setminus A} \rho \boldsymbol{u}_0 \cdot \frac{\partial^2 \boldsymbol{w}_0}{\partial t^2} dx dt dy + \int_{\Omega \times (0,T) \times Y \setminus A} \rho \boldsymbol{a}_0 \cdot \frac{\partial \boldsymbol{w}_0}{\partial t}(0) - \rho \boldsymbol{b}_0 \cdot \boldsymbol{w}_0(0) dx dy$$
$$+ \int_{\Omega \times (0,T)} \overline{\rho}_1 \boldsymbol{v} \cdot \frac{\partial^2 \boldsymbol{\psi}}{\partial t^2} dx dt + \int_\Omega \overline{\rho}_1 \boldsymbol{a}_0 \cdot \frac{\partial \boldsymbol{\psi}}{\partial t}(0) - \overline{\rho}_1 \boldsymbol{b}_0 \cdot \boldsymbol{\psi}(0) dx$$
$$+ \int_{\Omega \times (0,T) \times Y \setminus A} \boldsymbol{e}_y(\boldsymbol{u}_0) : \boldsymbol{\sigma}_{0y}(\boldsymbol{w}_0) dx dt dy + k \int_{\Omega \times (0,T)} \boldsymbol{e}_{x'}(\boldsymbol{v}') : \boldsymbol{\sigma}_{x'}(\boldsymbol{\psi}') dx dt$$
$$= \int_{\Omega \times (0,T) \times Y \setminus A} \rho \boldsymbol{f} \cdot \boldsymbol{w}_0 dx dt dy + \int_{\Omega \times (0,T)} \overline{\rho}_1 \boldsymbol{f} \cdot \boldsymbol{\psi} dx dt,$$

for all $(\boldsymbol{w}_0, \boldsymbol{\psi}) \in L^2(0, T; H)$ satisfying (6.2)-(6.5). We set (see (6.1))

$$\xi = (\boldsymbol{u}_0, \boldsymbol{v}), \quad \xi_0 = (\boldsymbol{a}_0, \boldsymbol{a}_0), \quad \xi_1 = (\boldsymbol{b}_0, \boldsymbol{b}_0), \quad h = (\boldsymbol{f}, \boldsymbol{f}),$$

$$V := \left\{ (\boldsymbol{w}_0, \boldsymbol{\psi}) \in H \,\middle|\, \begin{array}{l} \boldsymbol{w}_0 \in L^2(\Omega; H^1_\sharp(Y; \mathbb{R}^3)) \\ \psi_1, \psi_2 \in L^2(0, L; H^1_0(\Omega')) \end{array} \right\} \qquad \text{if } \vartheta > 0,$$

(6.27)
$$V := \left\{ (\boldsymbol{w}_0, \boldsymbol{\psi}) \in H \,\middle|\, \begin{array}{l} \boldsymbol{w}_0 \in L^2(\Omega; H^1_\sharp(Y; \mathbb{R}^3)), \\ \psi_1, \psi_2 \in L^2(0, L; H^1_0(\Omega')) \\ \boldsymbol{w}'_0(x, y) = \boldsymbol{\psi}'(x) \text{ on } \Omega \times \Sigma \\ \displaystyle\int_\Sigma w_{03}(., y) d\mathcal{H}^2(y) = \psi_3 \end{array} \right\} \qquad \text{if } r_\varepsilon \ll \varepsilon,$$

$$\bar{a}(\boldsymbol{v}, \boldsymbol{\psi}) := k \int_\Omega \boldsymbol{e}_{x'}(\boldsymbol{v}') : \boldsymbol{\sigma}_{x'}(\boldsymbol{\psi}') dx,$$

$$a((\boldsymbol{u}_0, \boldsymbol{v}), (\boldsymbol{w}_0, \boldsymbol{\psi})) := \int_{\Omega \times Y \setminus A} \boldsymbol{e}_y(\boldsymbol{u}_0) : \boldsymbol{\sigma}_y(\boldsymbol{w}_0) dx dy + \bar{a}(\boldsymbol{v}, \boldsymbol{\psi}),$$

$$((\boldsymbol{u}_0, \boldsymbol{v}), (\boldsymbol{w}_0, \boldsymbol{\psi}))_V := ((\boldsymbol{u}_0, \boldsymbol{v}), (\boldsymbol{w}_0, \boldsymbol{\psi}))_H + \bar{a}(\boldsymbol{v}, \boldsymbol{\psi}) + \int_{\Omega \times Y \setminus A} \boldsymbol{\nabla}_y \boldsymbol{u}_0 \cdot \boldsymbol{\nabla}_y \boldsymbol{w}_0 dx dy.$$

By (4.8), (5.11) and (5.48), we have $\xi = (\boldsymbol{u}_0, \boldsymbol{v}) \in L^2(0, T; V)$, $\frac{\partial \xi}{\partial t} \in L^2(0, T; H)$, thus by a density argument the variational formulation (6.26) is equivalent to (4.23). By (6.1), (6.27), and the following Korn's inequality (see [42], p. 14),

$$\int_{Y \setminus A} |\boldsymbol{w}|^2 + |\boldsymbol{\nabla}(\boldsymbol{w})|^2 dy \le C \int_{Y \setminus A} |\boldsymbol{w}|^2 + |\boldsymbol{e}(\boldsymbol{w})|^2 dy \quad \forall \boldsymbol{w} \in H^1(Y \setminus A; \mathbb{R}^3),$$

for all $\widetilde{\xi} = (\boldsymbol{w}_0, \boldsymbol{\psi}) \in V$, the following holds

(6.28)
$$\begin{aligned} ||\widetilde{\xi}||_V^2 &= |\widetilde{\xi}|_H^2 + \bar{a}(\psi, \psi) + ||\boldsymbol{\nabla}_y(\boldsymbol{w}_0)||^2_{L^2(\Omega \times Y \setminus A; \mathbb{R}^3)} \\ &\le C|\widetilde{\xi}|_H^2 + C||\boldsymbol{e}_y(\boldsymbol{w}_0)||^2_{L^2(\Omega \times Y \setminus A; \mathbb{R}^3)} + \bar{a}(\psi, \psi) \le C|\widetilde{\xi}|_H^2 + Ca(\widetilde{\xi}, \widetilde{\xi}), \end{aligned}$$

yielding (4.18). We deduce from Theorem 4.4 that $\xi = (\boldsymbol{u}_0, \boldsymbol{v})$ is the unique solution to (6.26). By (4.19), (4.20), (6.27), the following holds

(6.29)
$$\xi \in C([0, T]; V) \cap C^1([0, T]; H), \ \xi(0) = (\boldsymbol{a}_0, \boldsymbol{a}_0), \ \frac{\partial \xi}{\partial t}(0) = (\boldsymbol{b}_0, \boldsymbol{b}_0).$$

It follows from (6.29), from the following inequalities (deduced from (6.1), (6.27))

(6.30)
$$||\boldsymbol{w}_0||_{L^2(\Omega; H^1_\sharp(Y; \mathbb{R}^3))} + ||\boldsymbol{\psi}||_{L^2(\Omega; \mathbb{R}^3)} + ||\psi_1||_{L^2(0, L; H^1_0(\Omega'))}$$
$$+ ||\psi_2||_{L^2(0, L; H^1_0(\Omega'))} \le C||(\boldsymbol{w}_0, \boldsymbol{\psi})||_V \quad \forall (\boldsymbol{w}_0, \boldsymbol{\psi}) \in V,$$
$$||\boldsymbol{w}_0||_{L^2(\Omega \times Y; \mathbb{R}^3)} + ||\boldsymbol{\psi}||_{L^2(\Omega; \mathbb{R}^3)} \le C|(\boldsymbol{w}_0, \boldsymbol{\psi})|_H \quad \forall (\boldsymbol{w}_0, \boldsymbol{\psi}) \in H,$$

and the following implication, holding for any couple (E_1, E_2) of Banach spaces

$$\left. \begin{array}{l} A \in \mathcal{L}(E_1, E_2) \\ B \in C^k([0, T]; E_1) \end{array} \right\} \Rightarrow \left[A \circ B \in C^k([0, T]; E_2); \ \frac{d^s}{dt^s}(A \circ B) = A \circ \frac{d^s}{dt^s} B \ \forall s \le k \right],$$

applied with $B = \xi = (\boldsymbol{u}_0, \boldsymbol{v})$, $E_1 \in \{H, V\}$, $(A(\xi), E_2) \in \left\{ \left(\boldsymbol{u}_0, L^2(\Omega; H^1_\sharp(Y; \mathbb{R}^3)) \right), \right.$ $\left(\boldsymbol{v}, L^2(\Omega; \mathbb{R}^3) \right), \left(v_\alpha, L^2(0, L; H^1_0(\Omega')) \right), \left(\boldsymbol{u}_0, L^2(\Omega \times Y; \mathbb{R}^3) \right), \left(\boldsymbol{v}, L^2(\Omega) \right) \right\}$, that

$$
\begin{aligned}
&\boldsymbol{u}_0 \in C([0,T]; L^2(\Omega; H^1_\sharp(Y; \mathbb{R}^3))) \cap C^1([0,T]; L^2(\Omega \times Y; \mathbb{R}^3)), \\
&\boldsymbol{u}_0(0) = \boldsymbol{a}_0, \; \frac{\partial \boldsymbol{u}_0}{\partial t}(0) = \boldsymbol{b}_0, \\
&\boldsymbol{v} \in C^1([0,T]; L^2(\Omega; \mathbb{R}^3)), \; \boldsymbol{v}(0) = \boldsymbol{a}_0, \; \frac{\partial \boldsymbol{v}}{\partial t}(0) = \boldsymbol{b}_0, \\
&v_1, v_2 \in C([0,T]; L^2(0, L; H^1_0(\Omega'))) \cap C^1([0,T]; L^2(\Omega)).
\end{aligned}
\tag{6.31}
$$

Next we prove that the variational problem (6.26) is equivalent to (3.40). Setting $\boldsymbol{\psi} = 0$ in (6.26), noticing that $\boldsymbol{e}_y(\boldsymbol{u}_0) : \boldsymbol{\sigma}_{0y}(\boldsymbol{w}_0) = \boldsymbol{\sigma}_{0y}(\boldsymbol{u}_0) : \boldsymbol{\nabla}_y(\boldsymbol{w}_0)$, we get

$$
\begin{aligned}
&\int_{\Omega \times (0,T) \times Y \backslash A} \rho \boldsymbol{u}_0 \cdot \frac{\partial^2 \boldsymbol{w}_0}{\partial t^2} dxdtdy + \int_{\Omega \times Y \backslash A} \rho \boldsymbol{a}_0 \cdot \frac{\partial \boldsymbol{w}_0}{\partial t}(0) dxdy - \int_{\Omega \times Y \backslash A} \rho \boldsymbol{b}_0 \cdot \boldsymbol{w}_0(0) dxdy \\
&+ \int_{\Omega \times (0,T) \times Y \backslash A} \boldsymbol{\sigma}_{0y}(\boldsymbol{u}_0) : \boldsymbol{\nabla}_y(\boldsymbol{w}_0) dxdtdy = \int_{\Omega \times (0,T) \times Y \backslash A} \rho \boldsymbol{f} \cdot \boldsymbol{w}_0 dxdtdy,
\end{aligned}
\tag{6.32}
$$

and, letting $\boldsymbol{w}_0$ vary over $\mathcal{D}(\Omega \times (0,T) \times Y \backslash A; \mathbb{R}^3)$, deduce

$$
\rho \frac{\partial^2 \boldsymbol{u}_0}{\partial t^2} - \mathbf{div}_y(\boldsymbol{\sigma}_{0y}(\boldsymbol{u}_0)) = \rho \boldsymbol{f} \qquad \text{in} \; \Omega \times (0,T) \times Y \backslash A.
\tag{6.33}
$$

By integrating (6.32) by parts with respect to (t, y) for an arbitrary $\boldsymbol{w}_0$ satisfying (6.2), (6.3), (6.5), we infer from (6.33) that $\int_{\Omega \times (0,T) \times \partial Y} \boldsymbol{\sigma}_{0y}(\boldsymbol{u}_0) \boldsymbol{\nu} \cdot \boldsymbol{w}_0 dxdtd\mathcal{H}^2(y) = 0$ ($\boldsymbol{\nu} := $ outward pointing normal to ∂Y). Noticing that $\boldsymbol{\sigma}_{0y}(\boldsymbol{u}_0)\boldsymbol{\nu} = 0$ $\mathcal{H}^2$ a. e. on $\partial Y \cap \overline{A}$ (because if $\vartheta > 0$, then $A = B$ and, by (5.49), $\boldsymbol{\sigma}_{0y}(\boldsymbol{u}_0) = 0$ in B, whereas if $r_\varepsilon \ll \varepsilon$, then $A = \Sigma$ and $\mathcal{H}^2(\partial Y \cap \overline{\Sigma}) = 0$), we deduce

$$
\boldsymbol{\sigma}_{0y}(\boldsymbol{u}_0)\boldsymbol{\nu}(x, t, y) = -\boldsymbol{\sigma}_{0y}(\boldsymbol{u}_0)\boldsymbol{\nu}(x, t, -y) \quad \text{on} \quad \Omega \times (0,T) \times \partial Y.
\tag{6.34}
$$

Fixing $(\boldsymbol{w}_0, \boldsymbol{\psi}) \in L^2(0, T; H)$ satisfying (6.2), (6.3), we infer from the Y-periodicity of $\boldsymbol{w}_0$, (6.5), and (6.34), that (see (3.42))

$$
\begin{aligned}
-\int_{\Omega \times (0,T) \times \partial(Y \backslash A)} \boldsymbol{\sigma}_{0y}(\boldsymbol{u}_0)\boldsymbol{\nu} \cdot \boldsymbol{w}_0 dxdtd\mathcal{H}^2(y) &= -\int_{\Omega \times (0,T) \times \partial(Y \backslash A) \cap \overline{A}} \boldsymbol{\sigma}_{0y}(\boldsymbol{u}_0)\boldsymbol{\nu}_{Y \backslash A} \cdot \boldsymbol{\psi} dxdtd\mathcal{H}^2(y) \\
&= \int_{\Omega \times (0,T)} \boldsymbol{g}(\boldsymbol{u}_0) \cdot \boldsymbol{\psi} dxdt.
\end{aligned}
\tag{6.35}
$$

By multiplying (6.33) by $\boldsymbol{w}_0$ and by integrating it by parts over $\Omega \times (0,T) \times Y \backslash A$, thanks to (6.31), (6.34), (6.35) we obtain

$$
\begin{aligned}
&\int_{\Omega \times (0,T) \times Y \backslash A} \rho \, \boldsymbol{u}_0 \cdot \frac{\partial^2 \boldsymbol{w}_0}{\partial t^2} dxdtdy + \int_{\Omega \times Y \backslash A} \rho \, \boldsymbol{a}_0 \cdot \frac{\partial \boldsymbol{w}_0}{\partial t}(0) dxdy - \int_{\Omega \times Y \backslash A} \rho \, \boldsymbol{b}_0 \cdot \boldsymbol{w}_0(0) dxdy \\
&+ \int_{\Omega \times (0,T) \times Y \backslash A} \boldsymbol{e}_y(\boldsymbol{u}_0) : \boldsymbol{\sigma}_{0y}(\boldsymbol{w}_0) dxdtdy + \int_{\Omega \times (0,T)} \boldsymbol{g}(\boldsymbol{u}_0) \cdot \boldsymbol{\psi} dxdt = \int_{\Omega \times (0,T) \times Y \backslash A} \rho \boldsymbol{f} \cdot \boldsymbol{w}_0 dxdtdy.
\end{aligned}
\tag{6.36}
$$

By subtracting (6.36) from (6.26), we find

$$
\int_{\Omega \times (0,T)} \overline{\rho}_1 \boldsymbol{v} \cdot \frac{\partial^2 \boldsymbol{\psi}}{\partial t^2} dxdt - \int_{\Omega \times (0,T)} \boldsymbol{g}(\boldsymbol{u}_0) \cdot \boldsymbol{\psi} dxdt + k \int_{\Omega \times (0,T)} \boldsymbol{e}_{x'}(\boldsymbol{v}') : \boldsymbol{\sigma}_{x'}(\boldsymbol{\psi}') dxdt
$$

(6.37)

$$
+ \int_{\Omega} \overline{\rho}_1 \boldsymbol{a}_0 \cdot \frac{\partial \boldsymbol{\psi}}{\partial t}(0) dx - \int_{\Omega} \overline{\rho}_1 \boldsymbol{b}_0 \cdot \boldsymbol{\psi}(0) dx = \int_{\Omega \times (0,T)} \overline{\rho}_1 \boldsymbol{f} \cdot \boldsymbol{\psi} dxdt.
$$

Making $\boldsymbol{\psi}$ vary in $\mathcal{D}(\Omega \times (0,T); \mathbb{R}^3)$, we infer

(6.38)
$$
\overline{\rho}_1 \frac{\partial^2 \boldsymbol{v}}{\partial t^2} - k \mathbf{div} \boldsymbol{\sigma}_{x'}(\boldsymbol{v}') = \overline{\rho}_1 \boldsymbol{f} + \boldsymbol{g}(\boldsymbol{u}_0) \quad \text{in} \quad \Omega \times (0,T).
$$

By (6.31), (6.33), (6.34), (6.38), and Lemma 5.5, the couple $(\boldsymbol{u}_0, \boldsymbol{v})$ is a solution to (3.40), (3.45). Conversely, any solution to (3.40), (3.45) satisfies (6.26).
Case $k = +\infty$, $\kappa = 0$. We obtain

$$
\int_{\Omega \times (0,T) \times Y \setminus A} \rho \boldsymbol{u}_0 \cdot \frac{\partial^2 \boldsymbol{w}_0}{\partial t^2} dxdtdy + \int_{\Omega \times Y \setminus A} \rho \boldsymbol{a}_0 \cdot \frac{\partial \boldsymbol{w}_0}{\partial t}(0) - \rho \boldsymbol{b}_0 \cdot \boldsymbol{w}_0(0) dxdy
$$

(6.39)
$$
+ \int_{\Omega \times (0,T)} \overline{\rho}_1 v_3 \frac{\partial^2 \psi_3}{\partial t^2} dxdt + \int_{\Omega} \overline{\rho}_1 a_{03} \frac{\partial \psi_3}{\partial t}(0) - \overline{\rho}_1 b_{03} \psi_{03}(0) dx
$$

$$
+ \int_{\Omega \times (0,T) \times (Y \setminus A)} \boldsymbol{e}_y(\boldsymbol{u}_0) : \boldsymbol{\sigma}_{0y}(\boldsymbol{w}_0) dxdtdy = \int_{\Omega \times (0,T) \times Y \setminus A} \rho \boldsymbol{f} \cdot \boldsymbol{w}_0 dxdtdy + \int_{\Omega \times (0,T)} \overline{\rho}_1 f_3 \psi_3 dxdt.
$$

This variational formulation is satisfied for all $(\boldsymbol{w}_0, \boldsymbol{\psi}) \in L^2(0,T;H)$ verifying (6.2)-(6.5). We set (H and V being given by (6.1), (6.27))

(6.40)
$$
\begin{aligned}
&\xi = (\boldsymbol{u}_0, \boldsymbol{v}), \quad H^{(2)} := \{(\boldsymbol{w}_0, \boldsymbol{\psi}) \in H, \ \psi_1 = \psi_2 = 0\}, \\
&(.,.)_{H^{(2)}} := (.,.)_H, \quad V^{(2)} := V \cap H^{(2)}, \quad (.,.)_{V^{(2)}} := (.,.)_V, \\
&h^{(2)} := \left(\boldsymbol{f} \mathbb{1}_{Y \setminus A} + f_3 \boldsymbol{e}_3 \mathbb{1}_A, f_3 \boldsymbol{e}_3\right), \\
&a^{(2)}((\boldsymbol{u}_0, \boldsymbol{v}), (\boldsymbol{w}_0, \boldsymbol{\psi})) := \int_{\Omega \times Y \setminus A} \boldsymbol{e}_y(\boldsymbol{u}_0) : \boldsymbol{\sigma}_y(\boldsymbol{w}_0) dxdy, \\
&\xi_0^{(2)} := (\boldsymbol{a}_0 \mathbb{1}_{Y \setminus A} + a_{03} \boldsymbol{e}_3 \mathbb{1}_A, a_{03} \boldsymbol{e}_3), \quad \xi_1^{(2)} := (\boldsymbol{b}_0 \mathbb{1}_{Y \setminus A} + b_{03} \boldsymbol{e}_3 \mathbb{1}_A, b_{03} \boldsymbol{e}_3).
\end{aligned}
$$

By (4.8), (5.10), (5.48) and (5.80), we have $\xi \in L^2(0,T;V^{(2)})$ and $\xi' \in L^2(0,T;H^{(2)})$. Therefore, by a density argument, the variational problem (6.39) is equivalent to (4.23). By (6.28), (6.40), the estimate (4.18) is satisfied. We deduce from Theorem 4.4 that $\xi = (\boldsymbol{u}_0, \boldsymbol{v})$ is the unique solution to (6.39) and that $\xi \in C([0,T];V^{(2)}) \cap C^1([0,T];H^{(2)})$, $\xi(0) = \xi_0^{(2)}$, $\frac{\partial \xi}{\partial t}(0) = \xi_1^{(2)}$. Then, repeating the argument employed to prove (6.31), we infer from (6.30) and (6.40) that the initial-boundary conditions and regularity properties stated in (3.44), (3.46) are satisfied. Setting $\psi_3 = 0$ in (6.39), we get (6.32) and deduce (6.33), (6.34), (6.35), (6.36). Then, subtracting (6.36) from (6.39), taking (6.4) into account, we find

$$
\int_{\Omega \times (0,T)} \overline{\rho}_1 v_3 \frac{\partial^2 \psi_3}{\partial t^2} dxdt - \int_{\Omega \times (0,T)} (\boldsymbol{g}(\boldsymbol{u}_0))_3 \psi_3 dxdt + \int_{\Omega} \overline{\rho}_1 a_{03} \frac{\partial \psi_3}{\partial t}(0) dx
$$

$$
- \int_{\Omega} \overline{\rho}_1 b_{03} \psi_3(0) dx = \int_{\Omega \times (0,T)} \overline{\rho}_1 f_3 \psi_3 dxdt.
$$

Making ψ_3 vary in $\mathcal{D}(\Omega \times (0,T))$, we deduce that $\overline{\rho}_1 \frac{\partial^2 v_3}{\partial t^2} = \overline{\rho}_1 f_3 + (\boldsymbol{g}(\boldsymbol{u}_0))_3$ in $\Omega \times (0,T)$ and infer that $(\boldsymbol{u}_0, \boldsymbol{v})$ is solution to (3.40), (3.46).

Case $0 < \kappa < +\infty$. Passing to the limit as $\varepsilon \to 0$ in (6.16), we obtain

$$
(6.41)
\begin{aligned}
&\int_{\Omega \times (0,T) \times Y \backslash A} \rho \boldsymbol{u}_0 \cdot \frac{\partial^2 \boldsymbol{w}_0}{\partial t^2} dx dt dy + \int_{\Omega \times Y \backslash A} \rho \boldsymbol{a}_0 \cdot \frac{\partial \boldsymbol{w}_0}{\partial t}(0) - \rho \boldsymbol{b}_0 \cdot \boldsymbol{w}_0(0) dx dy \\
&+ \int_{\Omega \times (0,T)} \overline{\rho}_1 v_3 \frac{\partial^2 \psi_3}{\partial t^2} dx dt + \int_{\Omega} \overline{\rho}_1 a_{03} \frac{\partial \psi_3}{\partial t}(0) - \overline{\rho}_1 b_{03} \psi_{03}(0) dx \\
&+ \int_{\Omega \times (0,T) \times (Y \backslash A)} \boldsymbol{e}_y(\boldsymbol{u}_0) : \boldsymbol{\sigma}_{0y}(\boldsymbol{w}_0) dx dt dy + \frac{\kappa}{6} \int_{\Omega \times (0,T)} \boldsymbol{H}(v_3) : \boldsymbol{H}^{\boldsymbol{\sigma}}(\psi_3) dx dt \\
&= \int_{\Omega \times (0,T) \times Y \backslash A} \rho \boldsymbol{f} \cdot \boldsymbol{w}_0 dx dt dy + \int_{\Omega \times (0,T)} \overline{\rho}_1 f_3 \psi_3 dx dt,
\end{aligned}
$$

for all $(\boldsymbol{w}_0, \boldsymbol{\psi}) \in L^2(0,T;H)$ verifying (6.2)-(6.5). We set (see (6.27), (6.40), (7.6))

$$
(6.42)
\begin{aligned}
&H^{(3)} := H^{(2)}, \\
&V^{(3)} := \left\{ (\boldsymbol{w}_0, \psi_3 \boldsymbol{e}_3) \in V^{(2)} \left| \begin{array}{l} \psi_3 \in L^2(0,L;H_0^2(\Omega')) \\ \boldsymbol{w}_0(x,y) = \psi_3(x) \boldsymbol{e}_3 \text{ on } \Omega \times \Sigma \end{array} \right. \right\}, \\
&(((\boldsymbol{u}_0, v_3 \boldsymbol{e}_3), (\boldsymbol{w}_0, \psi_3 \boldsymbol{e}_3)))_{V^{(3)}} := (((\boldsymbol{u}_0, v_3 \boldsymbol{e}_3), (\boldsymbol{w}_0, \psi_3 \boldsymbol{e}_3)))_V \\
&\quad + \int_{\Omega} \left(\frac{\partial^2 v_3}{\partial x_1^2} \frac{\partial^2 \psi_3}{\partial x_1^2} + \frac{\partial^2 v_3}{\partial x_2^2} \frac{\partial^2 \psi_3}{\partial x_2^2} + \frac{\partial^2 v_3}{\partial x_1^2} \frac{\partial^2 \psi_3}{\partial x_2^2} + \frac{\partial^2 v_3}{\partial x_2^2} \frac{\partial^2 \psi_3}{\partial x_1^2} \right) dx,
\end{aligned}
$$

$$
(6.43)
\begin{aligned}
&\overline{a}^{(3)}(v_3 \boldsymbol{e}_3, \psi_3 \boldsymbol{e}_3)) := \frac{\kappa}{6} \int_{\Omega} \boldsymbol{H}(v_3) : \boldsymbol{H}^{\boldsymbol{\sigma}}(\psi_3) dx dt, \\
&a^{(3)}((\boldsymbol{u}_0, \boldsymbol{v}), (\boldsymbol{w}_0, \boldsymbol{\psi})) := \int_{\Omega \times (Y \backslash B)} \boldsymbol{e}_y(\boldsymbol{u}_0) : \boldsymbol{\sigma}_y(\boldsymbol{w}_0) dx dy + \overline{a}^{(3)}(\boldsymbol{v}, \boldsymbol{\psi}), \\
&\xi_0^{(3)} := \xi_0^{(2)}, \quad \xi_1^{(3)} := \xi_1^{(2)}, \quad h^{(3)} := h^{(2)}.
\end{aligned}
$$

By Corollary 5.6 (ii) and assertions (4.8), (5.10), (5.48), and (5.80), there holds $\xi = (\boldsymbol{u}_0, \boldsymbol{v}) \in L^2(0,T;V^{(3)})$ and $\frac{\partial \xi}{\partial t} \in L^2(0,T;H^{(3)})$ hence, by a density argument, the variational formulation (6.41) is equivalent to (4.23). By (6.27), (6.28), (6.40), (6.42), (6.43), and (7.6), for all $\widetilde{\xi} = (\boldsymbol{w}_0, \boldsymbol{\psi}) \in V^{(3)}$, we have

$$
\begin{aligned}
||\widetilde{\xi}||^2_{V^{(3)}} &\leq ||\widetilde{\xi}||^2_V + C\overline{a}^{(3)}(\boldsymbol{\psi}, \boldsymbol{\psi}) \leq C(|\widetilde{\xi}|_H + a(\widetilde{\xi}, \widetilde{\xi}) + \overline{a}^{(3)}(\boldsymbol{\psi}, \boldsymbol{\psi})) \\
&\leq C(|\widetilde{\xi}|_{H^{(3)}} + a^{(3)}(\widetilde{\xi}, \widetilde{\xi})),
\end{aligned}
$$

hence Assumption (4.18) is satisfied. We deduce from Theorem 4.4 that $\xi = (\boldsymbol{u}_0, \boldsymbol{v})$ is the unique solution to (6.41) and that $\xi \in C([0,T];V^{(3)}) \cap C^1([0,T];H^{(3)})$, $\xi(0) = \xi_0^{(3)}$, $\frac{\partial \xi}{\partial t}(0) = \xi_1^{(3)}$, yielding, by the inequality (6.30) joined with

$$
||\psi_3||_{L^2(0,L;H_0^2(\Omega'))} \leq C ||(\boldsymbol{w}_0, \boldsymbol{\psi})||_{V^{(3)}}, \forall (\boldsymbol{w}_0, \boldsymbol{\psi}) \in V^{(3)},
$$

the initial-boundary conditions and regularity properties stated in (3.44), (3.47). Repeating the argument for the case $0 < k < +\infty$, we set $\psi_3 = 0$ in (6.41), obtain (6.32), deduce (6.33), (6.34), (6.35), (6.36), subtract (6.36) from (6.41), and get

$$
\begin{aligned}
&\int_{\Omega\times(0,T)}\overline{\rho}_1 v_3\frac{\partial^2\psi_3}{\partial t^2}dxdt + \frac{\kappa}{6}\int_{\Omega\times(0,T)}\boldsymbol{H}(v_3):\boldsymbol{H^\sigma}(\psi_3)dxdt\\
&\quad -\int_{\Omega\times(0,T)}(\boldsymbol{g}(\boldsymbol{u}_0))_3\boldsymbol{\psi}_3 dxdt - \int_\Omega \overline{\rho}_1(\boldsymbol{b}_0)_3\psi_3(x,0)dx = \int_{\Omega\times(0,T)}\overline{\rho}_1 f_3\psi_3 dxdt.
\end{aligned}
$$
(6.44)

By (7.6), the following equation holds in the sense of distributions in $\mathcal{D}'(\Omega\times(0,T))$

$$
\frac{\kappa}{6}\langle\boldsymbol{H}(v_3):\boldsymbol{H^\sigma}(\psi_3)\rangle_{\mathcal{D}',\mathcal{D}} = \frac{\kappa}{3}\frac{l+1}{l+2}\left\langle\sum_{\alpha,\beta=1}^{2}\frac{\partial^4 v_3}{\partial x_\alpha^2\partial x_\beta^2},\psi_3\right\rangle_{\mathcal{D}',\mathcal{D}}.
$$

Making ψ_3 vary in $\mathcal{D}(\Omega\times(0,T))$ in (6.44), we infer

$$
\overline{\rho}_1\frac{\partial^2 v_3}{\partial t^2} + \frac{\kappa}{3}\frac{l+1}{l+2}\sum_{\alpha,\beta=1}^{2}\frac{\partial^4 v_3}{\partial x_\alpha^2\partial x_\beta^2} = \overline{\rho}_1 f_3 + (\boldsymbol{g}(\boldsymbol{u}_0))_3,\quad\text{in }\ \Omega\times(0,T),
$$

and deduce that $(\boldsymbol{u}_0,\boldsymbol{v})$ satisfies (3.40), (3.47).

Case $\kappa = +\infty$. By (7.5) we have $I_{3\varepsilon} = 0$. By passing to the limit as $\varepsilon\to 0$ in (6.16), we obtain (6.32) and, taking (5.80) into account, deduce in a similar manner that $(\boldsymbol{u}_0,\boldsymbol{v})$ satisfies (3.40), (3.48). The proof of Theorem 3.12 is achieved.

6.2. Proofs of theorems 3.1 and 3.7. Under the assumptions of Theorem 3.1, by (3.12) and (5.5), the sequence $(\boldsymbol{u}_\varepsilon)$ (resp. $\left(\frac{\partial\boldsymbol{u}_\varepsilon}{\partial t}\right)$) is bounded in $L^\infty(0,T;H_0^1(\Omega;\mathbb{R}^3))$ (resp. $L^\infty(0,T;L^2(\Omega;\mathbb{R}^3))$), therefore by the Aubin-Lions-Simon lemma (see [51, Corollary 6]), $(\boldsymbol{u}_\varepsilon)$ strongly converges in $L^2(0,T;L^2(\Omega;\mathbb{R}^3))$ and weakly* converges in $L^\infty(0,T;H_0^1(\Omega;\mathbb{R}^3))$, up to a subsequence, to some $\boldsymbol{u}\in L^\infty(0,T;H_0^1(\Omega;\mathbb{R}^3))$. In particular, assumption (5.14) of Lemma 5.3 is satisfied, hence $n\boldsymbol{u} = n\boldsymbol{v}$.

Under the assumptions of Theorem 3.7, by the apriori estimates (5.5), the sequence $(\boldsymbol{u}_\varepsilon)$ is bounded in $L^\infty(0,T;L^2(\Omega;\mathbb{R}^3))$, hence weakly* converges in $L^\infty(0,T;L^2(\Omega;\mathbb{R}^3))$, up to a subsequence, to some $\boldsymbol{u}\in L^\infty(0,T;L^2(\Omega;\mathbb{R}^3))$. By (3.26) and (3.27), Assumption (5.13) of Lemma 5.3 is satisfied, thus we also get $n\boldsymbol{u} = n\boldsymbol{v}$. Applying Corollary 5.6, we deduce in both cases from (5.11), (5.31), (5.80), and (7.5), that

$$
\begin{aligned}
&\boldsymbol{u}_\varepsilon m_\varepsilon \stackrel{\star}{\rightharpoonup} n\boldsymbol{u} &&\text{weakly* in }\ L^\infty(0,T;\mathcal{M}(\overline{\Omega};\mathbb{R}^3)),\\
&u_1,u_2 \in L^\infty(0,T;L_n^2(0,L;H_0^1(\Omega'))),\\
&\boldsymbol{e}_{x'}(\boldsymbol{u}'_\varepsilon)m_\varepsilon \stackrel{\star}{\rightharpoonup} n\boldsymbol{e}_{x'}(\boldsymbol{u}') &&\text{weakly* in }\ L^\infty(0,T;\mathcal{M}(\overline{\Omega};\mathbb{S}^3)),\\
&nu_1 = nu_2 = 0, &&\text{if }\ k = +\infty,\\
&u_3 \in L^\infty(0,T;L_n^2(0,L;H_0^2(\Omega'))) &&\text{if }\ \kappa > 0,\\
&n\boldsymbol{u} = 0 &&\text{if }\ \kappa = +\infty,\\
&\mathcal{I}_{n,k,\kappa}(\boldsymbol{v},\boldsymbol{\psi}) = \mathcal{I}_{n,k,\kappa}(\boldsymbol{u},\boldsymbol{\psi}).
\end{aligned}
$$
(6.45)

Let us check that

$$
\mathbb{1}_{\Omega\setminus B_\varepsilon} \stackrel{\star}{\rightharpoonup} 1 - \vartheta n\ \ \text{weakly* in }\ L^\infty(\Omega),
$$
(6.46)

where ϑ is defined by (3.10). If $\vartheta = 0$, (6.46) follows from the fact that $|B_\varepsilon| \to 0$. Otherwise, if $\vartheta > 0$, then the sequence $(\frac{\varepsilon}{r_\varepsilon}\mathbb{1}_{B_\varepsilon})$ is bounded in $L^\infty(\Omega)$ and, by (4.7), weakly* converges in $L^\infty(\Omega)$ to n. It then follows from (3.10) that $(\mathbb{1}_{B_\varepsilon})$ weakly* converges in $L^\infty(\Omega)$ to ϑn, yielding (6.46). Next, we check that

$$(6.47) \qquad \boldsymbol{u}_\varepsilon \mathbb{1}_{\Omega \setminus B_\varepsilon} \rightharpoonup \boldsymbol{u}(1 - \vartheta n) \quad \text{weakly in} \quad L^2(\Omega \times (0,T); \mathbb{R}^3).$$

If $\vartheta = 0$, Assertion (6.47) follows from the weak convergence of $(\boldsymbol{u}_\varepsilon)$ to $\boldsymbol{u}$ in $L^2(\Omega \times (0,T))$ and the convergence of $\mathcal{L}^3(B_\varepsilon)$ to 0. Otherwise, if $\vartheta > 0$, then $(\frac{\varepsilon}{r_\varepsilon}\boldsymbol{u}_\varepsilon \mathbb{1}_{B_\varepsilon})$ is bounded in $L^2(\Omega \times (0,T))$, and weakly converges, by (6.45), to $n\boldsymbol{u}$. Hence, by (3.10), $(\boldsymbol{u}_\varepsilon \mathbb{1}_{B_\varepsilon})$ weakly converges to $n\vartheta\boldsymbol{u}$, yielding (6.47).
We fix a field $\boldsymbol{\psi}$ verifying (6.2), (6.3), and

$$(6.48) \qquad n\psi_1 = n\psi_2 = 0 \quad \text{if} \quad k = +\infty; \quad n\boldsymbol{\psi} = 0 \quad \text{if} \quad \kappa = +\infty.$$

The sequence of test fields $(\boldsymbol{\phi}_\varepsilon)$ is defined by substituting $\boldsymbol{\psi}$ for $\boldsymbol{w}_0$ in (6.11), that is

$$(6.49) \qquad \boldsymbol{\phi}_\varepsilon(x,t) := \eta_\varepsilon(x)\widehat{\boldsymbol{\psi}}_\varepsilon(x,t) + (1 - \eta_\varepsilon(x))\boldsymbol{\psi}(x,t),$$

where $\widehat{\boldsymbol{\psi}}_\varepsilon(x,t)$ is described in Section 7, and η_ε satisfies (6.10), now with respect to the non-periodic sets B_ε, C_ε given by (3.4), (6.7). We assume that (see Remark 6.1)

$$(6.50) \qquad \begin{aligned} &\frac{r_\varepsilon}{\varepsilon} \ll \alpha_\varepsilon \ll 1 \quad \text{under the assumptions of Theorem 3.1,} \\ &\mu_{0\varepsilon} \ll \alpha_\varepsilon \ll 1 \quad \text{under the assumptions of Theorem 3.7.} \end{aligned}$$

By (6.10), (6.49), and (7.4), the following estimates hold in $\Omega \times (0,T)$ for $m \in \{1,2\}$:

$$(6.51) \qquad \left|\boldsymbol{\phi}_\varepsilon(x,t) - \boldsymbol{\psi}\left(x,t,\frac{x}{\varepsilon}\right)\right| + \left|\frac{\partial^m \boldsymbol{\phi}_\varepsilon}{\partial t^m}(x,t) - \frac{\partial^m \boldsymbol{\psi}}{\partial t^m}\left(x,t,\frac{x}{\varepsilon}\right)\right| \le Cr_\varepsilon.$$

We deduce from (6.8) and from the estimate $\boldsymbol{\sigma}_\varepsilon(\boldsymbol{\phi}_\varepsilon(x,t)) \le C\frac{\mu_{0\varepsilon}}{\alpha_\varepsilon}$ in $C_\varepsilon \setminus B_\varepsilon \times (0,T)$, obtained in a similar manner as (6.14), that

$$(6.52) \qquad \int_{C_\varepsilon \setminus B_\varepsilon \times (0,T)} |\boldsymbol{\sigma}_\varepsilon(\boldsymbol{\phi}_\varepsilon(x,t))|^2 \, dxdt \le C\frac{\mu_{\varepsilon0}^2 r_\varepsilon}{\alpha_\varepsilon \varepsilon}.$$

We multiply Equation (3.5) by $\boldsymbol{\phi}_\varepsilon$ and integrate it by parts to get (see (6.16), (6.20))

$$(6.53) \qquad \begin{aligned} &\int_{\Omega \times (0,T)} \rho_\varepsilon \boldsymbol{u}_\varepsilon \cdot \frac{\partial^2 \boldsymbol{\phi}_\varepsilon}{\partial t^2} dxdt + \int_\Omega \rho_\varepsilon \boldsymbol{a}_0 \cdot \frac{\partial \boldsymbol{\phi}_\varepsilon}{\partial t}(0)dx - \int_\Omega \rho_\varepsilon \boldsymbol{b}_0 \cdot \boldsymbol{\phi}_\varepsilon(0)dx \\ &\qquad\qquad + I_{1\varepsilon} + I_{2\varepsilon} + I_{3\varepsilon} = \int_{\Omega \times (0,T)} \rho_\varepsilon \boldsymbol{f} \cdot \boldsymbol{\phi}_\varepsilon dxdt. \end{aligned}$$

By the same argument as the one used to get (6.18), (6.19), splitting each term as in (6.17) and taking into account (3.7), (4.7), (6.46), (6.47), (6.51), we obtain

$$\lim_{\varepsilon \to 0} \int_{\Omega \times (0,T)} \rho_\varepsilon \boldsymbol{u}_\varepsilon \cdot \frac{\partial^2 \boldsymbol{\phi}_\varepsilon}{\partial t^2} dx dt = \int_{\Omega \times (0,T)} \rho(1 - \vartheta n) \boldsymbol{u} \cdot \frac{\partial^2 \boldsymbol{\psi}}{\partial t^2} dx dt + \int_{\Omega \times (0,T)} \overline{\rho}_1 \boldsymbol{u} \cdot \frac{\partial^2 \boldsymbol{\psi}}{\partial t^2} n dx dt,$$

(6.54)

$$\lim_{\varepsilon \to 0} \int_\Omega \rho_\varepsilon \boldsymbol{a}_0 \cdot \frac{\partial \boldsymbol{\phi}_\varepsilon}{\partial t}(0) dx = \int_\Omega \rho(1 - \vartheta n) \boldsymbol{a}_0 \cdot \frac{\partial \boldsymbol{\psi}}{\partial t}(0) dx + \int_\Omega \overline{\rho}_1 \boldsymbol{a}_0 \cdot \frac{\partial^2 \boldsymbol{\psi}}{\partial t^2}(0) n dx dt,$$

$$\lim_{\varepsilon \to 0} \int_\Omega \rho_\varepsilon \boldsymbol{b}_0 \cdot \boldsymbol{\phi}_\varepsilon(0) dx = \int_\Omega \rho(1 - \vartheta n) \boldsymbol{b}_0 \cdot \boldsymbol{\psi}(0) dx + \int_\Omega \overline{\rho}_1 \boldsymbol{b}_0 \cdot \boldsymbol{\psi}(0) n dx dt,$$

$$\lim_{\varepsilon \to 0} \int_{\Omega \times (0,T)} \rho_\varepsilon \boldsymbol{f} \cdot \boldsymbol{\phi}_\varepsilon dx dt = \int_{\Omega \times (0,T)} \rho(1 - \vartheta n) \boldsymbol{f} \cdot \boldsymbol{\psi} dx dt + \int_{\Omega \times (0,T)} \overline{\rho}_1 \boldsymbol{f} \cdot \frac{\partial^2 \boldsymbol{\psi}}{\partial t^2} n dx dt.$$

Under the assumptions of Theorem 3.1, by (3.12), (3.13) and (6.49) we have $\boldsymbol{\sigma}_\varepsilon(\boldsymbol{\phi}_\varepsilon) = \boldsymbol{\sigma}(\boldsymbol{\psi})$ in $\Omega \backslash C_\varepsilon \times (0,T)$, and by (3.11) and (6.7), $\lim_{\varepsilon \to 0} |C_\varepsilon| = 0$, therefore the sequence $(\boldsymbol{\sigma}_\varepsilon(\boldsymbol{\phi}_\varepsilon) \mathbb{1}_{\Omega \backslash C_\varepsilon})$ strongly converges to $\boldsymbol{\sigma}(\boldsymbol{\psi})$ in $L^2(\Omega \times (0,T); \mathbb{S}^3))$. We deduce from the weak* convergence of $(\boldsymbol{u}_\varepsilon)$ to $\boldsymbol{u}$ in $L^\infty(0,T; H_0^1(\Omega; \mathbb{R}^3))$ that

(6.55)
$$\lim_{\varepsilon \to 0} I_{1\varepsilon} = \int_{\Omega \times (0,T)} \boldsymbol{e}(\boldsymbol{u}) : \boldsymbol{\sigma}(\boldsymbol{\psi}) dx dt.$$

Under the assumptions of Theorem 3.7, noticing that $|\boldsymbol{\sigma}_\varepsilon(\boldsymbol{\phi}_\varepsilon) \mathbb{1}_{\Omega \backslash C_\varepsilon}| = |\boldsymbol{\sigma}_\varepsilon(\boldsymbol{\psi}) \mathbb{1}_{\Omega \backslash C_\varepsilon}| \leq C\mu_{0\varepsilon}$ and taking (3.26), (5.5), (6.20) into account, we get

(6.56)
$$\limsup_{\varepsilon \to 0} I_{1\varepsilon} \leq \limsup_{\varepsilon \to 0} C\mu_{0\varepsilon}^{\frac{1}{2}} \left(\int_{\Omega \times (0,T)} \mu_{0\varepsilon} |\boldsymbol{e}(\boldsymbol{u}_\varepsilon)(\tau)|^2 dx dt \right)^{\frac{1}{2}} = 0.$$

By (5.5), and (6.52), we have

$$I_{2\varepsilon} \leq \left(\int_{\Omega \times (0,T)} |\boldsymbol{e}(\boldsymbol{u}_\varepsilon)|^2 dx dt \right)^{\frac{1}{2}} \left(\int_{C_\varepsilon \backslash B_\varepsilon \times (0,T)} |\boldsymbol{\sigma}_\varepsilon(\boldsymbol{\phi}_\varepsilon(x,t))|^2 dx dt \right)^{\frac{1}{2}} \leq C \left(\frac{\mu_{0\varepsilon}}{\alpha_\varepsilon} \frac{r_\varepsilon}{\varepsilon} \right)^{\frac{1}{2}}.$$

Under the assumptions of Theorem 3.1 (resp. Theorem 3.7), we deduce from (3.12) and (6.50) that

(6.57)
$$\lim_{\varepsilon \to 0} I_{2\varepsilon} = 0.$$

Collecting (6.54), (6.55), (6.56), (6.57), and (7.5), by passing to the limit as $\varepsilon \to 0$ in (6.16), we obtain, under the assumptions of Theorem 3.1,

(6.58)
$$\int_{\Omega \times (0,T)} (\rho + \overline{\rho}_1 n) \boldsymbol{u} \cdot \frac{\partial^2 \boldsymbol{\psi}}{\partial t^2} dx dt + \int_{\Omega \times (0,T)} (\rho + \overline{\rho}_1 n) \left(\boldsymbol{a}_0 \cdot \frac{\partial \boldsymbol{\psi}}{\partial t}(0) - \boldsymbol{b}_0 \cdot \boldsymbol{\psi}(0) \right) dx$$
$$+ \int_{\Omega \times (0,T)} \boldsymbol{e}(\boldsymbol{u}) : \boldsymbol{\sigma}(\boldsymbol{\psi}) dx dt + \mathcal{I}_{n,k,\kappa}(\boldsymbol{u}, \boldsymbol{\psi}) = \int_{\Omega \times (0,T)} (\rho + \overline{\rho}_1 n) \boldsymbol{f} \cdot \boldsymbol{u} dx dt,$$

and, under the assumptions of Theorem 3.7,

$$\int_{\Omega\times(0,T)} (\rho(1-\vartheta n) + \overline{\rho}_1 n)\boldsymbol{u} \cdot \frac{\partial^2 \boldsymbol{\psi}}{\partial t^2} dxdt$$

$$(6.59) \qquad + \int_{\Omega\times(0,T)} (\rho(1-\vartheta n) + \overline{\rho}_1 n)\left(\boldsymbol{a}_0 \cdot \frac{\partial \boldsymbol{\psi}}{\partial t}(0) - \boldsymbol{b}_0 \cdot \boldsymbol{\psi}(0)\right) dx + \mathcal{I}_{n,k,\kappa}(\boldsymbol{u}, \boldsymbol{\psi})$$

$$= \int_{\Omega\times(0,T)} (\rho(1-\vartheta n) + \overline{\rho}_1 n)\boldsymbol{f} \cdot \boldsymbol{u}dxdt.$$

The variational formulation (6.59), joined with (6.45), is equivalent to (4.23), where

$$(6.60) \qquad H_{n,k,\kappa} = \left\{ \boldsymbol{\psi} \in L^2(\Omega; \mathbb{R}^3) \,\middle|\, \begin{array}{ll} n\psi_1 = n\psi_2 = 0 & \text{if } k = +\infty; \\ n\boldsymbol{\psi} = 0 & \text{if } \kappa = +\infty \end{array} \right\};$$

$$V_{n,k,\kappa} = \left\{ \boldsymbol{\psi} \in H_{n,k,\kappa} \,\middle|\, \begin{array}{ll} \psi_1, \psi_2 \in L_n^2(0,L; H_0^1(\Omega')) & \text{if } 0 < k \\ \psi_3 \in L_n^2(0,L; H_0^2(\Omega')) & \text{if } 0 < \kappa, \end{array} \right\};$$

$$(\boldsymbol{u}, \boldsymbol{\psi})_{H_{n,k,\kappa}} = \int_\Omega (\rho(1-\vartheta n) + \overline{\rho}_1 n)\boldsymbol{u} \cdot \boldsymbol{\psi}dx;$$

$$((\boldsymbol{u}, \boldsymbol{\psi}))_{V_{n,k,\kappa}} = (\boldsymbol{u}, \boldsymbol{\psi})_{H_{n,k,\kappa}} + \mathcal{I}_{n,k,\kappa}(\boldsymbol{u}, \boldsymbol{\psi}); \quad a_{n,k,\kappa}(\boldsymbol{u}, \boldsymbol{\psi}) = \mathcal{I}_{n,k,\kappa}(\boldsymbol{u}, \boldsymbol{\psi});$$

$$h_{n,k,\kappa} = \mathcal{H}_{n,k,\kappa}(\boldsymbol{f}), \ \xi_{0,n,k,\kappa} = \mathcal{H}_{n,k,\kappa}(\boldsymbol{a}_0), \ \xi_{1,n,k,\kappa} = \mathcal{H}_{n,k,\kappa}(\boldsymbol{b}_0),$$

$$(6.61) \qquad \mathcal{H}_{n,k,\kappa}(\boldsymbol{g}) := \begin{cases} \boldsymbol{g} & \text{if } 0 < k < +\infty, \\ (g_1 \mathbb{1}_{\{n=0\}}, g_2 \mathbb{1}_{\{n=0\}}, g_3), & \text{if } k = +\infty, \ \kappa < +\infty, \\ \boldsymbol{g} \mathbb{1}_{\{n=0\}} & \text{if } \kappa = +\infty. \end{cases}$$

The variational formulation (6.58), joined with (6.45), is equivalent to (4.23), with data deduced from (6.60), (6.61) by substituting $\widetilde{V}_{n,k,\kappa}$ and $\widetilde{a}_{n,k,\kappa}$ for $V_{n,k,\kappa}$ and $a_{n,k,\kappa}$, where

$$(6.62) \qquad \widetilde{V}_{n,k,\kappa} := V_{n,k,\kappa} \cap H_0^1(\Omega; \mathbb{R}^3); \ ((\boldsymbol{u}, \boldsymbol{\psi}))_{\widetilde{V}_{n,k,\kappa}} = ((\boldsymbol{u}, \boldsymbol{\psi}))_{V_{n,k,\kappa}} + \int_\Omega \nabla \boldsymbol{u} \cdot \nabla \boldsymbol{\psi}dx,$$

$$\widetilde{a}_{n,k,\kappa}(\boldsymbol{u}, \boldsymbol{\psi}) := a_{n,k,\kappa}(\boldsymbol{u}, \boldsymbol{\psi}) + \int_\Omega \boldsymbol{e}(\boldsymbol{u}) : \boldsymbol{\sigma}(\boldsymbol{\psi})dx.$$

The assumptions of Theorem 4.4 are satisfied in both cases, guaranteeing existence, uniqueness and regularity properties of the solution. Finally, by integrations by parts, it is easy to check that the variational problems (6.58), (6.59), associated with (6.45), are equivalent, under (3.17), to the problems announced in theorems 3.1, 3.7.

REMARK 6.1. *The assumption stated in the first line of (6.50) is employed to derive (6.25) and requires (3.11). The case $\mu_{0\varepsilon} = \mu > 0$, $\vartheta > 0$, $k = +\infty$ is open.*

REMARK 6.2 (Multiphase case). *Theorems 3.1, 3.7 can be extended to the case of m distributions $B_\varepsilon^{[s]}$ ($s \in \{1,..,m\}$) of parallel disjoint homothetical layers of thickness $r_\varepsilon^{[s]}$, Lamé coefficients $\lambda_{1\varepsilon}^{[s]}, \mu_{1\varepsilon}^{[s]}$, and mass density $\frac{\varepsilon}{r_\varepsilon^{[s]}}\overline{\rho}_1^{[s]}$, defined in terms of a finite subset $\omega_\varepsilon^{[s]}$ of $(0,L)$ and $r_\varepsilon^{[s]}$ by a formula like (3.4). The sets $\omega_\varepsilon^{[s]}$ are disjoint and their union $\omega_\varepsilon := \bigcup_{s=1}^m \omega_\varepsilon^{[s]}$ satisfies (3.2), which implies that the minimal distance between*

two distincts points of ω_ε is equal to ε. We suppose that $\varepsilon > r_\varepsilon^{[s]}(1+\delta)$, $\forall s \in \{1,..,m\}$ for some $\delta > 0$ and set $\vartheta^{[s]} := \lim_{\varepsilon \to 0} \frac{r_\varepsilon^{[s]}}{\varepsilon}$. The Lamé coefficients in $\Omega \setminus \bigcup_{s=1}^m B_\varepsilon^{[s]}$ are assume to be constant and denoted by $\lambda_{0\varepsilon}$, $\mu_{0\varepsilon}$.

When $\lambda_{0\varepsilon}$, $\mu_{0\varepsilon}$ satisfy (3.26) and each sequence $(n_\varepsilon^{[s]})$ strongly converges to $n^{[s]}$ in $L^1(\Omega)$, the solution to (3.5) weakly* converges in $L^\infty(0,T;L^2(\Omega;\mathbb{R}^3))$ to the unique solution to the problem (4.23), where the data are deduced from (6.60) as follows:

$$H := \bigcap_{s=1}^m H_{n^{[s]},k^{[s]},\kappa^{[s]}}; \quad (\boldsymbol{u},\boldsymbol{\psi})_H = \int_\Omega (\rho(1 - \sum_{s=1}^m \vartheta^{[s]} n^{[s]}) + \overline{\rho}_1^{[s]} n^{[s]})\boldsymbol{u}\cdot\boldsymbol{\psi} dx$$

$$V := \bigcap_{s=1}^m V_{n^{[s]},k^{[s]},\kappa^{[s]}}; \quad ((\boldsymbol{u},\boldsymbol{\psi}))_V = (\boldsymbol{u},\boldsymbol{\psi})_H + \sum_{s=1}^m \mathcal{I}_{n^{[s]},k^{[s]},\kappa^{[s]}}(\boldsymbol{u},\boldsymbol{\psi}),$$

(6.63)
$$a(\boldsymbol{u},\boldsymbol{\psi}) = \sum_{s=1}^m a_{n^{[s]},k^{[s]},\kappa^{[s]}}(\boldsymbol{u},\boldsymbol{\psi}),$$

$$h = \mathcal{H}(\boldsymbol{f}), \;\; \xi_0 = \mathcal{H}(\boldsymbol{a}_0), \;\; \xi_1 = \mathcal{H}(\boldsymbol{b}_0),$$

$$(\mathcal{H}(\boldsymbol{g})(x))_\alpha = \begin{cases} 0 & \text{if } \exists s \in \{1,..,m\},\, n^{[s]}(x) > 0 \;\; \text{and } k^{[s]} = +\infty, \\ g_\alpha & \text{otherwise}, \qquad (\alpha \in \{1,2\}), \end{cases}$$

$$(\mathcal{H}(\boldsymbol{g})(x))_3 = \begin{cases} 0 & \text{if } \exists s \in \{1,..,m\},\, n^{[s]}(x) > 0 \;\; \text{and } \kappa^{[s]} = +\infty, \\ g_3(x) & \text{otherwise}. \end{cases}$$

When $\lambda_{0\varepsilon}$, $\mu_{0\varepsilon}$ satisfy (3.12), and when $\vartheta^{[s]} = 0$ for each $s \in \{1,..,m\}$, the solution to (3.5) weakly* converges in $L^\infty(0,T; H_0^1(\Omega;\mathbb{R}^3))$ to the unique solution to (4.23), with data $\widetilde{H}$, $\widetilde{V}$, $\widetilde{a}$... deduced from H, V, a... defined in (6.63) as follows:

(6.64)
$$\widetilde{H} := H; \quad \widetilde{V} = V \cap H_0^1(\Omega;\mathbb{R}^3); \quad ((\boldsymbol{u},\boldsymbol{\psi}))_{\widetilde{V}} := ((\boldsymbol{u},\boldsymbol{\psi}))_V + \int_\Omega \boldsymbol{\nabla}\boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{\psi} dx;$$

$$\widetilde{a}(\boldsymbol{u},\boldsymbol{\psi}) = a(\boldsymbol{u},\boldsymbol{\psi}) + \int_\Omega \boldsymbol{e}(\boldsymbol{u}):\boldsymbol{\sigma}(\boldsymbol{\psi}) dx; \quad (\widetilde{h},\widetilde{\xi}_0,\widetilde{\xi}_1) := (h,\xi_0,\xi_1).$$

REMARK 6.3 (Elliptic case). When $\lambda_{0\varepsilon}$, $\mu_{0\varepsilon}$ satisfy (3.12), and when $\vartheta^{[s]} = 0$ for each $s \in \{1,..,m\}$, the solution $\boldsymbol{u}_\varepsilon$ to the equilibrium problem (3.32) is bounded in $H_0^1(\Omega;\mathbb{R}^3)$ and weakly converges to the unique field $\boldsymbol{u} \in \widetilde{V}$ satisfying $\widetilde{a}(\boldsymbol{u},\boldsymbol{\psi}) = (\boldsymbol{f},\boldsymbol{\psi})_{\widetilde{H}}$, $\forall\boldsymbol{\psi} \in \widetilde{V}$, where $\widetilde{V}$ is the Hilbert space and $\widetilde{a}(.,.)$ the continuous coercive bilinear form on $\widetilde{V}$ given by (6.64).

If $\lambda_{0\varepsilon}$, $\mu_{0\varepsilon}$ satisfy (3.26), each sequence $(n_\varepsilon^{[s]})$ strongly converges to $n^{[s]}$ in $L^2(\Omega)$, and $\boldsymbol{u}_\varepsilon$ is bounded in $L^2(\Omega;\mathbb{R}^3)$, then $\boldsymbol{u}_\varepsilon$ weakly converges, up to a subsequence, to some $\boldsymbol{u} \in V$ verifying $a(\boldsymbol{u},\boldsymbol{\psi}) = (\boldsymbol{f},\boldsymbol{\psi})_H$ $\forall\boldsymbol{\psi} \in V$, with $H,V,a(.,.)$ defined by (6.63). In this case, the non-negative bilinear form $a(.,.)$ may fail to be coercive on $L^2(\Omega;\mathbb{R}^3)$ and the sequence $\boldsymbol{u}_\varepsilon$ to be bounded in $L^2(\Omega;\mathbb{R}^3)$. These coercivity and boundedness are guaranteed by the existence of $s \in \{1,..,m\}$ and $c > 0$ such that $\kappa^{[s]} > 0$ and $n_\varepsilon^{[s]} \geq c$ a.e. in $\Omega_\varepsilon := \bigcup_{i\in Z_\varepsilon^{[s]}} \left(\varepsilon i - \frac{\varepsilon}{2}, \varepsilon i + \frac{\varepsilon}{2}\right]$ (see (3.15)). (Notice that if the second assumption in (3.2) is replaced by $\min_{j,j'\in J_\varepsilon, j\neq j'} |\omega_\varepsilon^j - \omega_\varepsilon^{j'}| = \eta\varepsilon$ for some arbitrarily fixed $\eta \in \left(0,\frac{1}{2}\right)$, our proofs are unchanged and $n_\varepsilon^{[s]} \geq c\mathbb{1}_{\Omega_\varepsilon}$ does not imply that $B_\varepsilon^{[s]}$ is ε-periodic).

Sketch of the proof. Let s be such that $\kappa^{[s]} > 0$. The bilinear form associated with (3.32), namely $a_\varepsilon(\boldsymbol{\varphi}, \boldsymbol{\psi}) = \int_\Omega \boldsymbol{e}(\boldsymbol{\varphi}) : \boldsymbol{\sigma}_\varepsilon(\boldsymbol{\psi}) dx \ \forall (\boldsymbol{\varphi}, \boldsymbol{\psi}) \in \left(H_0^1(\Omega; \mathbb{R}^3) \right)^2$, satisfies, by (3.5), (3.8), and (3.26)

$$(6.65) \qquad a_\varepsilon(\boldsymbol{\varphi}, \boldsymbol{\varphi}) \geq C \int_\Omega \varepsilon^2 \left| \boldsymbol{e}(\boldsymbol{\varphi}) \right|^2 dx + C \int_\Omega \left| \frac{1}{r_\varepsilon^{[s]}} \boldsymbol{e}(\boldsymbol{\varphi}) \right|^2 dm_\varepsilon^{[s]}.$$

Let $\boldsymbol{u}_\varepsilon$ be a sequence in $H_0^1(\Omega; \mathbb{R}^3)$, and let $m_\varepsilon^{[s]}$, $\hat{\boldsymbol{v}}_\varepsilon^{[s]}$, $\overline{\boldsymbol{v}}_\varepsilon^{[s]}$ be defined by substituting $\omega_\varepsilon^{[s]}$ for ω_ε in (3.39), (5.17), (5.20). We have, since $n_\varepsilon^{[s]} \geq c \mathbb{1}_{\Omega_\varepsilon}$,

$$\int_\Omega |\boldsymbol{u}_\varepsilon|^2 dx \leq \int_{\Omega \setminus \Omega_\varepsilon} |\boldsymbol{u}_\varepsilon|^2 dx + C \int_{\Omega_\varepsilon} \left| n_\varepsilon^{[s]} \boldsymbol{u}_\varepsilon - \overline{\boldsymbol{v}}_\varepsilon^{[s]} \right|^2 dx + C \int_\Omega \left| \overline{\boldsymbol{v}}_\varepsilon^{[s]} \right|^2 dx.$$

Looking back at (5.28), *and using the fact that* $\boldsymbol{u}_\varepsilon$ *vanishes on* $\partial\Omega$, *we obtain*

$$\int_{\Omega \setminus \Omega_\varepsilon} |\boldsymbol{u}_\varepsilon|^2 dx + C \int_{\Omega_\varepsilon} \left| n_\varepsilon^{[s]} \boldsymbol{u}_\varepsilon - \overline{\boldsymbol{v}}_\varepsilon^{[s]} \right|^2 dx$$

$$\leq C\varepsilon^2 \int_{\Omega \setminus \Omega_\varepsilon} \left| \frac{\partial \boldsymbol{u}_\varepsilon}{\partial x_3} \right|^2 (\tau) dx + C\varepsilon^2 \sum_{i \in Z_\varepsilon^{[s]}} \int_{\Omega' \times \left(\varepsilon i - \frac{\varepsilon}{2}, \varepsilon i + \frac{\varepsilon}{2} \right]} \left| \frac{\partial \boldsymbol{u}_\varepsilon}{\partial x_3} \right|^2 (\tau) dx$$

$$\leq C\varepsilon^2 \int_\Omega |\boldsymbol{\nabla} \boldsymbol{u}_\varepsilon|^2 dx \leq C\varepsilon^2 \int_\Omega |\boldsymbol{e}(\boldsymbol{u}_\varepsilon)|^2 dx,$$

yielding $\int_\Omega |\boldsymbol{u}_\varepsilon|^2 dx \leq C\varepsilon^2 \int_\Omega |\boldsymbol{e}(\boldsymbol{u}_\varepsilon)|^2 dx + C \int_\Omega \left| \overline{\boldsymbol{v}}_\varepsilon^{[s]} \right|^2 dx$. *On the other hand, by* (5.22), (5.18), (5.1),

$$\int_\Omega \left| \overline{\boldsymbol{v}}_\varepsilon^{[s]} \right|^2 dx = \int_\Omega \left| \hat{\boldsymbol{v}}_\varepsilon^{[s]} \right|^2 dm_\varepsilon^{[s]} \leq C \int_\Omega \left| \boldsymbol{u}_\varepsilon - \hat{\boldsymbol{v}}_\varepsilon^{[s]} \right|^2 dm_\varepsilon^{[s]} + C \int_\Omega |\boldsymbol{u}_\varepsilon|^2 dm_\varepsilon^{[s]}$$

$$\leq C\varepsilon r_\varepsilon^{[s]} \int_\Omega |\boldsymbol{e}(\boldsymbol{u}_\varepsilon)|^2 dm_\varepsilon^{[s]} + C \int_\Omega \left| \frac{1}{r_\varepsilon^{[s]}} \boldsymbol{e}(\boldsymbol{u}_\varepsilon) \right|^2 dm_\varepsilon^{[s]},$$

therefore, for all $\boldsymbol{u}_\varepsilon \in H_0^1(\Omega; \mathbb{R}^3)$, $\int_\Omega |\boldsymbol{u}_\varepsilon|^2 dx \leq Ca_\varepsilon(\boldsymbol{u}_\varepsilon, \boldsymbol{u}_\varepsilon)$. *In the particular case when* $\boldsymbol{u}_\varepsilon$ *is the solution to* (3.32), *we infer* $\int_\Omega |\boldsymbol{u}_\varepsilon|^2 dx \leq C \int_\Omega \boldsymbol{f} \cdot \boldsymbol{u}_\varepsilon dx \leq C \left(\int_\Omega |\boldsymbol{u}_\varepsilon|^2 dx \right)^{\frac{1}{2}}$, *hence* $(\boldsymbol{u}_\varepsilon)$ *is bounded in* $L^2(\Omega; \mathbb{R}^3)$. *We choose a smooth field* $\boldsymbol{\psi} \in V$ *and consider the associated sequence of test field* $\boldsymbol{\phi}_\varepsilon$ *used for the proof of the multiphase case, whose construction is similar to* (6.49). *Repeating the argument of [8, p. 40, (iii) $\Rightarrow$ (i)], we find that* $a(\boldsymbol{\psi}, \boldsymbol{\psi}) = \lim_{\varepsilon \to 0} a_\varepsilon(\boldsymbol{\phi}_\varepsilon, \boldsymbol{\phi}_\varepsilon) \geq c \lim_{\varepsilon \to 0} \int_\Omega |\boldsymbol{\phi}_\varepsilon|^2 dx = \int_\Omega |\boldsymbol{\psi}|^2 dx$. *By a density argument, we deduce* $\int_\Omega |\boldsymbol{\psi}|^2 dx \leq Ca(\boldsymbol{\psi}, \boldsymbol{\psi}) \ \forall \boldsymbol{\psi} \in V$.

7. Appendix. A common step in the proofs of theorems 3.1, 3.7, and 3.12 lies in the computation of the limit of the sequence $(I_{3\varepsilon})$ defined by (see (6.11), (6.20))

$$(7.1) \qquad I_{3\varepsilon} := \int_{B_\varepsilon \times (0,T)} \boldsymbol{e}(\boldsymbol{u}_\varepsilon) : \boldsymbol{\sigma}_\varepsilon(\widehat{\boldsymbol{\psi}_\varepsilon}) dx dt,$$

where $\boldsymbol{u}_\varepsilon$ is the solution to (3.5) and the oscillating test fields $\widehat{\boldsymbol{\psi}}_\varepsilon$ is defined bellow, in terms of $\boldsymbol{\psi} \in C^\infty([0,T]; \mathcal{D}(\Omega; \mathbb{R}^3))$ satisfying (6.4), of δ given by (3.3), and of the order of magnitude of the parameters k and κ. We introduce the field $\overline{\boldsymbol{\psi}}_\varepsilon$ given by

$$(7.2) \quad \overline{\boldsymbol{\psi}}_\varepsilon(x,t) := \sum_{j \in J_\varepsilon} \left(\fint_{\left(\omega_\varepsilon^j - \frac{r_\varepsilon}{2}, \omega_\varepsilon^j + \frac{r_\varepsilon}{2}\right)} \boldsymbol{\psi}(x_1, x_2, s_3, t) ds_3 \right) \mathbb{1}_{\left(\omega_\varepsilon^j - \frac{r_\varepsilon(1+\delta)}{2}, \omega_\varepsilon^j + \frac{r_\varepsilon(1+\delta)}{2}\right)}(x_3).$$

(i) If $0 < k \le +\infty$ and $\kappa = 0$, we set

$$(7.3) \quad \begin{aligned} \widehat{\boldsymbol{\psi}}_\varepsilon(x,t) &:= \overline{\boldsymbol{\psi}}_\varepsilon(x,t) + r_\varepsilon \boldsymbol{w}_{1\varepsilon}\left(x, t, \frac{y_\varepsilon(x_3)}{r_\varepsilon}\right), \\ \boldsymbol{w}_{1\varepsilon}(x,t,y_3) &:= \begin{pmatrix} -\frac{\partial \overline{\psi}_{\varepsilon 3}}{\partial x_1} y_3 \\ -\frac{\partial \overline{\psi}_{\varepsilon 3}}{\partial x_2} y_3 \\ \frac{-l_\varepsilon}{l_\varepsilon + 2}\left(\frac{\partial \overline{\psi}_{\varepsilon 1}}{\partial x_1} + \frac{\partial \overline{\psi}_{\varepsilon 2}}{\partial x_2}\right) y_3 \end{pmatrix}. \end{aligned}$$

where the function $y_\varepsilon(.)$ is defined by (4.9).
(ii) If $0 < \kappa \le +\infty$, we set

$$\begin{aligned} \widehat{\boldsymbol{\psi}}_\varepsilon(x,t) &:= \overline{\boldsymbol{\psi}}_\varepsilon(x,t) + r_\varepsilon \boldsymbol{w}_{1\varepsilon}\left(x, t, \frac{y_\varepsilon(x_3)}{r_\varepsilon}\right) + r_\varepsilon^2 \boldsymbol{w}_{2\varepsilon}\left(x, t, \frac{y_\varepsilon(x_3)}{r_\varepsilon}\right), \\ \boldsymbol{w}_{1\varepsilon}(x,t,y_3) &:= -\frac{\partial \overline{\psi}_{\varepsilon 3}}{\partial x_1} y_3 \boldsymbol{e}_1 - \frac{\partial \overline{\psi}_{\varepsilon 3}}{\partial x_2} y_3 \boldsymbol{e}_2, \\ \boldsymbol{w}_{2\varepsilon}(x,t,y_3) &:= \frac{l_\varepsilon}{2(l_\varepsilon + 2)}\left(\frac{\partial^2 \overline{\psi}_{\varepsilon 3}}{\partial x_1^2} + \frac{\partial^2 \overline{\psi}_{\varepsilon 3}}{\partial x_2^2}\right) y_3^2 \boldsymbol{e}_3. \end{aligned}$$

It is usefull to notice that $\widehat{\boldsymbol{\psi}}_\varepsilon$ is continuously differentiable in $C_\varepsilon \times (0,T)$ (see (6.7)), that $\widehat{\boldsymbol{\psi}}_\varepsilon = 0$ if $\kappa = +\infty$ (because then, by (6.4), $\boldsymbol{\psi} = 0$) and that for $m \in \{1,2\}$,

$$(7.4) \quad \begin{aligned} &\left(\left| \widehat{\boldsymbol{\psi}}_\varepsilon(x,t) - \boldsymbol{\psi}(x,t) \right| + \left| \frac{\partial^m \widehat{\boldsymbol{\psi}}_\varepsilon}{\partial t^m}(x,t) - \frac{\partial^m \boldsymbol{\psi}}{\partial t^m}(x,t) \right| \right) \mathbb{1}_{C_\varepsilon}(x) \le Cr_\varepsilon, \\ &\left| \nabla \widehat{\boldsymbol{\psi}}_\varepsilon(x,t) \right| \mathbb{1}_{C_\varepsilon} \le C. \end{aligned}$$

LEMMA 7.1. *Let $\boldsymbol{u}_\varepsilon$ be the solution to (3.5). Let $I_{3\varepsilon}$ be defined by (7.1) in terms of $\widehat{\boldsymbol{\psi}}_\varepsilon$ described above. Then,*

$$(7.5) \quad \lim_{\varepsilon \to 0} I_{3\varepsilon} = \mathcal{I}_{n,k,\kappa}(\boldsymbol{v}, \boldsymbol{\psi}) := \begin{cases} k \displaystyle\int_{\Omega \times (0,T)} \boldsymbol{e}_{x'}(\boldsymbol{v'}) : \boldsymbol{\sigma}_{x'}(\boldsymbol{\psi'}) n \, dx dt & \text{if } 0 < k < +\infty, \\ \dfrac{\kappa}{6} \displaystyle\int_{\Omega \times (0,T)} \boldsymbol{H}(v_3) : \boldsymbol{H^\sigma}(\psi_3) n \, dx dt & \text{if } 0 < \kappa < +\infty, \\ 0 & \text{if } (k,\kappa) = (+\infty, 0) \text{ or } \kappa = +\infty, \end{cases}$$

the operators $\boldsymbol{H}$, $\boldsymbol{H^\sigma}$ being defined by

$$\boldsymbol{H}(\psi) := \begin{pmatrix} \frac{\partial^2 \psi}{\partial x_1^2} & \frac{\partial^2 \psi}{\partial x_1 \partial x_2} & 0 \\ \frac{\partial^2 \psi}{\partial x_1 \partial x_2} & \frac{\partial^2 \psi}{\partial x_2^2} & 0 \\ 0 & 0 & 0 \end{pmatrix};$$

(7.6)

$$\boldsymbol{H^\sigma}(\psi) := \begin{pmatrix} 2\frac{l+1}{l+2}\frac{\partial^2 \psi}{\partial x_1^2} + \frac{l}{l+2}\frac{\partial^2 \psi}{\partial x_2^2} & \frac{\partial^2 \psi}{\partial x_1 \partial x_2} & 0 \\ \frac{\partial^2 \psi}{\partial x_1 \partial x_2} & \frac{l}{l+2}\frac{\partial^2 \psi}{\partial x_1^2} + 2\frac{l+1}{l+2}\frac{\partial^2 \psi}{\partial x_2^2} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Proof. **Case** $0 < k < +\infty$. We easily check that

(7.7)
$$\left\|\boldsymbol{\psi} - \overline{\boldsymbol{\psi}}_\varepsilon\right\|_{L^\infty(B_\varepsilon \times (0,T))} + \left\|\frac{\partial(\boldsymbol{\psi} - \overline{\boldsymbol{\psi}}_\varepsilon)}{\partial x_\alpha}\right\|_{L^\infty(B_\varepsilon \times (0,T))} \leq Cr_\varepsilon \quad (\alpha \in \{1,2\}),$$
$$\left\|\frac{\partial^2(\boldsymbol{\psi} - \overline{\boldsymbol{\psi}}_\varepsilon)}{\partial x_\alpha \partial x_\beta}\right\|_{L^\infty(B_\varepsilon \times (0,T))} \leq Cr_\varepsilon \quad (\alpha, \beta \in \{1,2\}).$$

A straightforward computation yields (see (3.5), (3.6))

$$\boldsymbol{\sigma}_\varepsilon(\widehat{\boldsymbol{\psi}}_\varepsilon)\mathbb{1}_{B_\varepsilon} =$$

(7.8)
$$\mu_{1\varepsilon}\begin{pmatrix} 2\frac{\partial\overline{\psi}_{\varepsilon 1}}{\partial x_1} + \frac{2l_\varepsilon}{l_\varepsilon+2}\left(\frac{\partial\overline{\psi}_{\varepsilon 1}}{\partial x_1} + \frac{\partial\overline{\psi}_{\varepsilon 2}}{\partial x_2}\right) & \frac{\partial\overline{\psi}_{\varepsilon 1}}{\partial x_2} + \frac{\partial\overline{\psi}_{\varepsilon 2}}{\partial x_1} & 0 \\ \frac{\partial\overline{\psi}_{\varepsilon 1}}{\partial x_2} + \frac{\partial\overline{\psi}_{\varepsilon 2}}{\partial x_1} & 2\frac{\partial\overline{\psi}_{\varepsilon 2}}{\partial x_2} + \frac{2l_\varepsilon}{l_\varepsilon+2}\left(\frac{\partial\overline{\psi}_{\varepsilon 1}}{\partial x_1} + \frac{\partial\overline{\psi}_{\varepsilon 2}}{\partial x_2}\right) & 0 \\ 0 & 0 & 0 \end{pmatrix}\mathbb{1}_{B_\varepsilon}$$
$$+ r_\varepsilon\mu_{1\varepsilon}O(1).$$

Since $\frac{r_\varepsilon}{\varepsilon}\mu_{1\varepsilon} \to k \in (0,+\infty)$ and $l_\varepsilon \to l \in (0,\infty)$ (see (3.6)), we infer from (7.7) that

$$\lim_{\varepsilon \to 0}\left|\frac{r_\varepsilon}{\varepsilon}\boldsymbol{\sigma}_\varepsilon(\widehat{\boldsymbol{\psi}}_\varepsilon) - k\boldsymbol{\sigma}_{x'}(\boldsymbol{\psi}')(x,t)\right|_{L^\infty(B_\varepsilon \times (0,T))} = 0,$$

where $\boldsymbol{\sigma}_{x'}(\boldsymbol{\psi}')$ is given by (2.1). By Corollary 5.6 (i), the convergences (5.11) are verified, thus the sequence $(\boldsymbol{e}_{x'}(\boldsymbol{u}'_\varepsilon)m_\varepsilon)$ weakly* converges in $L^\infty(0,T;\mathcal{M}(\overline{\Omega};\mathbb{S}^3))$ to $n\boldsymbol{e}_{x'}(\boldsymbol{v}')$. Taking (3.39), (5.5), and (7.1) into account, we deduce that

$$\lim_{\varepsilon \to 0} I_{3\varepsilon} = \lim_{\varepsilon \to 0}\int_{\Omega \times (0,T)} k\boldsymbol{e}_{x'}(\boldsymbol{u}_\varepsilon) : \boldsymbol{\sigma}_{x'}(\psi')dm_\varepsilon dt = k\int_{\Omega \times (0,T)} \boldsymbol{e}_{x'}(\boldsymbol{v}') : \boldsymbol{\sigma}_{x'}(\boldsymbol{\psi}')n dx dt.$$

Case $(k,\kappa) = (+\infty, 0)$. By (6.4), (7.8), we have $|\boldsymbol{\sigma}_\varepsilon(\widehat{\boldsymbol{\psi}}_\varepsilon)\mathbb{1}_{B_\varepsilon}| \leq C\mu_{1\varepsilon}r_\varepsilon$, thus, by (3.8), the second line of (5.5), and (7.1), there holds

$$I_{3\varepsilon} \leq C\mu_{1\varepsilon}r_\varepsilon\int_{B_\varepsilon \times (0,T)} |\boldsymbol{e}(\boldsymbol{u}_\varepsilon)|dx dt = C\mu_{1\varepsilon}\frac{r_\varepsilon^2}{\varepsilon}\int_{B_\varepsilon \times (0,T)} |\boldsymbol{e}(\boldsymbol{u}_\varepsilon)|dm_\varepsilon dt$$
$$\leq C\mu_{1\varepsilon}\frac{r_\varepsilon^2}{\varepsilon}\sqrt{\int_{B_\varepsilon \times (0,T)} |\boldsymbol{e}(\boldsymbol{u}_\varepsilon)|^2 dm_\varepsilon dt} \leq C\mu_{1\varepsilon}\frac{r_\varepsilon^2}{\varepsilon}\sqrt{\frac{\varepsilon}{r_\varepsilon\mu_{\varepsilon 1}}} \leq C\sqrt{\frac{r_\varepsilon^3}{\varepsilon}\mu_{1\varepsilon}} = o(1).$$

Case $0 < \kappa < +\infty$. A straightforward computation gives

$$
\frac{r_\varepsilon^2}{\varepsilon}\boldsymbol{\sigma}_\varepsilon(\widehat{\boldsymbol{\psi}}_\varepsilon)\mathbb{1}_{B_\varepsilon} =
$$

$$
- 2\frac{r_\varepsilon^3}{\varepsilon}\mu_{1\varepsilon}\frac{y_\varepsilon(x_3)}{r_\varepsilon}\begin{pmatrix} \frac{2(l_\varepsilon+1)}{l_\varepsilon+2}\frac{\partial^2\overline{\psi}_{\varepsilon 3}}{\partial x_1^2} + \frac{l_\varepsilon}{l_\varepsilon+2}\frac{\partial^2\overline{\psi}_{\varepsilon 3}}{\partial x_2^2} & \frac{\partial^2\overline{\psi}_{\varepsilon 3}}{\partial x_1\partial x_2} & 0 \\ \frac{\partial^2\overline{\psi}_{\varepsilon 3}}{\partial x_1\partial x_2} & \frac{l_\varepsilon}{l_\varepsilon+2}\frac{\partial^2\overline{\psi}_{\varepsilon 3}}{\partial x_1^2} + \frac{2(l_\varepsilon+1)}{l_\varepsilon+2}\frac{\partial^2\overline{\psi}_{\varepsilon 3}}{\partial x_2^2} & 0 \\ 0 & 0 & 0 \end{pmatrix}\mathbb{1}_{B_\varepsilon}
$$

$$
+ r_\varepsilon O\left(\frac{r_\varepsilon^3}{\varepsilon}\mu_\varepsilon\right).
$$

We deduce from (3.6), (3.8), (7.6), and (7.7), that

$$
(7.9) \qquad \lim_{\varepsilon\to 0}\left|\frac{r_\varepsilon^2}{\varepsilon}\boldsymbol{\sigma}_\varepsilon(\widehat{\boldsymbol{\psi}}_\varepsilon) + 2\kappa \boldsymbol{H}^{\boldsymbol{\sigma}}(\psi_3)\frac{y_\varepsilon(x_3)}{r_\varepsilon}\right|_{L^\infty(B_\varepsilon\times(0,T);\mathbb{S}^3)} = 0.
$$

By (3.39) and (7.1), we have

$$
(7.10) \qquad I_{3\varepsilon} = \frac{r_\varepsilon}{\varepsilon}\int_{\Omega\times(0,T)}\boldsymbol{e}(\boldsymbol{u}_\varepsilon):\boldsymbol{\sigma}_\varepsilon(\widehat{\boldsymbol{\psi}}_\varepsilon)dm_\varepsilon dt = \int_{\Omega\times(0,T)}\frac{1}{r_\varepsilon}\boldsymbol{e}(\boldsymbol{u}_\varepsilon):\frac{r_\varepsilon^2}{\varepsilon}\boldsymbol{\sigma}_\varepsilon(\widehat{\boldsymbol{\psi}}_\varepsilon)dm_\varepsilon dt.
$$

Taking Corollary 5.6 (ii) into account, we infer from (5.5), (5.30), (5.32), (7.9), (7.10), that

$$
\lim_{\varepsilon\to 0}I_{3\varepsilon} = \lim_{\varepsilon\to 0} -\int_{\Omega\times(0,T)}\frac{1}{r_\varepsilon}\boldsymbol{e}_{x'}(\boldsymbol{u}_\varepsilon):2\kappa \boldsymbol{H}^{\boldsymbol{\sigma}}(\psi_3)\frac{y_\varepsilon(x_3)}{r_\varepsilon}dm_\varepsilon dt
$$

$$
= -2\kappa\sum_{\alpha,\beta=1}^{2}\int_{\Omega\times(0,T)\times I}(\boldsymbol{H}^{\boldsymbol{\sigma}}(\psi_3))_{\alpha\beta}\left(\frac{1}{2}\left(\frac{\partial\xi_\alpha}{\partial x_\beta}+\frac{\partial\xi_\beta}{\partial x_\alpha}\right)(x,t) - \frac{\partial^2 v_3}{\partial x_\alpha\partial x_\beta}(x,t)y_3\right)y_3 n dx dt dy_3
$$

$$
= \frac{\kappa}{6}\int_{\Omega\times(0,T)}\boldsymbol{H}(v_3):\boldsymbol{H}^{\boldsymbol{\sigma}}(\psi_3)n dx dt.
$$

□

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